

## REMARKS ON ISOMORPHISMS OF TRANSFORMATION SEMIGROUPS RESTRICTED BY AN EQUIVALENCE RELATION

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ABSTRACT. Let  $T(X)$  be the full transformation semigroup on a set  $X$  and  $\sigma$  be an equivalence relation on  $X$ . Denote

$$E(X, \sigma) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \sigma \text{ implies } x\alpha = y\alpha\}.$$

Then  $E(X, \sigma)$  is a subsemigroup of  $T(X)$ . In this paper, we characterize two semigroups of type  $E(X, \sigma)$  when they are isomorphic.

### 1. Introduction and preliminaries

Let  $X$  be an arbitrary nonempty set. The semigroup  $T(X)$  of all transformations on  $X$  consists of the mappings from  $X$  into itself with composition as the semigroup operation. In [4], H. Pei studied subsemigroups of  $T(X)$  determined by an equivalence relation  $\sigma$  on  $X$ , defined by:

$$T(X, \sigma) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \sigma\}.$$

It is clear that if  $\sigma \in \{\Delta(X), X \times X\}$ , where  $\Delta(X)$  is the identity relation on  $X$ , then  $T(X, \sigma) = T(X)$ . He also discussed regularity of elements and Green's relations for  $T(X, \sigma)$ . Recently, R. P. Sullivan and S. Mendes-Gonçalves introduced a subsemigroup of  $T(X)$  defined by

$$E(X, \sigma) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \sigma \text{ implies } x\alpha = y\alpha\}$$

and called it the *semigroup of transformations restricted by the equivalence  $\sigma$*  in [3]. Then  $E(X, \sigma)$  is a subsemigroup of  $T(X, \sigma)$ . The authors characterized Green's relations on the largest regular subsemigroup of  $E(X, \sigma)$ . They also showed that if  $|X| \geq 2$  and  $\sigma \neq \Delta(X)$ , then  $E(X, \sigma)$  is not isomorphic to  $T(Z)$  for any set  $Z$ .

We easily get the following proposition which is a characterization of  $E(X, \sigma)$ .

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**Proposition 1.1.** *Let  $\sigma$  be an equivalence relation on a set  $X$ . Then the following statements hold.*

- (1)  $id_X \in E(X, \sigma)$  if and only if  $\sigma = \Delta(X)$  where  $id_X$  is the identity mapping on  $X$ .
- (2) If  $\sigma$  and  $\rho$  are equivalence relations on  $X$  with  $\rho \subseteq \sigma$ , then  $E(X, \sigma) \subseteq E(X, \rho)$ .
- (3)  $E(X, \sigma) = T(X, \sigma)$  if and only if  $\sigma = \Delta(X)$ . If this is the case, then  $E(X, \sigma) = T(X)$ .

J. Sanwong and W. Sommanee [6] introduced and studied the subsemigroup

$$T(X, Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$$

of  $T(X)$  where  $\emptyset \neq Y \subseteq X$ . We establish an embedding theorem for the semigroup  $E(X, \sigma)$  into the semigroup  $T(Y, Z)$ .

**Proposition 1.2.** *Let  $\sigma$  be an equivalence relation on a set  $X$ . Every semigroup  $E(X, \sigma)$  is embeddable in a semigroup  $T(Y, Z)$  for some sets  $Y$  and  $Z$  with  $Z \subseteq Y$ .*

*Proof.* Let  $Y = \sigma$  and  $Z = \Delta(X)$ . Then  $Z \subseteq Y$ . For each  $\alpha \in E(X, \sigma)$ , we define  $\beta_\alpha \in T(Y)$  by

$$(x, y)\beta_\alpha = (x\alpha, y\alpha) \text{ for all } (x, y) \in Y.$$

Since  $\alpha \in E(X, \sigma)$ , it then follows that  $Y\beta_\alpha \subseteq Z$ . Hence  $\beta_\alpha \in T(Y, Z)$ . Define  $\phi : E(X, \sigma) \rightarrow T(Y, Z)$  by

$$\alpha\phi = \beta_\alpha \text{ for all } \alpha \in E(X, \sigma).$$

Let  $\alpha_1, \alpha_2 \in E(X, \sigma)$ . To show that  $\beta_{\alpha_1\alpha_2} = \beta_{\alpha_1}\beta_{\alpha_2}$ , let  $(x, y) \in Y$ . Then

$$(x, y)\beta_{\alpha_1\alpha_2} = (x\alpha_1\alpha_2, y\alpha_1\alpha_2) = (x\alpha_1, y\alpha_1)\beta_{\alpha_2} = (x, y)\beta_{\alpha_1}\beta_{\alpha_2}.$$

Hence  $\phi$  is a homomorphism. Suppose that  $\alpha_1\phi = \alpha_2\phi$ . Then  $\beta_{\alpha_1} = \beta_{\alpha_2}$ . If  $x \in X$ , then  $(x, x) \in Y$  and

$$(x\alpha_1, x\alpha_1) = (x, x)\beta_{\alpha_1} = (x, x)\beta_{\alpha_2} = (x\alpha_2, x\alpha_2).$$

Hence  $x\alpha_1 = x\alpha_2$  for all  $x \in X$  which implies that  $\phi$  is injective.

Therefore the theorem is proved.  $\square$

Over the past, isomorphism theorems for semigroups have been widely considered, see [1, 2, 5, 7]. The purpose of this paper is to find necessary and sufficient conditions for two transformation semigroups restricted by an equivalence in order to be isomorphic.

### 2. Main results

For the fixed equivalence relation  $\sigma$  on a set  $X$  and  $a \in X$ , we write  $a\sigma$  for the set of all elements of  $X$  that are equivalent to  $a$ , that is,  $a\sigma = \{x \in X : (a, x) \in \sigma\}$ .

To obtain the main result, the following two lemmas are needed.

**Lemma 2.1.** *Let  $\alpha \in E(X, \sigma)$ . Then  $\alpha$  is a right zero element of  $E(X, \sigma)$  if and only if  $\alpha$  is constant.*

*Proof.* It is clear that if  $\alpha$  is constant, then  $\beta\alpha = \alpha$  for all  $\beta \in E(X, \sigma)$ .

Suppose that  $\alpha$  is nonconstant. Then there exist distinct elements  $a, b \in X\alpha$ . Thus  $a'\alpha = a$  and  $b'\alpha = b$  for some  $a', b' \in X$ . Since  $\alpha \in E(X, \sigma)$  and  $a'\alpha \neq b'\alpha$ , we deduce that  $(a', b') \notin \sigma$ . Define  $\beta \in T(X)$  by

$$x\beta = \begin{cases} a', & \text{if } x \in b'\sigma, \\ b', & \text{otherwise} \end{cases}$$

for all  $x \in X$ . It is clear that  $\beta \in E(X, \sigma)$ . Since  $b'\beta\alpha = a'\alpha = a$  and  $b'\alpha = b$ , it follows that  $\beta\alpha \neq \alpha$ . Consequently,  $\alpha$  is not a right zero element of  $E(X, \sigma)$ .  $\square$

Hence the corollary is an immediate consequence of Lemma 2.1.

**Corollary 2.2.**  *$E(X, \sigma)$  is a right zero semigroup if and only if  $\sigma = X \times X$ .*

*Proof.* Suppose that  $\sigma \neq X \times X$ . Then there exist  $a, b \in X$  such that  $(a, b) \notin \sigma$ . Thus  $a \neq b$ . Define  $\alpha \in E(X, \sigma)$  by

$$x\alpha = \begin{cases} a, & \text{if } x \in a\sigma, \\ b, & \text{otherwise} \end{cases}$$

for all  $x \in X$ . Then  $\alpha$  is nonconstant in  $E(X, \sigma)$ . By Lemma 2.1,  $\alpha$  is not a right zero element of  $E(X, \sigma)$ .

Conversely, assume that  $\sigma = X \times X$ . Then the semigroup  $E(X, \sigma)$  consists of all constant mappings in  $T(X)$ . By Lemma 2.1,  $E(X, \sigma)$  is a right zero semigroup.  $\square$

**Lemma 2.3.** *Let  $\alpha_1, \alpha_2 \in E(X, \sigma)$  and  $a \in X$ . If  $a\alpha_1\beta = a\alpha_2\beta$  for all  $\beta \in E(X, \sigma)$ , then  $(a\alpha_1, a\alpha_2) \in \sigma$ .*

*Proof.* Suppose that  $(a\alpha_1, a\alpha_2) \notin \sigma$ . Then  $a\alpha_1 \neq a\alpha_2$ . Define  $\beta \in T(X)$  by

$$x\beta = \begin{cases} a\alpha_1, & \text{if } x \in (a\alpha_1)\sigma, \\ a\alpha_2, & \text{otherwise} \end{cases}$$

for all  $x \in X$ . It is easy to see that  $\beta \in E(X, \sigma)$  and  $a\alpha_1\beta \neq a\alpha_2\beta$ .  $\square$

From now on, suppose that  $\sigma_1$  and  $\sigma_2$  are equivalence relations on sets  $X$  and  $Y$ , respectively. In what follows,  $|A|$  means the cardinality of the set  $A$ .

**Theorem 2.4.**  *$E(X, \sigma_1)$  and  $E(Y, \sigma_2)$  are isomorphic as semigroups if and only if there exists a bijection  $\theta : X \rightarrow Y$  such that  $(x\sigma_1)\theta = (x\theta)\sigma_2$  for all  $x \in X$ .*

*Proof.* Assume that  $E(X, \sigma_1)$  and  $E(Y, \sigma_2)$  are isomorphic. Let  $\varphi : E(X, \sigma_1) \rightarrow E(Y, \sigma_2)$  be an isomorphism.

For each  $a \in X$ , we define  $\alpha_a \in E(X, \sigma_1)$  by  $x\alpha_a = a$  for all  $x \in X$ . By Lemma 2.1,  $\alpha_a$  is a right zero element of  $E(X, \sigma_1)$  and hence

$$\alpha_a\varphi = (\beta\alpha_a)\varphi = (\beta\varphi)(\alpha_a\varphi) \text{ for all } \beta \in E(X, \sigma_1).$$

Since  $\varphi$  is a bijection, we deduce that  $\alpha_a\varphi$  is a right zero element of  $E(Y, \sigma_2)$ . Then from Lemma 2.1, there exists a unique  $y_a \in Y$  such that  $y(\alpha_a\varphi) = y_a$  for all  $y \in Y$ .

Define  $\theta : X \rightarrow Y$  by

$$x\theta = y_x \text{ for all } x \in X.$$

Clearly,  $\theta$  is well-defined. Let  $x_1, x_2 \in X$  be such that  $x_1\theta = x_2\theta$ . Then  $y_{x_1} = y_{x_2}$  which implies that  $\alpha_{x_1}\varphi = \alpha_{x_2}\varphi$ . Since  $\varphi$  is injective, it follows that  $\alpha_{x_1} = \alpha_{x_2}$  and hence  $x_1 = x_2$ . This shows that  $\alpha$  is injective.

To show that  $\theta$  is surjective, let  $y \in Y$ . Then there exists  $\beta_y \in E(Y, \sigma_2)$  such that  $z\beta_y = y$  for all  $z \in Y$ . Since  $\varphi^{-1}$  is an isomorphism and  $\beta_y$  is a right zero of  $E(Y, \sigma_2)$ , it follows that  $\beta_y\varphi^{-1}$  is a right zero of  $E(X, \sigma_1)$ . Then there exists an element  $x' \in X$  such that  $w(\beta_y\varphi^{-1}) = x' = w\alpha_{x'}$  for all  $w \in X$ . Since  $\alpha_{x'}\varphi = \beta_y\varphi^{-1}\varphi = \beta_y$ , we have  $y_{x'} = y$ . Therefore  $x'\theta = y$  and whence  $\theta$  is surjective.

Finally, we will show that  $(x\sigma_1)\theta = (x\theta)\sigma_2$  for all  $x \in X$ . Let  $x \in X$  and  $a \in (x\sigma_1)\theta$ . Then  $a = b\theta$  for some  $b \in x\sigma_1$  and thus  $(x, b) \in \sigma_1$ . It follows that  $\alpha_x\beta = \alpha_b\beta$  for all  $\beta \in E(X, \sigma_1)$ . Since  $\varphi$  is a homomorphism,

$$(\alpha_x\varphi)(\beta\varphi) = (\alpha_x\beta)\varphi = (\alpha_b\beta)\varphi = (\alpha_b\varphi)(\beta\varphi)$$

for all  $\beta \in E(X, \sigma_1)$ . Since  $\varphi$  is a bijection, it follows that

$$(\alpha_x\varphi)\gamma = (\alpha_b\varphi)\gamma \text{ for all } \gamma \in E(Y, \sigma_2).$$

We note here that if  $y \in Y$ , then  $y(\alpha_x\varphi)\gamma = y(\alpha_b\varphi)\gamma$  for all  $\gamma \in E(Y, \sigma_2)$ . By Lemma 2.3, we obtain that  $(y(\alpha_x\varphi), y(\alpha_b\varphi)) \in \sigma_2$ . Since  $(y(\alpha_x\varphi), y(\alpha_b\varphi)) = (y_x, y_b) = (x\theta, b\theta) = (x\theta, a)$ , we deduce  $a \in (x\theta)\sigma_2$ . This proves that  $(x\sigma_1)\theta \subseteq (x\theta)\sigma_2$ . For the reverse inclusion, let  $c \in (x\theta)\sigma_2$ . Then  $(c, x\theta) \in \sigma_2$ . Since  $\theta$  is surjective,  $c = d\theta$  for some  $d \in X$ . It follows that  $(\alpha_x\varphi)\beta = (\alpha_d\varphi)\beta$  for all  $\beta \in E(Y, \sigma_2)$ . Since  $\varphi^{-1}$  is a homomorphism,

$$((\alpha_x\varphi)\varphi^{-1})(\beta\varphi^{-1}) = (\alpha_x\varphi\beta)\varphi^{-1} = (\alpha_d\varphi\beta)\varphi^{-1} = ((\alpha_d\varphi)\varphi^{-1})(\beta\varphi^{-1})$$

for all  $\beta \in E(Y, \sigma_2)$ . It follows from the bijection of  $\varphi^{-1}$  that

$$d\alpha_x\gamma = d(\alpha_x\varphi)\varphi^{-1}\gamma = d(\alpha_d\varphi)\varphi^{-1}\gamma = d\alpha_d\gamma$$

for all  $\gamma \in E(X, \sigma_1)$ . By Lemma 2.3, we deduce that  $(x, d) = (d\alpha_x, d\alpha_d) \in \sigma_1$ , thus  $d \in x\sigma_1$ . This means that  $c = d\theta \in (x\sigma_1)\theta$ . Hence  $(x\theta)\sigma_2 \subseteq (x\sigma_1)\theta$  and the equality holds.

Conversely, suppose that  $\theta : X \rightarrow Y$  is a bijection such that  $(x\sigma_1)\theta = (x\theta)\sigma_2$  for all  $x \in X$ . Define  $\varphi : E(X, \sigma_1) \rightarrow E(Y, \sigma_2)$  by

$$\alpha\varphi = \theta^{-1}\alpha\theta \text{ for all } \alpha \in E(X, \sigma_1).$$

Let  $\alpha \in E(X, \sigma_1)$ . To show that  $\alpha\varphi \in E(Y, \sigma_2)$ , let  $(x, y) \in \sigma_2$ . Since  $\theta$  is surjective, we have  $x'\theta = x$  and  $y'\theta = y$  for some  $x', y' \in X$ . By assumption, we then have  $y'\theta \in (x'\theta)\sigma_2 = (x'\sigma_1)\theta$  which implies that  $(y', x') \in \sigma_1$ . Since  $\alpha \in E(X, \sigma_1)$ , it follows that  $y'\alpha = x'\alpha$ . Therefore

$$x\alpha\varphi = x\theta^{-1}\alpha\theta = x'\alpha\theta = y'\alpha\theta = y\theta^{-1}\alpha\theta = y\alpha\varphi.$$

This shows that  $\alpha\varphi \in E(Y, \sigma_2)$ , whence  $\varphi$  is well-defined. Let  $\alpha_1, \alpha_2 \in E(X, \sigma_1)$ . We see that

$$\begin{aligned} (\alpha_1\alpha_2)\varphi &= \theta^{-1}(\alpha_1\alpha_2)\theta \\ &= (\theta^{-1}\alpha_1\theta)(\theta^{-1}\alpha_2\theta) \\ &= (\alpha_1\varphi)(\alpha_2\varphi). \end{aligned}$$

Therefore  $\varphi$  is a homomorphism. It is easy to verify that  $\varphi$  is bijective.

The theorem is thereby proven. □

**Corollary 2.5.** *For positive integers  $m$  and  $n$ , let  $X$  and  $Y$  be sets such that  $|X| = |Y| = n$  and  $|X/\sigma_1| = |Y/\sigma_2| = m$ . If  $m \in \{1, n-1, n\}$ , then  $E(X, \sigma_1) \cong E(Y, \sigma_2)$ .*

*Proof.* Suppose that  $m \in \{1, n-1, n\}$ . Since  $|X| = |Y|$ , there exists a bijection  $\theta : X \rightarrow Y$ .

**Case 1.**  $m = 1$ . Then  $\sigma_1 = X \times X$  and  $\sigma_2 = Y \times Y$ . Thus  $(x\sigma_1)\theta = X\theta = Y = (x\theta)\sigma_2$  for all  $x \in X$ .

**Case 2.**  $m = n$ . Then  $\sigma_1 = \Delta(X)$  and  $\sigma_2 = \Delta(Y)$ . Thus  $(x\sigma_1)\theta = (x\theta)\sigma_2$  for all  $x \in X$ .

**Case 3.**  $m = n - 1$ . Then there exists a unique  $a_1\sigma_1 \in X/\sigma_1$  such that  $|a_1\sigma_1| = 2$  for some  $a_1 \in X$ , say that  $a_1\sigma_1 = \{a_1, a_2\}$  for some  $a_2 \in X$ . Similarly,  $\{b_1, b_2\} \in Y/\sigma_2$  for some  $b_1, b_2 \in Y$ . Thus

$$x\sigma_1 = \{x\} \text{ for all } x \in X \setminus \{a_1, a_2\}$$

and

$$y\sigma_2 = \{y\} \text{ for all } y \in Y \setminus \{b_1, b_2\}.$$

Since  $|X \setminus \{a_1, a_2\}| = |Y \setminus \{b_1, b_2\}|$ , there exists  $\varphi : X \setminus \{a_1, a_2\} \rightarrow Y \setminus \{b_1, b_2\}$  is a bijection. Define  $\theta : X \rightarrow Y$  by

$$x\theta = \begin{cases} b_i, & \text{if } x = a_i, \\ x\varphi, & \text{otherwise} \end{cases}$$

for all  $x \in X$ . It is clear that  $\theta$  is a bijection and each element  $x$  in  $X$ ,  $(x\sigma_1)\theta = (x\theta)\sigma_2$ .

From the three cases above,  $E(X, \sigma_1) \cong E(Y, \sigma_2)$  by Theorem 2.4.  $\square$

Note that if  $|X| \leq 3$  and  $\sigma$  is an equivalence on  $X$ , then  $|X/\sigma| \in \{1, 2, 3\}$ . The following corollary is a direct consequence of Corollary 2.5 and Theorem 2.4.

**Corollary 2.6.** *Let  $X$  and  $Y$  be sets such that  $|X| = |Y| \leq 3$ . Then  $E(X, \sigma_1) \cong E(Y, \sigma_2)$  if and only if  $|X/\sigma_1| = |Y/\sigma_2|$ .*

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