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Relations Between Ramanujan's Cubic Continued Fraction and a Continued Fraction of Order 12 and its Evaluations

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Abstract. In the present paper, we establish relationship between continued fraction $U(-q)$ of order 12 and Ramanujan's cubic continued fraction $G(-q)$ and $G(q^n)$ for $n = 1, 2, 3, 5$ and 7. Also we evaluate $U(q)$ and $U(-q)$ by using two parameters for Ramanujan's theta-functions and their explicit values.

1. Introduction

Throughout this paper, we assume that $|q| < 1$ and for any complex number a,

$$(a;q)_\infty := \lim_{n\to\infty} \prod_{k=0}^{n-1} (1 - aq^k),$$

where n is a positive integer. In Chapter 16 of his second notebook [5, p. 34], [21, p. 197], S. Ramanujan developed the theory of theta-functions and his theta-function is defined as follows:

$$f(a,b) := \sum_{n=\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1.$$

The three important special cases of $f(a,b)$ [5, p.36] are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

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$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q; q)_\infty.$$

After Ramanujan, define

$$\chi(q) := (-q; q^2)_\infty.$$

Recently H. M. Srivastava et. al. [25] proved two q-identities which provide relationship between $f(-q), \varphi(q)$ and $\psi(q)$, by using well known Jacobi's triple product identity. These q-identities are analogous to Ramanujan's identities.

The celebrated Rogers−Ramanujan continued fraction is defined as

$$R(q) := q^{\frac{1}{5}} \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)} = \frac{q^{\frac{1}{5}}}{1} \, \frac{q}{+1} \, \frac{q^2}{+1} \, \frac{q^3}{+1} \, {}_{+\ldots}. \tag{1.1}$$

On page 365 of his 'Lost' notebook [22], Ramanujan recorded five identities showing the relationships between $R(q)$ and five continued fractions $R(-q), R(q^2), R(q^3), R(q^4)$ and $R(q^5)$. He also recorded these identities at the scattered places of his Notebooks [21]. L. J. Rogers [23] established the modular equations relating $R(q)$ and $R(q^n)$ for $n = 2, 3, 5$ and 11. Recently, K. R. Vasuki and S. R. Swamy [28] found the modular equations relating $R(q)$ and $R(q^{11})$.

Recently C. Adiga et. al. [1] have established several modular relations for the Rogers−Ramanujan type functions of order eleven which analogous to Ramanuja's forty identities for Rogers−Ramanujan functions and also they established certain interesting partition-theoretic interpretation of some of the modular relations and H. M. Srivastava and M. P. Chaudhary [24] established a set of four new results which depict the interrelationships between q-product identities, continued fraction identities and combinatorial partition identities.

The Ramanujan's cubic continued fraction $G(q)$ is defined as

$$G(q) := \frac{q^{1/3}}{1} \, \frac{q + q^2}{+ \; 1} \, \frac{q^2 + q^4}{+ \; 1} \, \frac{q^3 + q^6}{+ \; 1} \, {}_{+\ldots}. \tag{1.2}$$

The continued fraction (1.2) was first introduced by Ramanujan in his second letter to G. H. Hardy [16]. He also recorded the continued fraction (1.2) on p. 365 of his 'Lost' notebook [22] and claimed that there are many results for $G(q)$ similar to the results obtained for Rogers−Ramanujan continued fraction (1.1). Motivated by Ramanujan's claim H. H. Chan [11] proved three identities giving the relations between $G(q)$ and three continued fractions $G(-q), G(q^2)$ and $G(q^3)$. Further N. D. Baruah [3], established the modular relations between $G(q)$ and $G(q^n)$ for $n = 5$ and 7.

The Ramanujan-Göllnitz-Gordon continued fraction [22, p. 44], [14, 15] is defined as

$$H(q) := q^{\frac{1}{2}} \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^4;q^8)_\infty} = \frac{q^{1/2}}{1+q}{}_{+}\frac{q^2}{1+q^3}{}_{+}\frac{q^4}{1+q^5}{}_{+\dots}. \tag{1.3}$$

H. H. Chan and S. S. Hang [12] and K. R. Vasuki and B. R. Srivatsa Kumar [27] established the relationships between $H(q)$ and $H(q^n)$ with $n = 3, 4, 5, 11$ by using the modular equations deduced by Ramanujan. Recently, B. Cho, J. K. Koo and Y. K. Park [13] extended the results cited above for the continued fraction (1.3) to all odd prime p by computing the affine models of modular curves $X(\Gamma)$ with $\Gamma = \Gamma_1(8) \bigcap \Gamma_0(16p)$.

The continued fraction

$$U(q) := \frac{qf(-q,-q^{11})}{f(-q^5,-q^7)} = \frac{q(1-q)}{(1-q^3)}{}_{+}\frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)}{}_{+}\frac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})}{}_{+\dots}, \tag{1.4}$$

was established by M. S. M. Naika et. al. [18] as a special case of fascinating continued fraction identity recorded by Ramanujan in his second notebook [21, p. 74] and they have established a modular relationship between the continued fraction $U(q)$ and $U(q^n)$ with $n = 3$ and 5. Recently K. R. Vasuki et. al. [26] have established a relationship between the continued fraction $U(q)$ and $U(q^n)$ with $n = 7, 9, 11$ and 13.

Also in his 'Lost' notebook [22], Ramanujan recorded the following continued fraction identity

$$\begin{aligned}\frac{G(aq,\lambda q,b;q)}{G(a,\lambda,b;q)} &= \frac{1}{1}{}_{+}\frac{aq+\lambda q}{1}{}_{+}\frac{bq+\lambda q^2}{1}{}_{+}\frac{aq^2+\lambda q^3}{1}{}_{+}\frac{bq^2+\lambda q^4}{1}{}_{+\dots}\\ &= \frac{1}{1+aq}{}_{+}\frac{\lambda q-abq^2}{1+aq^2+bq}{}_{+}\frac{\lambda q^2-abq^4}{1+aq^3+bq^2}{}_{+\dots},\end{aligned}$$

where

$$G(a,\lambda,b;q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}(-\lambda/a;q)_n a^n}{(q;q)_n(-bq;q)_n}.$$

For convenience, we use the following notations:

$$F_1(a,b,\lambda;q) := \frac{G(aq,\lambda q,b;q)}{G(a,\lambda,b;q)}$$

and

$$F_1(a,b,-b;q) =: F(a,b;q) := \frac{G(aq,-bq,b;q)}{G(a,-b,b;q)}.$$

C. Adiga et. al. [2], have established some relation between Ramanujan's continued fraction $F(a,b;q)$ and obtained three equivalent integral representations of $F(-1, 1; q)$ and some modular equations for the same. Also they found continued fraction representations for the Ramanujan-Weber class invariants. Further, they deduced some algebraic numbers and transcendental numbers involving $F(-1, 1; q) + 1$, the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ and the Dedekind eta function.

At the end of his brief communication [19, 20] announcing his proofs of the Rogers−Ramanujan identities, Ramanujan remarks, "I have now found an algebraic relation between $G(q)$ and $H(q)$, Viz,:

$$H(q)G^{11}(q) - q^2 G(q)H^{11}(q) = 1 + 11qG^6(q)H^6(q),$$

Another noteworthy formula is

$$H(q)G(q^{11}) - q^2 G(q)H(q^{11}) = 1.$$

Each of these formulae is the simplest of a large class". But Ramanujan has not shown how he had proved these two identities and these are two from the list of forty identities involving $G(q)$ and $H(q)$. Rogers [23] established ten of the identities and Watson [29] proved eight of the identities, but two of them in the group that Rogers had proved. These forty identities were first brought before the mathematical world by B. J. Birch [9] who found Watson's hand written copy of Ramanujan list of forty identities in the Oxford University Library. D. Bressoud [10], in his Ph. D. thesis proved fifteen from the list of forty identities. After the work of Rogers, Watson and Bressoud, nine remain to be proved. A. J. F. Biagioli [8] used modular forms to prove eight of them. In 2007, B. C. Berndt et. al. [6] have proved thirty five of the forty identities and also established several new identities involving the Rogers−Ramanujan functions by using modular equations found by Ramanujan. Recently C. Adiga et. al. [1] have established several modular relations for the Rogers−Ramanujan type functions of order eleven which analogous to Ramanujan's forty identities for Rogers−Ramanujan functions and also they established certain interesting partition-theoretic interpretation of some of the modular relations.

Next we introduce the Ramanujan−Weber class invariants. For $q = e^{-\pi\sqrt{n}}$, where n is a positive real number, define the two class invariants G_n and g_n by

$$G_n = 2^{-1/4} q^{-1/24} \chi(q)$$

and

$$g_n = 2^{-1/4} q^{-1/24} \chi(-q).$$

where $\chi(q)$ is as defined earlier. Ramanujan first introduced the above class invariants in his paper 'Modular equations and approximations to π'. Ramanujan [21] recorded the values of 107 class invariants in his first notebook. H. Weber [30] first constructed a table consisting of 50 values of class invariants earlier to Ramanujan. Weber preliminarily was motivated to calculate class so that he could construct Hilbert class fields. Ramanujan without the knowledge of Weber's work, independently calculated class invariants for different reasons. Furthermore, S. Bhargava,

K. R. Vasuki and B. R. Srivatsa Kumar [7], J. Yi [31], N. D. Baruah [4], M. S. M. naika and K. S. Bairy [17] have obtained several values for G_n and g_n. Recently J. Yi [31] defined two parametrization $h_{k,n}$ and $h'_{k,n}$ for the theta function φ and obtained some interesting properties of them for all real numbers k and n and they are defined as

$$h_{k,n} = \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})} \tag{1.5}$$

and

$$h'_{k,n} = \frac{\varphi(e^{-2\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-2\pi\sqrt{nk}})}.$$

Motivated by the above work, in the present paper, we establish the relationship between $U(-q)$ and the other continued fractions $G(-q)$ and $G(q^n)$ for $n = 1, 2, 3, 5$ and 7. Since the working method is monotonous, we skip the detailed proof of $U(-q)$ and $G(q^n)$ for $n = 3, 5$ and 7. Also we evaluate $U(q)$ and $U(-q)$ by using two parameters for Ramanujan's theta-functions and their explicit values.

2. Preliminary Results

Theorem 2.1 *We have*

$$8G^3(q) = 1 - \frac{\varphi^4(-q)}{\varphi^4(-q^3)}. \tag{2.1}$$

Proof. For a proof, see Chapter 20 [5, p. 345]. □

Theorem 2.2 *We have*

$$\frac{\varphi(q)}{\varphi(q^3)} = \frac{1+U(q)}{1-U(q)}. \tag{2.2}$$

Proof. For a proof, see [26]. □

Theorem 2.3 *We have*

$$G(q) + G(-q) + 2G^2(-q)G^2(q) = 0, \tag{2.3}$$

$$G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0 \tag{2.4}$$

and

$$G^3(q) = G(q^3)\frac{1 - G(q^3) + G^2(q^3)}{1 + 2G(q^3) + 4G^2(q^3)}. \tag{2.5}$$

Proof. For a proof, see [11]. □

Theorem 2.4 *If* $v = G(q)$ *and* $w = G(q^5)$, *then we have*

$$v^6 - vw + 5vw(v^3 + w^3)(1 - 2vw) + w^6 = v^2w^2(16v^3w^3 - 20v^2w^2 + 20vw - 5). \quad (2.6)$$

Proof. For a proof, see [3]. □

Theorem 2.5 *If* $v = G(q)$ *and* $w = G(q^7)$, *then we have*

$$\begin{aligned} v^8 - vw - 56v^3w^3(v^2 + w^2) + 7vw(v^3 + w^3)(1 - 8v^3w^3) \\ + 28v^2w^2(v^4 + w^4) \\ = -w^8 - v^4w^4(21 - 64v^3w^3). \end{aligned} \quad (2.7)$$

Proof. For a proof, see [3]. □

Theorem 2.6 *If* $a = \frac{\pi^{1/4}}{\Gamma(3/4)}$, *then*

(i) $\varphi(e^{-2\pi}) = a2^{-1/8}$,

(ii) $\varphi(e^{-4\pi}) = a2^{-7/16}(\sqrt{2} + 1)^{1/4}$,

(iii) $\varphi(e^{-6\pi}) = \frac{a(1+\sqrt{3}+\sqrt{2}\sqrt[4]{27})^{1/3}}{2^{11/24}3^{3/8}(\sqrt{3}-1)^{1/6}}$,

(iv) $\varphi(-e^{-12\pi}) = \frac{a2^{-19/48}3^{-3/8}(2-3\sqrt{2}+3^{5/4}+3^{3/4}+3^{3/4})^{1/3}}{(\sqrt{2}-1)^{1/12}(\sqrt{3}+1)^{1/6}(-1-\sqrt{3}+\sqrt{2}.3^{3/4})^{1/3}}$.

Proof. For a proof one can see [31]. □

3. Main Results

Theorem 3.1 *If* $u = U(-q)$ *and* $v = G(q)$, *then*

$$v^3(1 - u)^4 + u(1 + u^2) = 0. \quad (3.1)$$

Proof. Replacing q to $-q$ in (2.2) and then on raising to the power 4 we have

$$\frac{\varphi^4(-q)}{\varphi^4(-q^3)} = \left(\frac{1 + U(-q)}{1 - U(-q)}\right)^4.$$

On using the above in (2.1), we have the result. □

Theorem 3.2 *If* $u = U(-q)$ *and* $w = G(-q)$, *then*

$$w^3 = \frac{u(1 - u)^2}{(1 + u^2)^2}.$$

Proof. Eliminating $G(q)$ between (2.3) and (3.1) we have

$$(w^3 + 2w^3u^2 + w^3u^4 - u + 2u^2 - u^3)(u^4 - 2u^3 + 8w^3u^2 + 2u^2 - 2u + 1) = 0.$$

It follows from the definition of u and w that

$$u = U(-q) = 1 + q - q^5 - q^6 - q^7 + ... \tag{3.2}$$

and

$$w = G(-q) = (-q)^{1/3}(1 + q + 2q^3 - 2q^4 + q^5 + ...). \tag{3.3}$$

By using (3.2) and (3.3) in the above factors, we see that first factor becomes

$$-q(4 + 21q + 45q^2 + 80q^3 + 117q^4 + 97q^5 + ...)$$

and the second factor becomes

$$-q(8 + 38q + 78q^2 + 127q^3 + 184q^4 + 116q^5 + ...).$$

Thus, the second factor does not vanish. Hence by identity theorem, one can see that, the second factor does not tend to zero, whereas the first factor tends to zero in some neighbourhood of $q = 0$. Hence by analytic continuation in $|q| < 1$, we have

$$w^3 + 2w^3u^2 + w^3u^4 - u + 2u^2 - u^3 = 0. \qquad \square$$

Theorem 3.3 *If $u = U(-q)$ and $w = G(q^2)$, then*

$$w^3 = \frac{u^2}{(u^2 + 1)(u - 1)^2}.$$

Proof. On eliminating $G(q)$ between (2.4) and (3.1) we have

$$(2u^2w^3+u^4w^3-2uw^3-2u^3w^3+w^3-u^2)(u^4+8u^3w^3-16u^2w^3+2u^2+8uw^3+1) = 0.$$

Replacing $-q$ to q^2 in (3.3), we obtain

$$w = G(q^2) = q^{2/3}(1 - q^2 - 2q^6 - 2q^8 + ...).$$

By using (3.2) and the last relation in the same factors as above, we see that the first factor becomes

$$-1 - 2q - q^2 + 2q^4 + 4q^5 - q^6 - 2q^7 + ...$$

and the second factor becomes

$$4 + 8q + 8q^2 + 4q^3 + 9q^4 - 48q^6 - 60q^7 + ...$$

As discussed in the previous theorem, the second factor does not vanish in the neighbourhood of $q = 0$. Hence we must have

$$2u^2w^3 + u^4w^3 - 2uw^3 - 2u^3w^3 + w^3 - u^2 = 0. \qquad \square$$

Theorem 3.4 *If $u = U(-q)$ and $w = G(q^3)$, then*

$$(1+u^2)\big((u-1)^2 w + (1+4w^2)u\big) - w^2(1-w)(u-1)^4 = 0.$$

Proof. Eliminating $G(q)$ between (2.5) and (3.1), we have the result. □

Theorem 3.5 *If $u = U(-q)$ and $w = G(q^5)$, then*

$$\begin{aligned}(s^4 - s^3 - 24s^2 + 4s + 86)w^3 &- 5(3s^4 + 5s^3 - 144s^2 - 87s + 1532)w^6 \\ &+ 20(3s^4 + 3s^3 - 161s^2 - 32s + 1936)w^9 \\ &- 5(4s^4 - 295s^3 + 1524s^2 + 4336s - 25408)w^{12} \\ &- 2(45s^4 - 640s^3 + 1440s^2 + 11520s - 41728)w^{15} \\ &- (u(u^2+1)(u-1)^4 - (u-1)^{24})w^{18} \\ &+ u^6(u^2+1)^6 = 0.\end{aligned}$$

where $t = u + \frac{1}{u}$ and $s = t + \frac{4}{t}$.

Proof. The result directly follows from (2.6) and (3.1), by eliminating $G(q)$ using Maple. □

Theorem 3.6 *If $u = U(-q)$ and $w = G(q^7)$, then*

$$\begin{aligned}&w^3(s^6 - 3s^5 - 33s^4 + 71s^3 + 312s^2 - 348s - 552) \\ &- 7w^6(3s^6 - 39s^5 + 57s^4 + 1407s^3 - 5304s^2 - 12448s + 58336) \\ &+ 7w^9(21s^6 - 507s^5 + 2082s^4 + 13224s^3 - 74344s^2 - 81952s + 570368) \\ &- 7w^{12}(49s^6 - 1257s^5 + 5676s^4 + 23950s^3 - 154344s^2 - 65184s + 805760) \\ &- 28w^{15}(3s^6 - 975s^5 + 8686s^4 - 2224s^3 - 146400s^2 + 283520s + 91648) \\ &+ 28w^{18}(21s^6 + 556s^5 - 8936s^4 + 28432s^3 + 79808s^2 - 590080s + 920576) \\ &+ 8w^{21}(21s^6 + 84s^5 - 7392s^4 + 56000s^3 - 21504s^2 - 946176s + 2297856) \\ &+ t^{-1}(w^{24}(t-2)^{14} + u^8(u^2+1)^8(t-2)^{-2}) = 0.\end{aligned}$$

where $t = u + \frac{1}{u}$ and $s = t + \frac{4}{t}$.

Proof. By using (2.7) and (3.1), we arrive at the required result. □

4. Evaluations of $U(q)$

Theorem 4.1. *We have*

(i) $U(e^{-\pi}) = \dfrac{\sqrt[4]{3}\sqrt[4]{2\sqrt{3}-3}-1}{\sqrt[4]{3}\sqrt[4]{2\sqrt{3}-3}+1}$,

(ii) $U(e^{-\pi/3}) = \dfrac{\sqrt[4]{2\sqrt{3}+3}-1}{\sqrt[4]{2\sqrt{3}+3}+1}$,

(iii $U(e^{-\sqrt{3}\pi}) = \dfrac{1-\sqrt[3]{2}+\sqrt[3]{4}-\sqrt[4]{3}}{1-\sqrt[3]{2}+\sqrt[3]{4}+\sqrt[4]{3}}$,

(iv) $U(e^{-\pi/3\sqrt{3}}) = \dfrac{1+\sqrt[3]{2}-\sqrt[4]{3}}{1+\sqrt[3]{2}+\sqrt[4]{3}}$.

Proof. (i) Setting $k = n = 3$ in (1.5), we find that

$$h_{3,3} = \frac{\varphi(e^{-\pi})}{3^{1/4}\varphi(e^{-3\pi})}.$$

From (2.2), it is easy to see that

$$U(q) = \frac{\frac{\varphi(q)}{\varphi(q^3)}-1}{\frac{\varphi(q)}{\varphi(q^3)}+1}. \tag{4.1}$$

From Theorem 4.10(i) [31], we have

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = 3^{1/4}(2\sqrt{3}-3)^{1/4}. \tag{4.2}$$

Substituting $q = e^{-\pi}$ in (4.1), we arrive at

$$U(e^{-\pi}) = \frac{\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})}-1}{\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})}+1}.$$

Using (4.2) in the above, we complete the proof of (i).
(ii) Setting $k = 3, n = 1/3$ in (1.5) and then by using Theorem 4.10(ii) [31], we have

$$\frac{\varphi(e^{-\pi/3})}{\varphi(e^{-\pi})} = (2\sqrt{3}+3)^{1/4}.$$

Substituting $q = e^{-\pi/3}$ in (4.1) and using the above, we complete the proof of (ii).

(iii) Setting $k=3, n=9$ in (1.5) and from Theorem 4.10(iii) [31], we have

$$\frac{\varphi(e^{-\sqrt{3}\pi})}{\varphi(e^{-3\sqrt{3}\pi})} = \frac{1}{3^{1/4}}(1-\sqrt[3]{2}+\sqrt[3]{4}).$$

Using (4.1) with $q=e^{-\sqrt{3}\pi}$ and the above, we complete the proof of (iii).
(iv) Setting $k=3, n=1/9$ in (1.5) and using Theorem 4.10(iv) [31], we find that

$$\frac{\varphi(e^{-\pi/3\sqrt{3}})}{\varphi(e^{\pi/\sqrt{3}})} = \frac{1+\sqrt[3]{2}}{3^{1/4}}.$$

Using the above with $q=e^{-\pi/3\sqrt{3}}$ in (4.1), we complete the proof of (iv). □

Theorem 4.2. *We have*

(i) $U(e^{-\sqrt{5}\pi/\sqrt{3}}) = \dfrac{\sqrt[4]{3}\sqrt{\sqrt{5}-1}-\sqrt{2}}{\sqrt[4]{3}(\sqrt{\sqrt{5}-1}+\sqrt{2}}$,

(ii) $U(e^{-\pi/\sqrt{15}}) = \dfrac{\sqrt[4]{3}\sqrt{\sqrt{5}+1}-\sqrt{2}}{\sqrt[4]{3}\sqrt{\sqrt{5}+1}+\sqrt{2}}$,

(iii) $U(e^{-5\pi/\sqrt{3}}) = \dfrac{\sqrt[4]{3}\left(\sqrt[6]{5}-\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}\right)-\sqrt[3]{2}}{\sqrt[4]{3}\left(\sqrt[6]{5}-\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}\right)+\sqrt[3]{2}}$,

(iv) $U(e^{-\pi/5\sqrt{3}}) = \dfrac{\sqrt[4]{3}\left(\sqrt[6]{5}+\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}\right)-\sqrt[3]{2}}{\sqrt[4]{3}\left(\sqrt[6]{5}+\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}\right)+\sqrt[3]{2}}$.

Proof. (i) Setting $k=3$ and $n=5$ in (1.5), we have

$$h_{3,5} = \frac{\varphi(e^{-\sqrt{5}\pi/\sqrt{3}})}{3^{1/4}\varphi(e^{-\sqrt{15}\pi})}.$$

From Theorem 4.13(i) [31], we find that

$$\frac{\varphi(e^{-\sqrt{5}\pi/\sqrt{3}})}{\varphi(e^{-\sqrt{15}\pi})} = \frac{3^{1/4}(\sqrt{5}-1)^{1/2}}{\sqrt{2}}.$$

Now substituting $q=e^{-\sqrt{5}\pi/\sqrt{3}}$ in (4.1), and using the above, we deduce the result.
(ii) Setting $k=3$ and $n=1/5$ in (1.5), note that

$$h_{3,5} = \frac{\varphi(e^{-\sqrt{5}\pi/\sqrt{3}})}{3^{1/4}\varphi(e^{-\sqrt{15}\pi})}.$$

From Theorem 4.13(ii) [31], we have

$$\frac{\varphi(e^{-\pi/\sqrt{15}})}{\varphi(e^{-\sqrt{3}\pi/\sqrt{15}})} = 3^{1/4}\left(\frac{\sqrt{5}+1}{2}\right)^{1/2}.$$

Now by using the above with $q = e^{-\pi/\sqrt{15}}$ in (4.1) , we arrive at the required result.
(iii) Setting $k = 3$ and $n = 25$ in (1.5), we find that

$$h_{3,25} = \frac{\varphi(e^{-5\pi/\sqrt{3}})}{3^{1/4}\varphi(e^{-5\sqrt{3}\pi})}.$$

From Theorem 4.13(iii) [31], we have

$$\frac{\varphi(e^{-5\pi/\sqrt{3}})}{\varphi(e^{-5\sqrt{3}\pi})} = \frac{3^{1/4}}{2^{1/3}}\left(5^{1/6} - \sqrt{5^{1/3} - 2^{2/3}}.\right)$$

Now taking $q = e^{-5\pi/\sqrt{3}}$ in (4.1) and using the above, we directly obtain the result.
(iv) Setting $k = 3$ and $n = 1/25$ in (1.5), we deduce

$$h_{3,1/25} = \frac{\varphi(e^{-\pi/5\sqrt{3}})}{3^{1/4}\varphi(e^{-\sqrt{3}\pi/5)}}.$$

Again by Theorem 4.13(iv) [31], we have

$$\frac{\varphi(e^{-\pi/5\sqrt{3}})}{\varphi(e^{-\sqrt{3}\pi/5)}} = \frac{3^{1/4}}{2^{1/3}}\left(5^{1/6} + \sqrt{5^{1/3} - 2^{2/3}}\right).$$

Now taking $q = e^{-\pi/5\sqrt{3}}$ in (4.1) and using the above, we directly obtain the result. □

Theorem 3.9 *We have*

(i) $$U(-e^{-\pi/\sqrt{3}}) = \frac{\sqrt[4]{3}\sqrt[4]{(\sqrt{3}-1)} - \sqrt[4]{(5+3\sqrt{3})}}{\sqrt[4]{3}\sqrt[4]{(\sqrt{3}-1)} + \sqrt[4]{(5+3\sqrt{3})}},$$

(ii) $$U(-e^{-\pi\sqrt{2/3}}) = \frac{\sqrt[4]{(3\sqrt{2})} - \sqrt[4]{(4+3\sqrt{2})}}{\sqrt[4]{(3\sqrt{2})} + \sqrt[4]{(4+3\sqrt{2})}},$$

(iii) $$U(-e^{-2\pi}) = \frac{\sqrt[3]{2}\sqrt[8]{27}\sqrt[6]{\sqrt{3}-1} - \sqrt[3]{t+\sqrt{2}\sqrt[4]{27}}}{\sqrt[3]{2}\sqrt[8]{27}\sqrt[6]{\sqrt{3}-1} + \sqrt[3]{t+\sqrt{2}\sqrt[4]{27}}},$$

(iv) $$U(-e^{-4\pi}) = \frac{\sqrt[8]{27}\sqrt[12]{\sqrt{2}-1}\sqrt[4]{\sqrt{2}+1}\sqrt[6]{t}a - \sqrt[24]{2}b}{\sqrt[8]{27}\sqrt[12]{\sqrt{2}-1}\sqrt[4]{\sqrt{2}+1}\sqrt[6]{t}a + \sqrt[24]{2}b}.$$

where $a = \sqrt[3]{\sqrt{2}\sqrt[4]{27}) - t}$, $b = \sqrt[3]{2 - 3\sqrt{2}} + \sqrt[4]{27^2} + \sqrt[4]{27}$ *and* $t = \sqrt{3} + 1$.

Proof. (i) From Theorem 7.2(i) [3], we have

$$\frac{\varphi(-e^{-\pi/\sqrt{3}})}{\varphi(-e^{-\pi\sqrt{3}})} = \left[\frac{9 - 3(2+\sqrt{3})}{1 - 3(2+\sqrt{3})}\right]^{1/4}.$$

Replacing q to $-q$ in (4.1) and then using the above result, we arrive at (i).

(ii) Replacing q to $-q$ in (4.1), using Theorem 7.2(iii) [3] and on simplifying, we arrive at (ii).
(iii) Replacing q to $-q$ in (4.1), then by using Theorem 2.2 (i) and (iii) [4] and on simplification, we arrive at (iii).
(iv) Replacing q to $-q$ in (4.1), then by using Theorem 2.2 (ii) and (iv) [4] and on simplification, we arrive at (iv). □

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