KYUNGPOOK Math. J. 58(2018), 307-318
https://doi.org/10.5666/KMJ.2018.58.2.307
pISSN 1225-6951 eISSN 0454-8124

On the Fekete−Szegö Problem for a Certain Class of Meromorphic Functions Using $q-$Derivative Operator

Mohamed Kamal Aouf
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
e-mail : mkaouf127@yahoo.com

Halit Orhan*
Department of Mathematics, Faculty of Science, Ataturk University, Erzurum 25240, Turkey
e-mail : orhanhalit607@gmail.com

Abstract. In this paper, we obtain Fekete-Szegö inequalities for certain class of meromorphic functions $f(z)$ for which

$$-\frac{(1-\frac{\alpha}{q})qzD_qf(z)+\alpha qzD_q\left[zD_qf(z)\right]}{(1-\frac{\alpha}{q})f(z)+\alpha zD_qf(z)}\prec\varphi(z)(\alpha\in\mathbb{C}\backslash(0,1],\ 0<q<1).$$

Sharp bounds for the Fekete-Szegö functional $\left|a_1-\mu a_0^2\right|$ are obtained.

1. Introduction

The theory of $q-$analysis has important role in many areas of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, $q-$difference, $q-$integral equations and in $q-$transform analysis (see for instance [1, 6, 8, 9]). The study of $q-$calculus has gained momentum years mainly due to the pioneer work of M. E. H. Ismail et al. [7] in recent years; it was followed by such works as those by S. Kanas and D. Raducanu [10] and S. Sivasubramanian and M. Govindaraj [19]. Let Σ denote the class of meromorphic functions of the form:

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kz^k, \tag{1.1}$$

---

* Corresponding Author.
Received October 6, 2017; revised March 27, 2018; accepted March 29, 2018.
2010 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: Analytic, meromorphic, $q-$starlike and convex functions, Fekete-Szegö problem, convolution.

which are analytic in the open punctured unit disc

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}.$$

A function $f \in \Sigma$ is meromorphic starlike of order β, denoted by $\Sigma^*(\beta)$, if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \ (0 \le \beta < 1;\ z \in \mathbb{U}).$$

The class $\Sigma^*(\beta)$ was introduced and studied by Pommerenke [16] (see also Miller [14]). Let $\varphi(z)$ be an analytic function with positive real part on $\mathbb{U}$ satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\Sigma^*(\varphi)$ be the class of functions $f(z) \in \Sigma$ for which

$$-\frac{zf'(z)}{f(z)} \prec \varphi(z) \ (z \in \mathbb{U}).$$

The class $\Sigma^*(\varphi)$ was introduced and studied by Silverman et al. [18]. The class $\Sigma^*(\beta)$ is the special case of $\Sigma^*(\varphi)$ when $\varphi(z) = \dfrac{1 + (1 - 2\beta)z}{1 - z}$ $(0 \le \beta < 1)$. Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disc $\mathbb{U}$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are analytic and univalent in $\mathbb{U}$. Ma and Minda [13] introduced and studied the class $\mathcal{S}^*(\varphi)$ which consists of functions $f(z) \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \ (z \in \mathbb{U}),$$

and the class $\mathcal{C}(\varphi)$ consists of functions $f(z) \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \ (z \in \mathbb{U}).$$

Following Ma and Minda [13], Shanmugam and Sivasubramanian [17] defined a more general class $\mathcal{M}_\alpha(\varphi)$ consists of functions $f(z) \in \mathcal{S}$ for which

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha z f'(z)} \prec \varphi(z) \ (\alpha \ge 0).$$

Analogous to the class $\mathcal{M}_\alpha(\varphi)$, Aouf et al. [4] defined the class $\mathcal{F}^*_\alpha(\varphi)$ as follows: For $\alpha \in \mathbb{C}\backslash(0, 1]$, let $\mathcal{F}^*_\alpha(\varphi)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion:

$$-\frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha z f'(z)} \prec \varphi(z).$$

For a function $f(z) \in \Sigma$ given by (1.1) and $0 < q < 1$, the $q-$derivative of a function $f(z)$ is defined by (see Gasper and Rahman [6])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \text{ if } z \in \mathbb{U}^*. \tag{1.2}$$

From (1.2), we deduce that $D_q f(z)$ for a function $f(z)$ of the form (1.1) is given by

$$D_q f(z) = -\frac{1}{qz^2} + \sum_{k=0}^{\infty} [k]_q \ a_k z^{k-1} \ (z \neq 0),$$

where

$$[i]_q = \frac{1-q^i}{1-q}.$$

As $q \to 1^-, \ [k]_q \to k$, we have

$$\lim_{q \to 1^-} \ D_q f(z) = f'(z).$$

Making use of the $q-$derivative D_q, we introduce the subclass $\mathcal{F}^*_{q,\alpha}(\varphi)$ as follows: For $\alpha \in \mathbb{C} \backslash (0, 1]$, $0 < q < 1$, a function $f(z) \in \Sigma$ is said to be in the class $\mathcal{F}^*_{q,\alpha}(\varphi)$, if and only if

$$-\frac{(1-\frac{\alpha}{q})qzD_q f(z) + \alpha qz D_q \left[z D_q f(z) \right]}{(1-\frac{\alpha}{q}) f(z) + \alpha z D_q f(z)} \prec \varphi(z) \ (z \in \mathbb{U}). \tag{1.3}$$

We note that:

(i) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,\alpha}(\varphi) = \mathcal{F}^*_{\alpha}(\varphi)$ (see Aouf et al. [4]);

(ii) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0}(\varphi) = \Sigma^*(\varphi)$ (see Silverman et al. [18] and Ali and Ravichandran [2]);

(iii) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0} \left(\frac{1+z}{1-z} \right) = \mathcal{F}^*(1) = \mathcal{F}^*$ (see Aouf [3, with $b = 1$]);

(iv) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0} \left(\frac{1+(1-2\beta)z}{1-z} \right) = \Sigma^*(\beta) \ (0 \leq \beta < 1)$ (see Pommerenke [16]);

(v) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0} \left(\frac{1+\beta(1-2\gamma\eta)z}{1+\beta(1-2\gamma)z} \right) = \Sigma(\eta, \beta, \gamma) \ (0 \leq \eta < 1, \ 0 < \beta \leq 1, \ \frac{1}{2} \leq \gamma \leq 1)$ (see Kulkarni and Joshi [12]);

(vi) $\lim\limits_{q \to 1^-} \mathcal{F}^*_{q,0} \left(\frac{1+Az}{1+Bz} \right) = K_1(A, B) \ (0 \leq B < 1, \ -B < A < B)$ (see Karunakaran [11]).

2. Fekete-Szegö Problem

To prove our results, we need the following lemmas.

Lemma 1.([13]) *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathbb{U}$ and μ is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \le 2\max\{1; |2\mu - 1|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

Lemma 2.([13]) *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in $\mathbb{U}$, then*

$$\left|c_2 - \nu c_1^2\right| \le \begin{cases} -4\nu + 2 & \text{if } \nu \le 0, \\ 2 & \text{if } 0 \le \nu \le 1, \\ 4\nu - 2 & \text{if } \nu \ge 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p_1(z) = \dfrac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p_1(z) = \dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z} \ (0 \le \lambda \le 1),$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if

$$\frac{1}{p_1(z)} = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z} \ (0 \le \lambda \le 1),$$

or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2 \le 2 \ \left(0 < \nu \le \frac{1}{2}\right),$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \le 2 \ \left(\frac{1}{2} < \nu < 1\right).$$

Unless otherwise mentioned, we assume throughout this paper that $\alpha \in \mathbb{C}\backslash(0,1]$ and $0 < q < 1$.

Theorem 1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{F}^*_{q,\alpha}(\varphi)$ and μ is a complex number, then*

$$\text{(2.1)}\quad \text{(i)}\quad \left|a_1 - \mu a_0^2\right| \leq \frac{1}{1+q}\left|\frac{(q-2\alpha)\,B_1}{(q-\alpha+\alpha q)}\right| \times \max\left\{1, \left|\frac{B_2}{B_1} - \left[1 - \mu\frac{(q-2\alpha)\,(q-\alpha+\alpha q)\,(q+1)}{(q-\alpha)^2}\right]B_1\right|\right\}\ (B_1 \neq 0),$$

and

$$\text{(2.2)}\quad \text{(ii)}\qquad |a_1| \leq \frac{1}{1+q}\left|\frac{(q-2\alpha)\,B_2}{(q-\alpha+\alpha q)}\right|\ (B_1 = 0).$$

The result is sharp.

Proof. If $f(z) \in \mathcal{F}^*_\alpha(\varphi)$, then there is a Schwarz function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ and such that

$$\text{(2.3)}\qquad -\frac{(1-\frac{\alpha}{q})qzD_q f(z) + \alpha q z D_q\left[zD_q f(z)\right]}{(1-\frac{\alpha}{q})f(z) + \alpha z D_q f(z)} = \varphi(w(z)).$$

Define the function $p_1(z)$ by

$$\text{(2.4)}\qquad p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots.$$

Since $w(z)$ is a Schwarz function, we see that $\Re\{p_1(z)\} > 0$ and $p_1(0) = 1$. Define

$$\text{(2.5)}\qquad p(z) = -\frac{(1-\frac{\alpha}{q})qzD_q f(z) + \alpha q z D_q\left[zD_q f(z)\right]}{(1-\frac{\alpha}{q})f(z) + \alpha z D_q f(z)} = 1 + b_1 z + b_2 z^2 + \ldots.$$

In view of (2.3), (2.4) and (2.5), we have

$$\text{(2.6)}\qquad p(z) = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right).$$

Since

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right)z^3 + \ldots\right].$$

Therefore, we have

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \ldots,$$

and from this equation and (2.6), we obtain

$$b_1 = \frac{1}{2}B_1 c_1,$$

and

$$b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2.$$

Then, from (2.5) and (1.1), we see that

$$b_1 = -\left(\frac{q-\alpha}{q-2\alpha}\right)a_0,$$

and

$$b_2 = \left(\frac{q-\alpha}{q-2\alpha}\right)^2 a_0^2 - \frac{(q+1)\,(q-\alpha+\alpha q)}{q-2\alpha}a_1,$$

or, equivalently, we have

$$a_0 = -\left(\frac{q-2\alpha}{q-\alpha}\right), \tag{2.7}$$

and

$$a_1 = -\frac{(q-2\alpha)\,B_1}{2(1+q)\,(q-\alpha+\alpha q)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{B_2}{B_1} + B_1\right)\right]. \tag{2.8}$$

Therefore

$$a_1 - \mu a_0^2 = -\frac{(q-2\alpha)\,B_1}{2(1+q)\,(q-\alpha+\alpha q)}\left\{c_2 - \nu c_1^2\right\},$$

where

$$\nu = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + B_1 - \mu\frac{(q-2\alpha)\,(q-\alpha+\alpha q)\,(q+1)B_1}{(q-\alpha)^2}\right]. \tag{2.9}$$

Now, the result (2.1) follows by an application of Lemma 1. Also, if $B_1 = 0$, then

$$a_0 = 0 \text{ and } a_1 = -\frac{(q-2\alpha)\,B_2c_1^2}{4(1+q)\,(q-\alpha+\alpha q)}.$$

Since $p(z)$ has positive real part, $|c_1| \le 2$ (see Nehari [15]), so that

$$|a_1| \le \frac{1}{1+q}\left|\frac{(q-2\alpha)\,B_2}{(q-\alpha+\alpha q)}\right|,$$

this proving (2.2). The result is sharp for the functions

$$-\frac{(1-\frac{\alpha}{q})qzD_qf(z) + \alpha qzD_q\left[D_qf(z)\right]}{(1-\frac{\alpha}{q})f(z) + \alpha zD_qf(z)} = \varphi(z^2),$$

and

$$-\frac{(1-\frac{\alpha}{q})qzD_qf(z) + \alpha qzD_q\left[D_qf(z)\right]}{(1-\frac{\alpha}{q})f(z) + \alpha zD_qf(z)} = \varphi(z).$$

This completes the proof of Theorem 1. □

Remark 1.

(i) For $q \to 1^-$ in Theorem 1, we obtain the result obtained by Aouf et al. [4, Theorem 2.1];

(ii) For $q \to 1^-$ and $\alpha = 0$ in Theorem 1, we obtain the result obtained by Silverman et al. [18, Theorem 2.1].

By using Lemma 2, we can obtain the following theorem.

Theorem 2. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + ...$ $\left(B_i > 0,\ i \in \{1,2\}, 0 < \alpha < \frac{q}{1+q}\right)$. *If* $f(z)$ *given by (1.1) belongs to the class* $\mathcal{F}^*_{q,\alpha}(\varphi)$, *then*

(2.10)

$$\left|a_1 - \mu a_0^2\right| \leq \begin{cases} \frac{(q-2\alpha)B_1^2}{(1+q)(q-\alpha+\alpha q)}\left\{-B_2 + \left[1 - \mu\frac{[q-\alpha(1+q)](1+q)(q-2\alpha)}{q(1-\alpha)^2}\right]B_1^2\right\} & \text{if } \mu \leq \sigma_1, \\ \frac{(q-2\alpha)B_1}{(1+q)(q-\alpha+\alpha q)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{(q-2\alpha)B_1^2}{(1+q)(q-\alpha+\alpha q)}\left\{B_2 - \left[1 - \mu\frac{[q-\alpha(1+q)](1+q)(q-2\alpha)}{q(1-\alpha)^2}\right]B_1^2\right\} & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{(q-\alpha)^2\left[-B_1 - B_2 + B_1^2\right]}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2} \text{ and } \sigma_2 = \frac{(q-\alpha)^2\left(B_1 - B_2 + B_1^2\right)}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2}.$$

The result is sharp. Further, let

$$\sigma_3 = \frac{(q-\alpha)^2\left[-B_2 + B_1^2\right]}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2}.$$

(i) *If* $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$\begin{aligned} &\left|a_1 - \mu a_0^2\right| + \frac{(q-\alpha)^2}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2} \\ &\times\left\{(B_1 + B_2) + \left[\mu\frac{(1+q)(q-2\alpha)(q-\alpha+\alpha q)}{q(1-\alpha)^2} - 1\right]B_1^2\right\}|a_0|^2 \\ &\leq \frac{(q-2\alpha)B_1}{(1+q)(q-\alpha+\alpha q)}. \end{aligned} \tag{2.11}$$

(ii) *If* $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$\begin{aligned} &\left|a_1 - \mu a_0^2\right| + \frac{(q-\alpha)^2}{(q-2\alpha)(q-\alpha+\alpha q)(1+q)B_1^2} \\ &\times\left\{(B_1 - B_2) + \left[1 - \mu\frac{(1+q)(q-2\alpha)(q-\alpha+\alpha q)}{q(1-\alpha)^2}\right]B_1^2\right\}|a_0|^2 \\ &\leq \frac{(q-2\alpha)B_1}{(1+q)(q-\alpha+\alpha q)}. \end{aligned} \tag{2.12}$$

Proof. First, let $\mu \le \sigma_1$. Then

$$\begin{aligned}\left|a_1-\mu a_0^2\right| &\le \frac{(q-2\alpha)\,B_1}{(1+q)\,(q-\alpha+\alpha q)}\left\{-\frac{B_2}{B_1}+\left[1-\mu\frac{[q-\alpha(1+q)]\,(1+q)\,(q-2\alpha)}{q(1-\alpha)^2}\right]B_1\right\}\\ &\le \frac{(q-2\alpha)\,B_1^2}{(1+q)\,(q-\alpha+\alpha q)}\left\{-B_2+\left[1-\mu\frac{[q-\alpha(1+q)]\,(1+q)\,(q-2\alpha)}{q(1-\alpha)^2}\right]B_1^2\right\}.\end{aligned}$$

Let, now $\sigma_1 \le \mu \le \sigma_2$. Then, using the above calculations, we obtain

$$\left|a_1-\mu a_0^2\right| \le \frac{(q-2\alpha)\,B_1}{(1+q)\,(q-\alpha+\alpha q)}.$$

Finally, if $\mu \ge \sigma_2$, then

$$\begin{aligned}\left|a_1-\mu a_0^2\right| &\le \frac{(q-2\alpha)\,B_1}{(1+q)\,(q-\alpha+\alpha q)}\left\{\frac{B_2}{B_1}-\left[1-\mu\frac{[q-\alpha(1+q)]\,(1+q)\,(q-2\alpha)}{q(1-\alpha)^2}\right]B_1\right\}\\ &\le \frac{(q-2\alpha)\,B_1^2}{(1+q)\,(q-\alpha+\alpha q)}\left\{B_2-\left[1-\mu\frac{[q-\alpha(1+q)]\,(1+q)\,(q-2\alpha)}{q(1-\alpha)^2}\right]B_1^2\right\}.\end{aligned}$$

To show that the bounds are sharp, we define the functions $K_{\varphi n}$ $(n \ge 2)$ by

$$-\frac{(1-\frac{\alpha}{q})qzD_qK_{\varphi n}(z)+\alpha qzD_q\left[zD_qK_{\varphi n}(z)\right]}{(1-\frac{\alpha}{q})K_{\varphi n}(z)+\alpha zD_qK_{\varphi n}(z)}=\varphi(z^{n-1}),$$

$$z^2K_{\varphi n}(z)\mid_{z=0}=0=-z^2K'_{\varphi n}(z)\mid_{z=0}-1,$$

and the functions F_γ and G_γ $(0 \le \gamma \le 1)$ by

$$-\frac{(1-\frac{\alpha}{q})qzD_qF_\gamma(z)+\alpha qzD_q\left[zD_qF_\gamma(z)\right]}{(1-\frac{\alpha}{q})F_\gamma(z)+\alpha zD_qF_\gamma(z)}=\varphi\left(\frac{z(z+\gamma)}{1+\gamma z}\right),$$

$$z^2F_\gamma(z)\mid_{z=0}=0=-z^2F'_\gamma(z)\mid_{z=0}-1,$$

and

$$-\frac{(1-\frac{\alpha}{q})qzD_qG_\gamma(z)+\alpha qzD_q\left[zD_qG_\gamma(z)\right]}{(1-\frac{\alpha}{q})G_\gamma(z)+\alpha zD_qG_\gamma(z)}=\varphi\left(-\frac{z(z+\gamma)}{1+\gamma z}\right),$$

$$z^2G_\gamma(z)\mid_{z=0}=0=-z^2G'_\gamma(z)\mid_{z=0}-1.$$

Cleary the functions $K_{\varphi n}$, F_γ and $G_\gamma \in \mathcal{F}^*_{q,\alpha}(\varphi)$. Also we write $K_\varphi = K_{\varphi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if $f(z)$ is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if $f(z)$ is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f(z)$ is F_γ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f(z)$ is G_γ or one of its rotations. This completes the proof of Theorem 2. □

Remark 2.

(i) For $q \to 1^-$ in Theorem 2, we obtain the result obtained by Aouf et al. [4, Theorem 2];

(ii) Putting $q \to 1^-$ and $\alpha = 0$ in Theorem 2, we obtain the result obtained by Ali and Ravichandran [2, Theorem 5.1].

3. Applications to Functions Defined by $q-$Bessel Function

We recall some definitions of $q-$calculus which we will be used in our paper. For any complex number α, the $q-$shifted factorials are defined by

$$(\alpha;q)_0 = 1;\ (\alpha;q)_n = \prod_{k=0}^{n-1}\left(1-\alpha q^k\right)\ \ (n \in \mathbb{N} = \{1,2,...\}). \tag{3.1}$$

If $|q| < 1$, the definition (3.1) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha;q)_\infty = \prod_{j=0}^{\infty}\left(1-\alpha q^j\right).$$

In terms of the analogue of the gamma function

$$(q^\alpha;q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}\ \ (n>0),$$

where the $q-$gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty}\ \ (0<q<1).$$

We note that

$$\lim_{q\to 1^-}\frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n,$$

where

$$(\alpha)_n = \begin{cases} 1 & \text{if } n=0, \\ \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1) & \text{if } \ n\in\mathbb{N}. \end{cases}$$

Now, consider the $q-$analoge of Bessel fnction defined by (Jackson [8])

$$\mathfrak{J}_\upsilon^{(1)}(z;q) = \frac{(q^{\upsilon+1};q)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{(q;q)_k(q^{\upsilon+1};q)_k}\left(\frac{z}{2}\right)^{2k+\nu}\ \ (0<q<1).$$

Also, let us define

$$\begin{aligned}\mathcal{L}_\upsilon(z;q) &= \frac{2^\upsilon(q;q)_\infty}{(q^{\upsilon+1};q)_\infty(1-q)^\upsilon z^{\upsilon/2+1}}\mathfrak{J}_\upsilon^{(1)}\left(z^{1/2}(1-q);q\right) \\ &= \frac{1}{z}+\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(1-q)^{2(k+1)}}{4^{(k+1)}\ (q;q)_{k+1}(q^{\upsilon+1};q)_{k+1}}z^k\ \ (z\in\mathbb{U}).\end{aligned}$$

By using the Hadamard product (or convolution), we define the linear operator $\mathcal{L}_{q,\upsilon} : \Sigma \to \Sigma$, as follows:

$$\begin{aligned} (\mathcal{L}_{q,\upsilon} f)(z) &= \mathcal{L}_{\upsilon}(z;q) * f(z) \\ &= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(1-q)^{2(k+1)}}{4^{(k+1)}\ (q;q)_{k+1}(q^{\upsilon+1};q)_{k+1}} a_k z^k. \end{aligned}$$

As $q \to 1^-$, the linear operator $\mathcal{L}_{q,\upsilon}$ reduces to the operator $\mathcal{L}_{\upsilon}$ introduced and studied by Aouf et al. [5]. For $0 < q < 1$ and $\alpha \in \mathbb{C}\backslash(0,1]$, let $\mathcal{F}^*_{q,\alpha,\upsilon}(\varphi)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1.1) and satisfies the analytic criterion:

$$-\frac{(1-\frac{\alpha}{q})qzD_q(\mathcal{L}_{q,\upsilon} f) + \alpha qzD_q\left[D_q(\mathcal{L}_{q,\upsilon} f)\right]}{(1-\frac{\alpha}{q})(\mathcal{L}_{q,\upsilon} f) + \alpha zD_q(\mathcal{L}_{q,\upsilon} f)} \prec \varphi(z)\ (z \in \mathbb{U}).$$

Using similar arguments to those in the proof of the above theorems, we obtain the following theorems.

Theorem 3. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{F}^*_{q,\alpha,\upsilon}(\varphi)$ and μ is a complex number, then*

(i) $$\left|a_1 - \mu a_0^2\right| \leq \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})}{(1-q)^2}\left|\frac{B_1\,(q-2\alpha)}{q+\alpha q-\alpha}\right| \times \max\left\{1, \left|\frac{B_2}{B_1} - \left[1 - \mu\frac{(q-2\alpha)\,(1-q^{\upsilon+1})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}\right] B_1\right|\right\}\ (B_1 \neq 0),$$

(ii) $$|a_1| \leq \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})}{(1-q)^2}\left|\frac{B_2\,(q-2\alpha)}{q+\alpha q-\alpha}\right|\ (B_1 = 0).$$

The result is sharp.

Theorem 4. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ...,(B_i > 0,\ i \in \{1,2\},\ \alpha > 0)$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{F}^*_{q,\alpha,\upsilon}(\varphi)$, then*

$$\left|a_1 - \mu a_0^2\right| \leq \begin{cases} \dfrac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,(q-2\alpha)\,B_1^2}{(1-q)^2\,(q+\alpha q-\alpha)} \\ \quad \times \left\{-B_2 + \left[1 - \mu\dfrac{(q-2\alpha)\,(1-q^{\upsilon+1})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}\right] B_1^2\right\} & \text{if } \mu \leq \sigma_1^*, \\ \dfrac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,[q-\alpha(q+1)]\,B_1}{q(1-q)^2} & \text{if } \sigma_1^* \leq \mu \leq \sigma_2^*, \\ \dfrac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,[q-\alpha(q+1)]}{q(1-q)^2} \\ \quad \times \left\{B_2 - \left[1 - \mu\dfrac{(q-2\alpha)\,(1-q^{\upsilon+1})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}\right] B_1^2\right\} & \text{if } \mu \geq \sigma_2^*, \end{cases}$$

where

$$\sigma_1^* = \frac{(q-\alpha)^2(1-q^{\upsilon+2})\left[-B_1-B_2+B_1^2\right]}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2},$$

and

$$\sigma_2^* = \frac{(q-\alpha)^2(1-q^{\upsilon+2})\left[B_1-B_2+B_1^2\right]}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2}.$$

The result is sharp. Further, let

$$\sigma_3^* = \frac{(q-\alpha)^2(1-q^{\upsilon+2})\left[-B_2+B_1^2\right]}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2}.$$

(i) *If* $\sigma_1^* \le \mu \le \sigma_3^*$, *then*

$$\begin{aligned}
&\left|a_1-\mu a_0^2\right| + \frac{(1-q^{\upsilon+2})(q-\alpha)^2}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2}\\
&\times\left\{(B_1+B_2)+\left[\mu\frac{(q-2\alpha)\,(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}-1\right]B_1^2\right\}|a_0|^2\\
\le\;& \frac{4^2\,(q-2\alpha)\,(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2\,(q-\alpha+\alpha q)}.
\end{aligned}$$

(ii) *If* $\sigma_3^* \le \mu \le \sigma_2^*$, *then*

$$\begin{aligned}
&\left|a_1-\mu a_0^2\right| + \frac{(1-q^{\upsilon+2})(q-\alpha)^2}{(q-2\alpha)\,(q-\alpha+\alpha q)\,(1-q^{\upsilon+1})B_1^2}\\
&\times\left\{(B_1-B_2)+\left[1-\mu\frac{(q-2\alpha)\,(1-q^{\upsilon+1})(1-q^{\upsilon+2})\,(q-\alpha+\alpha q)}{(1-q^{\upsilon+2})(q-\alpha)^2}\right]B_1^2\right\}|a_0|^2\\
\le\;& \frac{4^2\,(q-2\alpha)\,(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2\,(q-\alpha+\alpha q)}.
\end{aligned}$$

References

[1] M. H. Abu-Risha, M. H. Annaby, M. E. H. Ismail and Z. S. Mansour, *Linear q-difference equations*, Z. Anal. Anwend., **26**(2007), 481–494.

[2] R. M. Ali and V. Ravichandran, *Classes of meromorphic α−convex functions*, Taiwanese J. Math., **14(4)**(2010), 1479–1490.

[3] M. K. Aouf, *Coefficient results for some classes of meromorphic functions*, J. Natur. Sci. Math., **27(2)**(1987), 81–97.

[4] M. K. Aouf, R. M. El-Ashwah and H. M. Zayed, *Fekete–Szego inequalities for certain class of meromorphic functions*, J. Egyptian Math. Soc., **21**(2013), 197–200.

[5] M. K. Aouf, A. O. Mostafa and,H. M. Zayed, *Convolution properties for some subclasses of meromorphic functions of complex order*, Abstr. Appl. Anal., Art. ID 973613 (2015), 6 pp.

[6] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[7] M. E. H. Ismail, E. Merkes and D. Styer, *A generalization of starlike functions*, Complex Variables Theory Appl., **14**(1990), 77–84.

[8] F. H. Jackson, *The application of basic numbers to Bessel's and Legendre's functions*, (Second paper), Proc. London Math. Soc., **3(2)**(1904-1905), 1–23.

[9] V. G. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.

[10] S. Kanas and D. Raducanu, *Some class of analytic functions related to conic domains*, Math. Slovaca, **64(5)**(2014) 1183–1196.

[11] V. Karunakaran, *On a class of meromorphic starlike functions in the unit disc*, Math. Chronicle, **4(2-3)**(1976), 112-121.

[12] S. R. Kulkarni and S. S. Joshi, *On a subclass of meromorphic univalent functions with positive coefficients*, J. Indian Acad. Math., **24(1)**(2002), 197-205.

[13] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994, 157–169.

[14] J. E. Miller, *Convex meromorphic mappings and related functions*, Proc. Amer. Math. Soc., **25**(1970), 220-228.

[15] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, 1952.

[16] Ch. Pommerenke, *On meromorphic starlike functions*, Pacific J. Math., **13**(1963), 221-235.

[17] T. Shanmugam and S. Sivasubramanian, *On the Fekete-Szegö problem for some subclasses of analytic functions*, J. Inequal. Pure Appl. Math., **6(3)**(2005), Article 71, 6 pp.

[18] H. Silverman, K. Suchithra, B. A. Stephen and A. Gangadharan, *Coefficient bounds for certain classes of meromorphic functions*, J. Inequal. Appl., (2008), Art. ID 931981, 9 pp.

[19] S. Sivasubramanian and M.Govindaraj, *On a class of analytic functions related to conic domains involving q-calculus*, Anal. Math., **43(3)**(2017), 475–487.