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First Order Differential Subordinations for Carathéodory Functions

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Abstract. The well-known theory of differential subordination developed by Miller and Mocanu is applied to obtain several inclusions between Carathéodory functions and starlike functions. These inclusions provide sufficient conditions for normalized analytic functions to belong to certain class of Ma-Minda starlike functions.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f with $f(0) = 0$ and $f'(0) = 1$, defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the subset of $\mathcal{A}$ containing one-to-one functions. Let f and g be analytic in $\mathbb{D}$. The function f is subordinate to g, written as $f \prec g$, if there is a Schwarz function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent in $\mathbb{D}$ then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Let $\mathcal{P}$ be the class of Carathéodory functions $p : \mathbb{D} \to \mathbb{C}$ of the form $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for all z in $\mathbb{D}$ such that $\text{Re}(p(z)) > 0$. These functions are analytic in $\mathbb{D}$. A Carathéodory function f maps $\mathbb{D}$ into the right-half plane. The function $p(z) = (1 + z)/(1 - z)$ is a prominent

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example of Carathéodory function which maps $\mathbb{D}$ conformally onto the right-half plane. The first two important results involving first-order differential implications were given in 1935 by Goluzin and in 1947 by Robinson. Goluzin [7] proved that if $zp'(z)$ is subordinate to a convex function h then $p(z) \prec \int_0^z h(t)t^{-1}dt$. After this basic result, many authors investigated several aspects of first order differential subordination. This first order differential subordination has many applications in the theory of univalent functions. Miller and Mocanu [15] discussed the general theory of differential subordination.

Several classes of starlike and convex functions are characterized by the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ and unified classes of starlike and convex functions using concept of Hadamard product and subordination. Further, Shanmugam [22] studied the class $\mathcal{S}^*_g(\omega)$ of all $f \in \mathcal{A}$ satisfying $z(f * g)'/(f * g) \prec \omega$, where ω is a convex function, g is a fixed function in $\mathcal{A}$. For $g(z) = z/(1-z)^\alpha$ the class was investigated by Padmanabhan and Parvatham [19]. When g is $z/(1-z)$ and $z/(1-z)^2$, the subclass $\mathcal{S}^*_g(\omega)$ reduces to $\mathcal{S}^*(\omega)$ and $\mathcal{K}(\omega)$ respectively. In 1992, Ma and Minda [13] considered the class Ω consisting of analytic univalent functions ω with the positive real part mapping $\mathbb{D}$ onto domains symmetric with respect to real axis and starlike with respect to $\omega(0) = 1$ and $\omega'(0) > 0$. For such an $\omega \in \Omega$, Ma and Minda [13] proved distortion, growth, and covering theorems. The class $\mathcal{S}^*(\omega)$ generalizes many subclasses of $\mathcal{A}$, for example, $\mathcal{S}^*[A, B] := \mathcal{S}^*((1 + Az)/(1 + Bz))$ $(-1 \leq B < A \leq 1)$ [10], $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+z})$ [25], $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$ [14], $\mathcal{S}^*_S := \mathcal{S}^*(\varphi_S(z))$ [4], $\mathcal{S}^*_C := \mathcal{S}^*(\varphi_C(z))$ [23], $\mathcal{S}^*_R := \mathcal{S}^*(\varphi_k(z))$ [11], and $\mathcal{S}^*_{\mathfrak{C}} := \mathcal{S}^*(\varphi_{\mathfrak{C}}(z))$ [20] where $\varphi_C(z) := 1 + \frac{4}{3}z + \frac{2}{3}z^2$, $\varphi_S(z) := 1 + \sin z$, $\varphi_k(z) := 1 + \frac{z}{k}\frac{(k+z)}{(k-z)}$ $(k = \sqrt{2} + 1)$ and $\varphi_{\mathfrak{C}}(z) := z + \sqrt{1 + z^2}$ respectively.

In 1989, Nunokawa *et al.* [17] showed that $zp'(z) \prec z$ implies $p(z) - 1 \prec z$ and further authors [18] have shown some sufficient condition for Carathéodory functions. Further, Ali *et al.* [2] obtained the estimates on β for which $1 + \beta zp'(z)/p^j(z) \prec (1 + Dz)/(1 + Ez)$ $(j = 0, 1, 2)$ implies $p(z) \prec (1 + Az)/(1 + Bz)$, where $A, B, D, E \in [-1, 1]$. In 2013, Sivaprasad Kumar *et al.* [24] determined the bound on β with $-1 < E < 1$ and $|D| \leq 1$ such that $1 + \beta zp'(z)/p^j(z) \prec (1 + Dz)/(1 + Ez)$ $(j = 0, 1, 2)$ implies $p(z) \prec \sqrt{1+z}$. Recently, Kumar and Ravichandran [12] determined certain sufficient conditions for first order differential subordinations to imply the corresponding analytic solution is subordinate to a function with positive real part. For related results, see [1, 3, 5, 6, 8, 9, 21, 26].

Motivated by these earlier works, in the next section, we obtain the sharp bound on β so that the Carathéodory function p is subordinate to a starlike function with positive real part like $\sqrt{1+z}$, e^z, $(1 + Az)/(1 + Bz)$, whenever $1 + \beta zp'(z)/p^j(z)$ $((j = 0, 1, 2)$ is subordinate to certain well known starlike functions. Our estimates on β are sharp. Several sufficient conditions for functions to belong to the above defined classes can be obtained as an application of the subordination results for Carathéodory functions.

2. Main Results

The first result of this section gives the sharp bound on β such that each of the first order differential subordination $1+\beta zp'(z) \prec \varphi_{\mathbb{C}}(z)$, $1+\beta zp'(z)/p(z) \prec \varphi_{\mathbb{C}}(z)$, $1+\beta zp'(z)/p^2(z) \prec \varphi_{\mathbb{C}}(z)$ implies p is subordinated to some well known starlike functions.

Theorem 2.1. *Let p be analytic function defined in $\mathbb{D}$ with $p(0)=1$ satisfying the subordination $1+\beta zp'(z)/p^j(z) \prec \phi_{\mathbb{C}}$ $((j=0,1,2)$. Then*

(a) *$p(z) \prec e^z$ holds for $\beta \geq \beta_j$ $(j=0,1,2)$, where*

$$\beta_0 = \frac{e(2-\sqrt{2}-\log 2+\log(1+\sqrt{2}))}{e-1} \approx 1.22447,$$
$$\beta_1 = \sqrt{2}+\log(2)-\log(1+\sqrt{2}) \approx 1.22599$$

and

$$\beta_2 = \frac{e(\sqrt{2}+\log 2-\log(1+\sqrt{2}))}{e-1} \approx 1.93948.$$

(b) *$p(z) \prec \varphi_{\mathbb{C}}(z)$ holds for $\beta \geq \beta_j$ $(j=0,1,2)$, where*

$$\beta_0 = \frac{2-\sqrt{2}-\log 2+\log(1+\sqrt{2})}{2-\sqrt{2}} \approx 1.32132,$$
$$\beta_1 = \frac{\sqrt{2}+\log 2-\log(1+\sqrt{2})}{\log(1+\sqrt{2})} \approx 1.391$$

and

$$\beta_2 = \frac{2+\sqrt{2}+(1+\sqrt{2})\log 2-(1+\sqrt{2})\log(1+\sqrt{2}))}{\sqrt{2}} \approx 2.09289.$$

(c) *$p(z) \prec \varphi_S(z)$ holds for $\beta \geq \beta_j$ $(j=0,1,2)$, where*

$$\beta_0 = (\sqrt{2}+\log 2-\log(1+\sqrt{2}))\csc(1) \approx 1.45696,$$
$$\beta_1 = \frac{\sqrt{2}+\log 2-\log(1+\sqrt{2})}{\log(1+\sin(1))} \approx 2.00796$$

and

$$\beta_2 = (\sqrt{2}+\log 2-\log(1+\sqrt{2}))(1+\csc(1)) \approx 2.68294.$$

(d) *$p(z) \prec \varphi_k(z)$ holds for $\beta \geq \beta_j$ $(j = 0, 1, 2)$ where*

$$\beta_0 = 2 + \sqrt{2} - (3 + 2\sqrt{2})(\log 2 - \log(1 + \sqrt{2})) \approx 4.51128,$$
$$\beta_1 = \frac{\sqrt{2} - 2 + \log 2 - \log(1 + \sqrt{2})}{\log(2 + \sqrt{2}) - \log(3 + 2\sqrt{2})} \approx 4.11214,$$

and

$$\beta_2 = 2(\sqrt{2} - (1 + \sqrt{2})(\log 2 - \log(1 + \sqrt{2})) \approx 3.73726.$$

(e) *$p(z) \prec \varphi_C(z)$ holds for $\beta \geq \beta_j$ $(j = 0, 1, 2)$ where*

$$\beta_0 = \frac{3}{2}(2 - \sqrt{2} - \log 2 + \log(1 + \sqrt{2})) \approx 1.16102,$$
$$\beta_1 = \frac{\sqrt{2} + \log 2 - \log(1 + \sqrt{2})}{\log 3} \approx 1.11594,$$

and

$$\beta_2 = \frac{3}{2}(\sqrt{2} + \log 2 - \log(1 + \sqrt{2})) \approx 1.83898.$$

The bounds on β are best possible.

The following lemma will be used in our investigation.

Lemma 2.2.([16, Theorem 3.4h, p. 132]) *Let q be analytic in $\mathbb{D}$ and let ψ and ν be analytic in a domain U containing $q(\mathbb{D})$ with $\psi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) := zq'(z)\psi(q(z))$ and $h(z) := \nu(q(z)) + Q(z)$. Suppose that (i) either h is convex, or Q is starlike univalent in $\mathbb{D}$ and (ii)* $\operatorname{Re}(zh'(z)/Q(z)) > 0$ *for $z \in \mathbb{D}$. If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq U$ and $\nu(p(z)) + zp'(z)\psi(p(z)) \prec \nu(q(z)) + zq'(z)\psi(q(z))$, then $p(z) \prec q(z)$, and q is best dominant.*

Proof of Theorem 2.1 (a) Case (i)($j = 0$) The function $q_\beta : \mathbb{D} \to \mathbb{C}$ defined by

$$q_\beta(z) = 1 + \frac{1}{\beta}\left(z + \sqrt{1 + z^2} - \log(1 + \sqrt{1 + z^2}) - 1 + \log 2\right)$$

is analytic in $\mathbb{D}$ and satisfies the differential equation $1 + \beta z q_\beta'(z) = \varphi_{\mathbb{C}}(z)$ for $z \in \mathbb{D}$. Consider the functions $\nu(w) = 1$ and $\psi(w) = \beta$. The function $Q : \mathbb{D} \to \mathbb{C}$ defined by $Q(z) = zq_\beta'(z)\psi(q_\beta(z)) = \beta z q_\beta'(z) = \varphi_{\mathbb{C}}(z) - 1$ is starlike in $\mathbb{D}$. If the function $h : \mathbb{D} \to \mathbb{C}$ defined by $h(z) := \nu(q_\beta(z)) + Q(z) = 1 + Q(z)$, then $zh'(z)/Q(z) = zQ'(z)/Q(z)$ is a function with positive real part. By making use of Lemma 2.2, we see that the subordination

$$1 + \beta z p'(z) \prec 1 + \beta z q_\beta'(z) \text{ implies } p(z) \prec q_\beta(z).$$

The desired subordination $p(z) \prec e^z$ holds if the subordination $q_\beta(z) \prec e^z$ holds. Since q_β is an increasing function on $(-1,1)$, $q_\beta(z) \prec e^z$ holds if

$$e^{-1} \leq q_\beta(-1) \leq q_\beta(1) \leq e.$$

This gives a necessary condition for $p(z) \prec e^z$ to hold. Surprisingly, this necessary condition is also sufficient. This can be seen by looking at the graph of the exponential function. The inequalities $q_\beta(-1) \geq e^{-1}$ and $q_\beta(1) \leq e$ reduces to $\beta \geq \beta_0$ and $\beta \geq \beta_0^*$, where

$$\beta_0 = \frac{2e - \sqrt{2}e - e\log(2) + e\log(1+\sqrt{2})}{e-1} \quad \text{and} \quad \beta_0^* = \frac{\sqrt{2} + \log(2) - \log(1+\sqrt{2})}{e-1}$$

respectively. Thus, the subordination $q_\beta(z) \prec e^z$ holds if $\beta \geq \max\{\beta_0, \beta_0^*\} = \beta_0$.

Case (ii)$(j=1)$ The analytic function

$$q_\beta(z) = \exp\left(\frac{1}{\beta}\left(z + \sqrt{1+z^2} - \log\left(1+\sqrt{1+z^2}\right) - 1 + \log 2\right)\right)$$

satisfies $1 + \beta z q_\beta'(z)/q_\beta(z) = \varphi_{\mathbb{C}}(z)$. Consider the functions $\nu(w) = 1$ and $\psi(w) = \beta/w$. The function $Q : \mathbb{D} \to \mathbb{C}$ defined by $Q(z) = zq_\beta'(z)\psi(q_\beta(z)) = \varphi_{\mathbb{C}}(z) - 1$ is starlike in $\mathbb{D}$. The function $h(z) := \nu(q(z)) + Q(z) = 1 + Q(z)$ satisfies $\mathrm{Re}(zh'(z)/Q(z)) > 0$ in $\mathbb{D}$. By Lemma 2.2, the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq_\beta'(z)}{q_\beta(z)} \text{ implies } p(z) \prec q_\beta(z).$$

Thus as in the proof of case(i), the subordination $p(z) \prec e^z$ holds if $\beta \geq \sqrt{2} + \log 2 - \log(1+\sqrt{2})$.

Case (iii)$(j=2)$. The analytic function

$$q_\beta(z) = \left(1 - \frac{1}{\beta}\left(z + \sqrt{1+z^2} - \log\left(1+\sqrt{1+z^2}\right) - 1 + \log 2\right)\right)^{-1}$$

satisfies the differential equation $1 + \beta z q_\beta'(z)/q_\beta^2(z) = \varphi_{\mathbb{C}}(z)$. Consider the functions $\nu(w) = 1$ and $\psi(w) = \beta/w^2$. The function $Q : \mathbb{D} \to \mathbb{C}$ defined by $Q(z) = zq'(z)\psi(q(z)) = zq'(z)/q^2(z) = \varphi_{\mathbb{C}}(z) - 1$ is starlike in $\mathbb{D}$. The function $h(z) := \nu(q_\beta(z)) + Q(z) = 1 + Q(z)$ satisfies $\mathrm{Re}(zh'(z)/Q(z)) > 0$ in $\mathbb{D}$. Therefore, by use of Lemma 2.2, it follows that the subordination

$$1 + \beta\frac{zp'(z)}{p^2(z)} \prec 1 + \beta\frac{zq_\beta'(z)}{q_\beta^2(z)} \text{ implies } p(z) \prec q_\beta(z).$$

As in proof of case(i), the subordination $p(z) \prec e^z$ holds if

$$\beta \geq \frac{e(\sqrt{2} + \log 2 - \log(1+\sqrt{2}))}{(e-1)}.$$

The proofs of part (b)-(e) of this theorem are similar to the proof of part (a). □

By applying Theorem 2.1(a) to the function $p(z) = zf'(z)/f(z)$ we see that any one of the following is a sufficient condition for f to be in $\mathcal{S}^*_e$.

Let $\beta_0 = \frac{e(2-\sqrt{2}-\log 2+\log(1+\sqrt{2}))}{e-1} \approx 1.22447$, $\beta_1 = \sqrt{2} + \log(2) - \log(1+\sqrt{2}) \approx 1.22599$, and $\beta_2 = \frac{e(\sqrt{2}+\log 2-\log(1+\sqrt{2}))}{e-1} \approx 1.93948$. If $f \in \mathcal{A}$ satisfying the following subordinations

$$1 + \beta \left(\frac{zf'(z)}{f(z)}\right)^i \left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \prec \varphi_{\complement}(z) \text{ for } \beta \geq \beta_{i+1},\ (i = -1, 0, 1),$$

then $f \in \mathcal{S}^*_e$.

Theorem 2.3. *Let $-1 < B < A < 1$ and p be analytic in $\mathbb{D}$ with $p(0) = 1$. Anyone of the following subordination conditions is sufficient for $p(z) \prec (1 + Az)/(1 + Bz)$:*

(a) $1 + \beta z p'(z) \prec \varphi_{\complement}(z)$ *for* $\beta \geq \max\{\beta_1, \beta_2\}$, *where*

$$\beta_1 = \frac{(1+B)(\sqrt{2} + \log 2 - \log(1+\sqrt{2}))}{A - B}$$

and

$$\beta_2 = \frac{(1-B)(2 - \sqrt{2} - \log 2 + \log(1+\sqrt{2}))}{A - B};$$

(b) $1 + \beta \frac{zp'(z)}{p(z)} \prec \varphi_{\complement}(z)$ *for* $\beta \geq \max\{\beta_3, \beta_4\}$, *where*

$$\beta_3 = \frac{\sqrt{2} - 2 + \log 2 - \log(1+\sqrt{2})}{\log(1-A) - \log(1-B)}$$

and

$$\beta_4 = \frac{\sqrt{2} + \log 2 - \log(1+\sqrt{2})}{\log(1+A) - \log(1+B)};$$

(c) $1 + \beta \frac{zp'(z)}{p^2(z)} \prec \varphi_{\complement}(z)$ *for* $\beta \geq \max\{\beta_5, \beta_6\}$, *where*

$$\beta_5 = \frac{(1+A)(\sqrt{2} + \log 2 - \log(1+\sqrt{2}))}{A - B}$$

and

$$\beta_6 = \frac{(1-A)(2 - \sqrt{2} - \log 2 + \log(1+\sqrt{2}))}{A - B}.$$

Proof. (a) We take the functions $q_\beta(z)$, ν, ψ, $Q(z)$, and $h(z)$ as in the proof of case (i) of the Theorem 2.1(a). By an application of Lemma 2.2, the subordination $1+\beta zp'(z) \prec 1+\beta zq'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. Similar reasoning as in the proof of Theorem 2.1(a), the necessary subordination $p(z) \prec (1+Az)/(1+Bz)$ holds if $\beta \geq \max\{\beta_1, \beta_2\}$.

Let $C_0 = 1-\sqrt{2}-\log 2/(1+\sqrt{2})$. A simple calculation gives that if $B \geq C_0$, then $\beta \geq \beta_1$ or $B \leq C_0$, then $\beta \geq \beta_2$.

(b) By defining the functions $q_\beta(z)$, ν, ψ, $Q(z)$, and $h(z)$ as in case (ii) of the Theorem 2.1(a), Lemma 2.2 is applied. Therefore, the subordination $1+\beta zp'(z)/p(z) \prec 1+\beta zq'(z)//q_\beta(z)$ implies $p(z) \prec q_\beta(z)$. By the similar reasoning as in the proof of Theorem 2.1(b), the necessary subordination $p(z) \prec (1+Az)/(1+Bz)$ holds if $\beta \geq \max\{\beta_3, \beta_4\}$.

(c) Consider the functions $q_\beta(z)$, ν, ψ, $Q(z)$, and $h(z)$ as in case (iii) of the Theorem 2.1(a). By the application of Lemma 2.2, the subordination $1+\beta zp'(z)/p^2(z) \prec 1+\beta zq'(z)//q_\beta^2(z)$ implies $p(z) \prec q_\beta(z)$. Similar reasoning as in the proof of Theorem 2.1(c), the necessary subordination $p(z) \prec (1+Az)/(1+Bz)$ holds if $\beta \geq \max\{\beta_5, \beta_6\}$.

By taking $C_1 = (1+\log(1+\sqrt{2}))/((1+\sqrt{2})+\log 2)$, it can be seen that if $B \geq C_1$, then $\beta \geq \beta_5$ or $B \leq C_1$, then $\beta \geq \beta_6$. □

For a function $f \in \mathcal{A}$, by applying Theorem 2.3 to $p(z) = zf'(z)/f(z)$ to , we see that any one of the following is a sufficient condition for $f \in \mathcal{S}^*[A, B]$:

$$1+\beta\left(\frac{zf'(z)}{f(z)}\right)\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right) \prec \varphi_{\text{☾}}(z) \text{ for } \beta \geq \max\{\beta_1, \beta_2\},$$

$$1+\beta\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right) \prec \varphi_{\text{☾}}(z) \text{ for } \beta \geq \max\{\beta_2, \beta_4\},$$

$$1+\beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right) \prec \varphi_{\text{☾}}(z) \text{ for } \beta \geq \max\{\beta_5, \beta_6\}.$$

Next result provides the bound on β such that the subordination $p(z) \prec \varphi_{\text{☾}}(z)$ holds whenever $1+\beta zp'(z)$, $1+\beta zp'(z)/p(z)$, or $1+\beta zp'(z)/p^2(z)$ is subordinate to some well known starlike functions.

Theorem 2.4. *Let p be analytic in $\mathbb{D}$ with $p(0)=1$. Anyone of following subordination conditions is sufficient for $p(z) \prec \varphi_{\text{☾}}(z)$:*

(a) $1+\beta\frac{zp'(z)}{p^j(z)} \prec e^z$ *for* $\beta \geq \beta_j$ $(j=0,1,2)$ *where*

$$\beta_0 = \frac{1}{2-\sqrt{2}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!n} \approx 1.35988, \quad \beta_1 = \frac{1}{\log(\sqrt{2}+1)}\sum_{n=1}^{\infty}\frac{1}{n!n} \approx 1.49528$$

and

$$\beta_2 = \frac{\sqrt{2}+1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{n!n} \approx 2.24979;$$

(b) $1+\beta\frac{zp'(z)}{p^j(z)} \prec \varphi_S(z)$ *for* $\beta \geq \beta_j$ $(j=0,1,2)$, *where*

$$\beta_0 = \frac{1}{2-\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)} \approx 1.61506,$$

$$\beta_1 = \frac{1}{\log(\sqrt{2}+1)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)} \approx 1.07342$$

and

$$\beta_2 = \frac{\sqrt{2}+1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)} \approx 1.61506;$$

(c) $1+\beta\frac{zp'(z)}{p^j(z)} \prec \varphi_k(z)$ *for* $\beta \geq \beta_j$ $(j=0,1,2)$, *where*

$$\beta_0 = \frac{(-1-(2\sqrt{2}+2)\log\frac{\sqrt{2}}{\sqrt{2}+1}}{2(\sqrt{2}+1)} \approx 0.327693,$$

$$\beta_1 = \frac{(-1-(2\sqrt{2}+2)\log\frac{\sqrt{2}}{\sqrt{2}+1}}{(\sqrt{2}+1)\log(\sqrt{2}+1)} \approx 0.743597$$

and

$$\beta_2 = \frac{(-1-(2\sqrt{2}+2)\log\frac{\sqrt{2}}{\sqrt{2}+1}}{\sqrt{2}} \approx 1.11881;$$

(d) $1+\beta\frac{zp'(z)}{p^j(z)} \prec \varphi_C(z)$ *for* $\beta \geq \beta_j$ $(j=0,1,2)$ *where*

$$\beta_0 = \frac{1}{2-\sqrt{2}} \approx 1.70711, \;\; \beta_1 = \frac{5}{3(\log(\sqrt{2}+1))} \approx 1.89099$$

and

$$\beta_2 \geq \frac{5(\sqrt{2}+1)}{3\sqrt{2}} \approx 2.84518.$$

Proof. (a) Case (i) $(j=0)$ The function

$$q_\beta(z) = 1 + \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{z^n}{n!n}$$

is analytic and satisfies the differential equation $1+\beta z q_\beta'(z) = e^z$ for $z \in \mathbb{D}$. Consider the functions ν, ψ as in Theorem 2.1(a). The function $Q(z) = \beta z q'(z) = e^z - 1$ is starlike and the function $h(z) = \nu(q_\beta(z)) + Q(z)$ satisfies an inequality $\operatorname{Re}(zh'(z)/Q(z))) > 0$ in $\mathbb{D}$. By making use of Lemma 2.2, the subordination $1+\beta zp'(z) \prec 1+\beta z q_\beta'(z)$ implies $p(z) \prec q_\beta(z)$. As in the proof of Theorem 2.1(a), the necessary subordination $p(z) \prec \varphi_{\leftmoon}(z)$ holds if $\beta \geq \beta_0$.

Case (ii) ($j=1$) Define the function q_β as:

$$q_\beta(z) = \exp\left(\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{z^n}{n!n}\right),$$

which is analytic in $\mathbb{D}$. The function $q_\beta(z)$ satisfies the differential equation $1+\beta z q'_\beta(z)/q_\beta(z) = e^z$. Consider the functions ν and ψ as in Theorem 2.1(a). Note that the function $Q(z) = \beta z q'_\beta(z)/q_\beta(z) = e^z - 1$ is starlike and the function $h(z) = \nu(q_\beta(z)) + Q(z)$ satisfies an inequality $\operatorname{Re}(zh'(z)/Q(z))) > 0$ in $\mathbb{D}$. From the view of Lemma 2.2, the subordination $1+\beta zp'(z)/p(z) \prec 1+\beta zq'_\beta(z)/q_\beta(z)$ implies $p(z) \prec q_\beta(z)$. As in the proof of Theorem 2.1(a), the necessary subordination $p(z) \prec \varphi_{\mathbb{C}}(z)$ holds if $\beta \geq \beta_1$.

Case (iii) ($j=2$) Define the function q_β as:

$$q_\beta(z) = \left(1 - \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{z^n}{n!n}\right)^{-1}$$

which is analytic in $\mathbb{D}$ and satisfies the differential equation $1+\beta zq'_\beta(z)/q^2_\beta(z) = e^z$ for $z \in \mathbb{D}$. We take the functions ν and ψ as in Theorem 2.1(a). The function $Q(z) = \beta zq'_\beta(z)/q^2_\beta(z) = e^z - 1$ is starlike and the function $h(z) = 1+Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z))) > 0$ in $\mathbb{D}$. From Lemma 2.2, the subordination $1+\beta zp'(z)/p^2(z) \prec 1+\beta zq'_\beta(z)/q^2_\beta(z)$ implies $p(z) \prec q_\beta(z)$. The desired subordination $p(z) \prec \varphi_{\mathbb{C}}(z)$ holds if $\beta \geq \beta_2$ as in the proof of Theorem 2.1(a).

(b) The analytic functions

$$q_{1\beta}(z) = 1 + \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!(2n+1)}, \quad q_{2\beta}(z) = \exp\left(\frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!(2n+1)}\right)$$

and

$$q_{3\beta}(z) = \left(1 - \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!(2n+1)}\right)^{-1}.$$

satisfies the differential equations $1+\beta zq'(z)/q^j_{i_\beta}(z) = \varphi_S(z)$ for $z \in \mathbb{D}$ where $i, j = 0, 1, 2$. The required subordination $p(z) \prec \varphi_{\mathbb{C}}(z)$ holds for all three cases if $\beta \geq \beta_0,\ \beta_1$, and β_2 as in Theorem 2.4(a) respectively.

(c) The functions

$$q_{1_\beta}(z) = 1 - \frac{1}{\beta k}\left(z + 2k\log\left(1 - \frac{z}{k}\right)\right), \quad q_{2_\beta}(z) = \exp\left(-\frac{1}{\beta k}\left(z + 2k\log\left(1 - \frac{z}{k}\right)\right)\right)$$

and

$$q_{3_\beta}(z) = \left(1 + \frac{1}{\beta k}\left(z + 2k\log\left(1 - \frac{z}{k}\right)\right)\right)^{-1}$$

satisfies the differential equations $1+\beta zq'(z)/q^{j}_{i_\beta}(z)=\varphi_k(z)$ for $z\in\mathbb{D}$, where $i,j=0,1,2$. For all three cases, the required subordination $p(z)\prec\varphi_{\mathbb{C}}(z)$ holds if $\beta\geq\beta_0,\ \beta_1$, and β_2 as in Theorem 2.4(a) respectively.

(d) The analytic functions

$$q_{1_\beta}(z)=1+\frac{1}{\beta}\left(\frac{4}{3}z+\frac{1}{3}z^2\right)\quad q_{2_\beta}(z)=\exp\left(\frac{1}{\beta}\left(\frac{4}{3}z+\frac{1}{3}z^2\right)\right)$$

and

$$q_{3_\beta}(z)=\left(1-\frac{1}{\beta}\left(\frac{4}{3}z+\frac{1}{3}z^2\right)\right)^{-1}$$

satisfy the differential equations $1+\beta zq'(z)/q^{j}_{i_\beta}(z)=\varphi_C(z)$ for $z\in\mathbb{D}$, where $i,j=0,1,2$. Proceeding as in part (a) of this theorem, we get the subordination $p(z)\prec\varphi_{\mathbb{C}}(z)$ if $\beta\geq\beta_0,\ \beta_1$, and β_2 respectively. □

Theorem 2.5. *Let $-1<B<A<1$, $B\neq 0$ and $p(z)$ be the analytic function with $p(0)=1$. Then each of the following subordination is sufficient for $p(z)\prec\varphi_S(z)$:*

(a) $1+\beta zp'(z)\prec\frac{1+Az}{1+Bz}$ *for* $\beta\geq\max\{\beta_1,\ \beta_2\}$ *where*

$$\beta_1=\frac{(A-B)\log(1-B)^{-1}}{B\sin(1)}\ \ \textit{and}\ \ \beta_2=\frac{(A-B)\log(1+B)}{B\sin(1)}.$$

(b) $1+\beta\frac{zp'(z)}{p(z)}\prec\frac{1+Az}{1+Bz}$ *for* $\beta\geq\max\{\beta_3,\ \beta_4\}$ *where*

$$\beta_3=\frac{(A-B)\log(1-B)^{-1}}{B\log(1-\sin(1))^{-1}}\ \ \textit{and}\ \ \beta_4=\frac{(A-B)\log(1+B)}{B\log(1+\sin(1))}.$$

(c) $1+\beta\frac{zp'(z)}{p^2(z)}\prec\frac{1+Az}{1+Bz}$ *for* $\beta\geq\max\{\beta_5,\ \beta_6\}$ *where*

$$\beta_5=\frac{(A-B)(1-\sin(1))\log(1-B)^{-1}}{B\sin(1)}$$

and

$$\beta_6=\frac{(A-B)(1+\sin(1))\log(1+B)}{B\sin(1)}.$$

The estimates on β are sharp.

Proof. (a) Define the analytic function

$$q_\beta(z)=1+\frac{(A-B)\log(1+Bz)}{B\beta}.$$

Consider the functions ν and ψ as in previous theorem. The function $Q : \mathbb{D} \to \mathbb{C}$ defined by $Q(z) = zq'_\beta(z)\psi(q_\beta(z)) = (A-B)z/(1+Bz)$. Since $1+\beta zq'_\beta(z) = (1+Az)/(1+Bz)$ and $(1+Az)/(1+Bz)-1$ is starlike in $\mathbb{D}$, then Q is starlike in $\mathbb{D}$. The function $h : \mathbb{D} \to \mathbb{C}$ defined by $h(z) := \nu(q_\beta(z)) + Q(z) = 1 + Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z)) > 0$ in $\mathbb{D}$. Therefore, by making use of Lemma 2.2, it is easy to see that the subordination $1+\beta zp'(z) \prec 1+\beta zq'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. As in Theorem 2.1(a), the subordination $p(z) \prec \varphi_S(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\}$.

(b) Let the function $q_\beta(z)$ be defined by

$$q_\beta(z) = \exp\left(\frac{(A-B)\log(1+Bz)}{B\beta}\right).$$

Consider the functions ν, ψ as in Theorem 2.1. The function $Q : \mathbb{D} \to \mathbb{C}$ defined by $Q(z) = zq'_\beta(z)\psi(q_\beta(z)) = \beta zq'_\beta(z)/q_\beta(z) = (A-B)z/(1+Bz)$ is starlike in $\mathbb{D}$ and the function $h(z) := \nu(q(z)) + Q(z) = 1 + Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z)) > 0$ in $\mathbb{D}$. Hence, from Lemma 2.2, the subordination $p(z) \prec q_\beta(z)$ holds, if $1+\beta zp'(z)/p(z) \prec 1+\beta zq'_\beta(z)/q_\beta(z)$ holds. As in Theorem 2.1(a), the required subordination $p(z) \prec \varphi_S(z)$ holds if $\beta \geq \max\{\beta_3, \beta_4\}$.

(c) Let the function $q_\beta(z)$ be defined by

$$q_\beta(z) = \left(1 - \frac{(A-B)\log(1+Bz)}{B\beta}\right)^{-1}.$$

Consider the functions ν, ψ as in Theorem 2.1. The function $Q : \mathbb{D} \to \mathbb{C}$ defined by $Q(z) = zq'_\beta(z)\psi(q_\beta(z)) = \beta zq'_\beta(z)/q^2_\beta(z) = (A-B)z/(1+Bz)$ is starlike in $\mathbb{D}$ and the function $h(z) := \nu(q_\beta(z)) + Q(z) = 1 + Q(z)$ satisfies an inequality $\operatorname{Re}(zh'(z)/Q(z)) > 0$ in $\mathbb{D}$. Hence, from Lemma 2.2, the subordination $p(z) \prec q_\beta(z)$ holds, if subordination $1+\beta zp'(z)/p^2(z) \prec 1+\beta zq'_\beta(z)/q^2_\beta(z)$ holds. As in Theorem 2.1(a), the required subordination $p(z) \prec \varphi_S(z)$ holds if $\beta \geq \max\{\beta_5, \beta_6\}$.

Let $C_1 = \log(1-B^2)^{-1} + \sin(1)\log((1-B)(1+B)^{-1})$. A simple calculation gives that if $B \geq 0$, then $\beta \geq \beta_1$ or $B \leq 0$, then $\beta \geq \beta_2$. Further, if $B \geq C_1$, then $\beta \geq \beta_5$ or $B \leq C_1$, then $\beta \geq \beta_6$. □

Next, Theorem 2.6–2.9 provides the sharp estimates on β so that the subordination $1+\beta\dfrac{zp'(z)}{p^j(z)} \prec \dfrac{1+Az}{1+Bz}$ implies the subordination $p(z) \prec \varphi_C(z)$, $\phi_{\mathbb{Q}}(z)$ and $\frac{1+Cz}{1+Dz}$, e^z. Proofs of these theorems are omitted as it is similar to Theorem 2.5.

Theorem 2.6. *Let $-1 < B < A < 1$, $B \neq 0$ and $p(z)$ be an analytic function with $p(0) = 1$. Then each of the following subordination is sufficient for $p(z) \prec \varphi_C(z)$:*

(a) $1+\beta zp'(z) \prec \frac{1+Az}{1+Bz}$ *for $\beta \geq \max\{\beta_1, \beta_2\}$, where*

$$\beta_1 = \frac{3(A-B)\log(1-B)^{-1}}{2B} \quad \text{and} \quad \beta_2 = \frac{(A-B)\log(1+B)}{2B}.$$

(b) $1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Az}{1+Bz}$ *for* $\beta \geq \max\{\beta_3,\, \beta_4\}$, *where*

$$\beta_3 = \frac{(A-B)\log(1-B)^{-1}}{B\log 3} \quad \text{and} \quad \beta_4 = \frac{(A-B)\log(1+B)}{B\log 3}.$$

(c) $1+\beta\frac{zp'(z)}{p^2(z)} \prec \frac{1+Az}{1+Bz}$ *for* $\beta \geq \max\{\beta_5,\, \beta_6\}$, *where*

$$\beta_5 = \frac{(A-B)\log(1-B)^{-1}}{2B} \quad \text{and} \quad \beta_6 = \frac{3(A-B)\log(1+B)}{2B}.$$

The estimates on β *are sharp.*

Theorem 2.7. *Let* $-1 < B < A < 1$, $B \neq 0$ *and* $p(z)$ *be an analytic function with* $p(0) = 1$. *Then each of the following subordination is sufficient for* $p(z) \prec \varphi_{\mathbb{C}}(z)$:

(a) $1+\beta zp'(z) \prec \frac{1+Az}{1+Bz}$ *for* $\beta \geq \max\{\beta_1,\, \beta_2\}$, *where*

$$\beta_1 = \frac{(A-B)\log(1-B)^{-1}}{B(2-\sqrt{2})} \quad \text{and} \quad \beta_2 = \frac{(A-B)\log(1+B)}{\sqrt{2}B}.$$

(b) $1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Az}{1+Bz}$ *for* $\beta \geq \max\{\beta_3,\, \beta_4\}$, *where*

$$\beta_3 = \frac{(A-B)\log(1-B)^{-1}}{B\log(\sqrt{2}-1)^{-1}} \quad \text{and} \quad \beta_4 = \frac{(A-B)\log(1+B)}{B\log(\sqrt{2}+1)}.$$

(c) $1+\beta\frac{zp'(z)}{p^2(z)} \prec \frac{1+Az}{1+Bz}$ *for* $\beta \geq \max\{\beta_5,\, \beta_6\}$, *where*

$$\beta_5 = \frac{(A-B)\log(1-B)^{-1}}{\sqrt{2}B} \quad \text{and} \quad \beta_6 = \frac{(\sqrt{2}+1)(A-B)\log(1+B)}{\sqrt{2}B}.$$

The estimates on β *are sharp.*

Theorem 2.8. *Let* $-1 < B < A < 1$, $-1 < C < D < 1$, $B \neq 0$ *and* $p(z)$ *be an analytic function with* $p(0) = 1$. *Then each of the following subordinations is sufficient for* $p(z) \prec (1+Cz)/(1+Dz)$:

(a) $1+\beta zp'(z) \prec \frac{1+Az}{1+Bz}$ *for* $\beta \geq \max\{\beta_1,\, \beta_2\}$, *where*

$$\beta_1 = \frac{(A-B)(1-D)\log(1-B)^{-1}}{B(C-D)} \quad \text{and} \quad \beta_2 = \frac{(A-B)(1+D)\log(1+B)}{B(C-D)}.$$

(b) $1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Az}{1+Bz}$ *for* $\beta \geq \max\{\beta_3,\, \beta_4\}$ *where*

$$\beta_3 = \frac{(A-B)\log(1-B)^{-1}}{B(\log(1-D)-\log(1-C))} \quad \text{and} \quad \beta_4 = \frac{(A-B)\log(1+B)}{B(\log(1+C)-\log(1+D))}.$$

(c) $1+\beta\frac{zp'(z)}{p^2(z)} \prec \frac{1+Az}{1+Bz}$ *for* $\beta \geq \max\{\beta_5, \beta_6\}$ *where*

$$\beta_5 = \frac{(A-B)(C-D)\log(1-B)^{-1}}{B(1-C)} \quad \text{and} \quad \beta_6 = \frac{(A-B)(1+C)\log(1+B)}{B(C-D)}.$$

The estimates on β *are sharp.*

Theorem 2.9. *Let* $-1 < B < A < 1$, $B \neq 0$. *If the subordination*

$$1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Az}{1+Bz} \quad \text{for} \quad \beta \geq \frac{(A-B)\log(1-B)^{-1}}{B} \quad \text{holds},$$

then $p(z) \prec e^z$.

The subordination result in Theorem 2.9 was also investigated by the authors in [14, Theorem 2.16, p. 1019], which was not sharp.

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