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Certain Subclasses of $k-$uniformly Functions Involving the Generalized Fractional Differintegral Operator

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Abstract. We introduce several $k-$uniformly subclasses of $p-$valent functions defined by the generalized fractional differintegral operator and investigate various inclusion relationships for these subclasses. Some interesting applications involving certain classes of integral operators are also considered.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \,(p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (see [8] and [9]).

For $0 \leq \gamma, \eta < p, k \geq 0$ and $z \in \mathbb{U}$, we define $US_p^*(k; \gamma), UC_p(k; \gamma), UK_p(k; \gamma, \eta)$ and $UK_p^*(k; \gamma, \eta)$ the $k-$uniformly subclasses of $\mathcal{A}_p$ consisting of all analytic functions which are, respectively, $p-$valent starlike of order γ, $p-$valent convex of order γ, $p-$valent close-to-convex of order γ, and type η and $p-$valent quasi-convex of order γ, and type η as follows:

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$$US_p^*(k;\gamma)=\left\{f\in\mathcal{A}_p:\Re\left(\frac{zf'(z)}{f(z)}-\gamma\right)>k\left|\frac{zf'(z)}{f(z)}-p\right|\right\}, \tag{1.2}$$

$$UC_p(k;\gamma)=\left\{f\in\mathcal{A}_p:\Re\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right)>k\left|1+\frac{zf''(z)}{f'(z)}-p\right|\right\}, \tag{1.3}$$

(1.4)
$$UK_p(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in US_p^*(k;\eta),\Re\left(\frac{zf'(z)}{g(z)}-\gamma\right)>k\left|\frac{zf'(z)}{g(z)}-p\right|\right\},$$

(1.5)
$$UK_p^*(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in UC_p(k;\eta),\Re\left(\frac{\left(zf'(z)\right)'}{g'(z)}-\gamma\right)>k\left|\frac{\left(zf'(z)\right)'}{g'(z)}-p\right|\right\}.$$

These subclasses were introduced and studied by Al-Kharsani [1]. We note that

(i) $US_1^*(k;\gamma)=US^*(k;\gamma)$ and $UC_1(k;\gamma)=UC(k;\gamma)\,(0\le\gamma<1)$ (see [6] and [20]);

(ii) $US_p^*(0;\gamma)=S_p^*(\gamma)$ $(0\le\gamma<p)$ (see [12] and [15]);

(iii) $UC_p(0;\gamma)=C_p(\gamma)\,(0\le\gamma<p)$ (see [12]);

(iv) $UK_p(0;\gamma,\eta)=K_p(\gamma,\eta)\,(0\le\gamma,\eta<p)$ (see [2]);

(v) $UK_p^*(0;\gamma,\eta)=K_p^*(\gamma,\eta)\,(0\le\gamma,\eta<p)$ (see [10]).

Corresponding to a conic domain $\Omega_{p,k,\gamma}$ defined by

$$\Omega_{p,k,\gamma}=\left\{u+iv:u>k\sqrt{(u-p)^2+v^2}+\gamma\right\}, \tag{1.6}$$

we define the function $q_{p,k,\gamma}(z)$ which maps $\mathbb{U}$ onto the conic domain $\Omega_{p,k,\gamma}$ such that $1\in\Omega_{p,k,\gamma}$ as the following (see [1]):

(1.7)
$$q_{p,k,\gamma}(z)=\begin{cases}\dfrac{p+(p-2\gamma)z}{1-z} & (k=0),\\[2ex] \frac{p-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}\left(\cos^{-1}k\right)i\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{k^2p-\gamma}{1-k^2} & (0<k<1),\\[2ex] p+\dfrac{2(p-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & (k=1),\\[2ex] \frac{p-\gamma}{k^2-1}\sin\left\{\dfrac{\pi}{2\zeta(k)}\int_0^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^2\sqrt{1-k^2t^2}}}\right\}+\frac{k^2p-\gamma}{k^2-1} & (k>1).\end{cases}$$

where $u(z) = \frac{z-\sqrt{x}}{1-\sqrt{x}z}, x \in (0,1)$ and $\zeta(k)$ is such that $k = \cosh \frac{\pi\zeta'(z)}{4\zeta(z)}$. By virtue of the properties of the conic domain $\Omega_{p,k,\gamma}$, we have

$$\Re\{q_{p,k,\gamma}(z)\} > \frac{kp+\gamma}{k+1}. \tag{1.8}$$

Making use of the principal of subordination between analytic functions and the definition of $q_{p,k,\gamma}(z)$, we may rewrite the subclasses $US_p^*(k;\gamma)$, $UC_p(k;\gamma)$, $UK_p(k;\gamma,\beta)$ and $UK_p^*(k;\gamma,\beta)$ as the following:

$$US_p^*(k;\gamma) = \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{f(z)} \prec q_{p,k,\gamma}(z) \right\}, \tag{1.9}$$

$$UC_p(k;\gamma) = \left\{ f \in \mathcal{A}_p : 1 + \frac{zf''(z)}{f'(z)} \prec q_{p,k,\gamma}(z) \right\}, \tag{1.10}$$

$$UK_p(k;\gamma,\eta) = \left\{ f \in \mathcal{A}_p : \exists\, g \in US_p^*(k;\eta), \frac{zf'(z)}{g(z)} \prec q_{p,k,\gamma}(z) \right\}, \tag{1.11}$$

$$UK_p^*(k;\gamma,\eta) = \left\{ f \in \mathcal{A}_p : \exists\, g \in UC_p(k;\eta), \frac{\left(zf'(z)\right)'}{g'(z)} \prec q_{p,k,\gamma}(z) \right\}. \tag{1.12}$$

Srivastava et al. [23] introduced the following generalized fractional integral and generalized fractional derivative operators as follows(see also [16] and [19]):

Definition 1.1.([23]) For real numbers $\lambda > 0, \mu$ and η, the Saigo hypergeometric fractional integral operator $I_{0,z}^{\lambda,\mu,\eta} : \mathcal{A}_p \to \mathcal{A}_p$ is defined by

$$I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} \;\; {}_2F_1\left(\lambda+\mu, -\eta; \lambda; 1-\frac{t}{z}\right) f(t)\; dt, \tag{1.13}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z-$plane containing the origin, with the order

$$f(z) = O\left(|z|^{\varepsilon}\right) \quad (z \to 0; \varepsilon > \max\{0, \mu-\lambda\} - 1),$$

and the multiplying of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 1.2.([23]) Under the hypotheses of Definition 1.1, Saigo hypergeometric fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}:\mathcal{A}_p\to\mathcal{A}_p$ is defined by

(1.14)
$$J_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\left\{z^{\lambda-\mu}\int_0^z(z-t)^{-\lambda}\right.\\ \left.{}_2F_1\left(\mu-\lambda,1-\eta;1-\lambda;1-\frac{t}{z}\right)f(t)\ dt\right\} & (0\le\lambda<1),\\ \frac{d^n}{dz^n}J_{0,z}^{\lambda-\mu,\mu,\eta}f(z) & (n\le\lambda<n+1;n\in\mathbb{N}),\end{cases}$$

where the multiplying of $(z-t)^{-\lambda}$ is removed as in Definition 1.1.

We note that

$$I_{0,z}^{\lambda,-\lambda,\eta}f(z)=D_z^{-\lambda}f(z)\quad(\lambda>0)\quad\text{and}\quad J_{0,z}^{\lambda,\lambda,\eta}f(z)=D_z^{\lambda}f(z)\quad(0\le\lambda<1),$$

where $D_z^{-\lambda}$ denotes fractional integral operator and D_z^{λ} denotes fractional derivative operator studied by Owa [11].

Recently, Goyal and Prajapat [7] (see also [17] and [18]) introduced the generalized fractional differintegral operator $S_{0,z}^{\lambda,\mu,\eta}:\mathcal{A}_p\to\mathcal{A}_p\,(p\in\mathbb{N},\eta\in\mathbb{R},\mu<p+1)$ by

$$S_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\frac{\Gamma(1+p-\mu)\ \Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}z^{\mu}J_{0,z}^{\lambda,\mu,\eta} & (0\le\lambda<\eta+p+1),\\ \frac{\Gamma(1+p-\mu)\ \Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}z^{\mu}I_{0,z}^{-\lambda,\mu,\eta} & (-\infty<\lambda<0).\end{cases}\tag{1.15}$$

It is easily seen from a function f of the form (1.1), we have

$$\begin{aligned}S_{0,z}^{\lambda,\mu,\eta}f(z) &= z^p\ {}_3F_2(1,1+p,1+p+\eta-\mu;1+p-\mu,1+p+\eta-\lambda;z)*f(z)\\ &= z^p+\sum_{n=1}^{\infty}\frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n}a_{n+p}z^{n+p}\\ & \qquad (z\in\mathbb{U};p\in\mathbb{N};\mu,\eta\in\mathbb{R};\mu<p+1;-\infty<\lambda<\eta+p+1),\end{aligned}\tag{1.16}$$

where ${}_qF_s\,(q\le s+1;q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is well known generalized hypergeometric function (see, for details, [13, 22]) and $(v)_n$ is the Pochhammer symbol defined, in terms of Gamma function, by

$$(v)_n=\frac{\Gamma(v+n)}{\Gamma(v)}=\begin{cases}1 & (n=0)\\ v(v+1)\dots(v+n-1) & (n\in\mathbb{N}).\end{cases}$$

We note that

$$S_{0,z}^{0,0,0}f(z)=f(z)\,,S_{0,z}^{1,0,0}f(z)=\frac{zf'(z)}{p}$$

and

$$S_{0,z}^{\lambda,\lambda,0}f(z) = \Omega_z^{(\lambda,p)}f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)}z^{\lambda}D_z^{\lambda}f(z)$$
$$(-\infty \leq \lambda < p+1; p \in \mathbb{N}; z \in \mathbb{U}),$$

where the extended fractional differintegral operator $\Omega_z^{(\lambda,p)}$ was introduced and studied by Patel and Mishra [14]. The fractional differential operator $\Omega_z^{(\lambda,p)}$ with $0 \leq \lambda < 1$ was investigated by Srivastava and Aouf [21]. The operator $\Omega_z^{(\lambda,1)} = \Omega_z^{\lambda}$ was introduced by Owa and Srivastava [13];

Upon setting

$$G_{p,\eta,\mu}^{\lambda}(z) = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+p+\eta-\mu)_n}{(1+p-\mu)_n (1+p+\eta-\lambda)_n} z^{n+p}$$
$$(z \in \mathbb{U}; p \in \mathbb{N}; \mu, \eta \in \mathbb{R}; \mu < p+1; -\infty < \lambda < \eta+p+1), \tag{1.17}$$

we define a new function $\left[G_{p,\mu,\eta}^{\lambda}(z)\right]^{-1}$ by means of the Hadamard product (or convolution):

$$G_{p,\eta,\mu}^{\lambda}(z) * \left[G_{p,\eta,\mu}^{\lambda}(z)\right]^{-1} = \frac{z^p}{(1-z)^{\delta+p}} \, (\delta > -p; z \in \mathbb{U}). \tag{1.18}$$

Tang et al. [24] introduced the linear operator $H_{p,\eta,\mu}^{\lambda,\delta} : \mathcal{A}_p \to \mathcal{A}_p$ as follows:

$$H_{p,\eta,\mu}^{\lambda,\delta}f(z) = \left[G_{p,\eta,\mu}^{\lambda}(z)\right]^{-1} * f(z). \tag{1.19}$$

For $f \in \mathcal{A}_p$ given by (1.1), then from (1.19), we have

$$H_{p,\eta,\mu}^{\lambda,\delta}f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\delta+p)_n (1+p-\mu)_n (1+p+\eta-\lambda)_n}{n!\,(1+p)_n (1+p+\eta-\mu)_n} a_{n+p}\, z^{n+p} \tag{1.20}$$

by using (1.20), we get

$$z\left(H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)\right)' = (p+\eta-\lambda)\, H_{p,\eta,\mu}^{\lambda,\delta}f(z) - (\eta-\lambda)\, H_{p,\eta,\mu}^{\lambda+1,\delta}f(z) \tag{1.21}$$

and

$$z\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)' = (\delta+p)\, H_{p,\eta,\mu}^{\lambda,\delta+1}f(z) - \delta H_{p,\eta,\mu}^{\lambda,\delta}f(z). \tag{1.22}$$

Next, using the operator $H_{p,\eta,\mu}^{\lambda,\delta}$, we introduce the following $k-$uniformly subclasses of $p-$valent functions for $\eta \in \mathbb{R}, \mu < p+1, -\infty < \lambda < \eta+p+1, \delta > -p, p \in \mathbb{N}, k \geq 0$ and $0 \leq \gamma, \rho < p$:

$$US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma) = \left\{f \in \mathcal{A}_p : H_{p,\eta,\mu}^{\lambda,\delta}f(z) \in US_p^*(k;\gamma);\, z \in \mathbb{U}\right\}, \tag{1.23}$$

$$UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)=\left\{f\in\mathcal{A}_p : H_{p,\eta,\mu}^{\lambda,\delta}f(z)\in UC_p(k;\gamma);z\in\mathbb{U}\right\}, \tag{1.24}$$

$$UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)=\left\{f\in\mathcal{A}_p : H_{p,\eta,\mu}^{\lambda,\delta}f(z)\in UK_p(k;\gamma,\rho);z\in\mathbb{U}\right\}, \tag{1.25}$$

$$UQ_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)=\left\{f\in\mathcal{A}_p : H_{p,\eta,\mu}^{\lambda,\delta}f(z)\in UK_p^{*}(k;\gamma,\rho);z\in\mathbb{U}\right\}. \tag{1.26}$$

We also note that

$$f\in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)\Leftrightarrow\frac{zf'}{p}\in UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma), \tag{1.27}$$

and

$$f\in UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)\Leftrightarrow\frac{zf'}{p}\in UQ_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho). \tag{1.28}$$

In this paper, we investigate several inclusion properties of the classes $US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$, $UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$, $UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)$ and $UQ_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)$ associated with the operator $H_{p,\eta,\mu}^{\lambda,\delta}$. Some applications involving integral operators are also considered.

2. Inclusion Properties Involving the Operator $H_{p,\eta,\mu}^{\lambda,\delta}$

In order to prove the main results, we shall need The following lemmas.

Lemma 2.1.([5]) *Let $h(z)$ be convex univalent in $\mathbb{U}$ with $\Re\{\alpha h(z)+\beta\}>0$ $(\alpha,\beta\in\mathbb{C})$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=h(0)$, then*

$$p(z)+\frac{zp'(z)}{\alpha p(z)+\beta}\prec h(z) \tag{2.1}$$

implies

$$p(z)\prec h(z). \tag{2.2}$$

Lemma 2.2.([8]) *Let $h(z)$ be convex univalent in $\mathbb{U}$ and let w be analytic in $\mathbb{U}$ with $\Re\{w(z)\}\geq 0$. If $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=h(0)$, then*

$$p(z)+w(z)zp'(z)\prec h(z) \tag{2.3}$$

implies

$$p(z)\prec h(z). \tag{2.4}$$

Theorem 2.3. *Let $\delta(k+1)+kp+\gamma>0$ and $(\eta-\lambda)(k+1)+kp+\gamma>0$. Then,*

$$US_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma)\subset US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)\subset US_{p,\eta,\mu}^{\lambda+1,\delta}(k;\gamma). \tag{2.5}$$

Proof. We first prove that $US_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma)\subset US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$. Let $f\in US_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma)$ and set

$$p(z)=\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}\quad(z\in\mathbb{U}), \tag{2.6}$$

where the function $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$. Using (1.22), (2.5) and (2.6), we have

$$\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)}=p(z)+\frac{zp'(z)}{p(z)+\delta}\prec q_{p,k,\gamma}(z). \tag{2.7}$$

Since $\delta(k+1)+kp+\gamma>0$, we see that

$$\Re\{q_{p,k,\gamma}(z)+\delta\}>0\quad(z\in\mathbb{U}). \tag{2.8}$$

Applying Lemma 2.1 to (2.7), it follows that $p(z)\prec q_{p,k,\gamma}(z)$, that is, $f\in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$. To prove the right part, let $f\in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$ and consider

$$h(z)=\frac{z\left(H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)\right)'}{H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)}\quad(z\in\mathbb{U}),$$

where the function $h(z)$ is analytic in $\mathbb{U}$ with $h(0)=p$. Then, by using the arguments similar to those detailed above, together with (1.21), it follows that $p(z)\prec q_{p,k,\gamma}(z)$, which implies that $f\in US_{p,\eta,\mu}^{\lambda+1,\delta}(k;\gamma)$. Therefore, we complete the proof of Theorem 2.3. □

Theorem 2.4. *Let $\delta(k+1)+kp+\gamma>0$ and $(\eta-\lambda)(k+1)+kp+\gamma>0$. Then,*

$$UC_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma)\subset UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)\subset UC_{p,\eta,\mu}^{\lambda+1,\delta}(k;\gamma). \tag{2.9}$$

Proof. Applying (1.27) and Theorem 2.3, we observe that

$$\begin{aligned}f\in UC_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma)\quad&\Longleftrightarrow\frac{zf'}{p}\in US_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma)\\&\Longrightarrow\frac{zf'}{p}\in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)\quad(\text{ by Theorem 2.3 }),\\&\Longleftrightarrow f\in UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)\end{aligned}$$

and

$$\begin{aligned}f\in UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)\quad&\Longleftrightarrow\frac{zf'}{p}\in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)\\&\Longrightarrow\frac{zf'}{p}\in US_{p,\eta,\mu}^{\lambda+1,\delta}(k;\gamma)\quad(\text{ by Theorem 2.3 }),\\&\Longleftrightarrow f\in UC_{p,\eta,\mu}^{\lambda+1,\delta}(k;\gamma),\end{aligned}$$

which evidently proves Theorem 2.4. □

Next, by using Lemma 2.2, we obtain the following inclusion relation for the class $UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)$.

Theorem 2.5. *Let* $\delta(k+1)+kp+\rho>0$ *and* $(\eta-\lambda)(k+1)+kp+\rho>0$. *Then,*

$$UK_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma,\rho)\subset UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)\subset UK_{p,\eta,\mu}^{\lambda+1,\delta}(k;\gamma,\rho). \tag{2.10}$$

Proof. We begin by proving that $UK_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma,\rho)\subset UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)$. Let $f\in UK_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma,\rho)$. Then, from the definition of $UK_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma,\rho)$, there exists a function $r(z)\in US_p(k;\gamma)$ such that

$$\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)\right)'}{r(z)}\prec q_{p,k,\gamma}(z). \tag{2.11}$$

Choose the function g such that $H_{p,\eta,\mu}^{\lambda,\delta+1}g(z)=r(z)$. Then, $g\in US_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma)$ and

$$\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta+1}g(z)}\prec q_{p,k,\gamma}(z). \tag{2.12}$$

Now let

$$p(z)=\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}\quad(z\in\mathbb{U}), \tag{2.13}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$. Since $g\in US_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma)$, by Theorem 2.3, we know that $g\in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$. Let

$$t(z)=\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}g(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}\quad(z\in\mathbb{U}), \tag{2.14}$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\}>\dfrac{kp+\rho}{k+1}$. Also, from (2.13), we note that

$$H_{p,\eta,\mu}^{\lambda,\delta}zf'(z)=H_{p,\eta,\mu}^{\lambda,\delta}g(z)\;p(z). \tag{2.15}$$

Differentiating both sides of (2.15) with respect to z, we obtain

$$\begin{aligned}\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}zf'(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}&=\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}g(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}p(z)+zp'(z)\\&=t(z)\,p(z)+zp'(z).\end{aligned}\tag{2.16}$$

Now using the identity (1.22) and (2.14), we obtain

$$\begin{aligned}
\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)\right)^{'}}{H_{p,\eta,\mu}^{\lambda,\delta+1}g(z)} &= \frac{H_{p,\eta,\mu}^{\lambda,\delta+1}zf^{'}(z)}{H_{p,\eta,\mu}^{\lambda,\delta+1}g(z)} = \frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}zf^{'}(z)\right)^{'}+\delta H_{p,\eta,\mu}^{\lambda,\delta}zf^{'}(z)}{z\left(H_{p,\eta,\mu}^{\lambda,\delta}g(z)\right)^{'}+\delta H_{p,\eta,\mu}^{\lambda,\delta}g(z)} \\
&= \frac{\dfrac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}zf^{'}(z)\right)^{'}}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}+\delta\dfrac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)^{'}}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}}{\dfrac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}g(z)\right)^{'}}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)}+\delta} \\
&= \frac{t(z)\,p(z)+zp^{'}(z)+\delta p(z)}{t(z)+\delta} \\
&= p(z)+\frac{zp^{'}(z)}{t(z)+\delta}. \qquad (2.17)
\end{aligned}$$

Since $\delta(k+1)+kp+\rho>0$ and $\Re\{t(z)\}>\dfrac{kp+\rho}{k+1}$, we see that

$$\Re\{t(z)+\delta\}>0 \quad (z\in\mathbb{U}).$$

Hence, applying Lemma 2.2, we can show that $p(z)\prec q_{p,k,\gamma}(z)$ so that $f\in UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)$. For the second part, by using the arguments similar to those detailed above with (1:15), we obtain

$$UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)\subset UK_{p,\eta,\mu}^{\lambda+1,\delta}(k;\gamma,\rho).$$

Therefore, we complete the proof of Theorem 2.5. □

Theorem 2.6. *Let $\delta(k+1)+kp+\rho>0$ and $(\eta-\lambda)(k+1)+kp+\rho>0$ Then,*

$$UQ_{p,\eta,\mu}^{\lambda,\delta+1}(k;\gamma,\rho)\subset UQ_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)\subset UQ_{p,\eta,\mu}^{\lambda+1,\delta}(k;\gamma,\rho).$$

Proof. Just as we derived Theorem 2.4 as consequence of Theorem 2.3 by using the equivalence (1.27), we can also prove Theorem 2.6 by using Theorem 2.5 and the equivalence (1.28). □

3. Inclusion Properties Involving the Integral Operator $F_{c,p}$

In this section, we present several integral-preserving properties of the p-valent function classes introduced here. We consider the generalized Libera integral operator $F_{c,p}(f)$ (see [4] and [3]) defined by

$$F_{c,p}(f)(z)=\frac{c+p}{z^{c}}\int t^{c-1}f(z)\,dt \quad (c>-p). \qquad (3.1)$$

Theorem 3.1. *Let* $c(k+1)+kp+\gamma \geq 0$. *If* $f \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$, *then* $F_{c,p}(f) \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$.

Proof. Let $f \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$ and set

$$p(z)=\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(f)(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(f)(z)} \quad (z \in \mathbb{U}), \tag{3.2}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$.

From (3.1), we have

$$z\left(H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(f)(z)\right)'=(c+p)H_{p,\eta,\mu}^{\lambda,\delta}f(z)-cH_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we obtain

$$(c+p)\frac{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}{H_{p,\eta,\mu}^{\lambda,\delta}F_{c}(f)(z)}=p(z)+c. \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}f(z)}=p(z)+\frac{zp'(z)}{p(z)+c}\prec q_{k,\gamma}(z) \quad (z \in \mathbb{U}). \tag{3.5}$$

Hence, by virtue of Lemma 2.1, we conclude that $p(z) \prec q_{k,\gamma}(z)$ in $\mathbb{U}$, which implies that $F_{c,p}(f) \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$. □

Next, we derive an inclusion property involving $F_{c,p}(f)$, which is given by the following.

Theorem 3.2. *Let* $c(k+1)+kp+\gamma \geq 0$. *If* $f \in UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$, *then* $F_{c,p}(f) \in UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$.

Proof. By applying Theorem 2.5, it follows that

$$\begin{aligned} f \in UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma) &\Longleftrightarrow \frac{zf'}{p} \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma) \\ &\Longrightarrow F_{c,p}\left(\frac{zf'}{p}\right) \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma) \quad (\text{ by Theorem 3.1 }) \\ &\Longleftrightarrow \frac{z\left(F_{c,p}(f)\right)'}{p} \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma) \\ &\Longleftrightarrow F_{c,p}(f) \in UC_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma), \end{aligned}$$

which proves Theorem 3.2. □

Theorem 3.3. *Let $c(k+1)+kp+\rho \geq 0$. If $f \in UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)$, then $F_{c,p}(f) \in UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)$.*

Proof. Let $f \in UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)$. Then, in view of the definition of the class $UK_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma,\rho)$, there exists a function $g \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$ such that

$$\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)} \prec q_{k,\gamma}(z). \tag{3.6}$$

Thus, we set

$$p(z) = \frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(f)(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)} \quad (z \in \mathbb{U}), \tag{3.7}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$. Since $g \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$, we see from Theorem 3.1 that $F_{c,p}(g) \in US_{p,\eta,\mu}^{\lambda,\delta}(k;\gamma)$. Let

$$t(z) = \frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)} \quad (z \in \mathbb{U}), \tag{3.8}$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\} > \frac{kp+\eta}{k+1}$. Also, from (3.7), we note that

$$H_{p,\eta,\mu}^{\lambda,\delta}zF_{c,p}'(f)(z) = H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)\ .\ p(z). \tag{3.9}$$

Differentiating both sides of (3.9) with respect to z, we obtain

$$\begin{aligned}\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}zF_{c,p}'(f)(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)} &= \frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)}p(z)+zp'(z)\\ &= t(z)p(z)+zp'(z).\end{aligned} \tag{3.10}$$

Now using the identity (3.3) and (3.10), we obtain

$$\begin{aligned}\frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}g(z)} &= \frac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}zF_{c,p}'(f)(z)\right)'+cH_{p,\eta,\mu}^{\lambda,\delta}zF_{c,p}'(f)(z)}{z\left(H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)\right)'+cH_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)}\\ &= \frac{\dfrac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}zF_{c,p}'(f)(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)}+c\dfrac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(f)(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)}}{\dfrac{z\left(H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)\right)'}{H_{p,\eta,\mu}^{\lambda,\delta}F_{c,p}(g)(z)}+c}\\ &= \frac{t(z)p(z)+zp'(z)+cp(z)}{t(z)+c}\\ &= p(z)+\frac{zp'(z)}{t(z)+c}.\end{aligned} \tag{3.11}$$

Since $c(k+1)+kp+\rho \geq 0$ and $\Re\{t(z)\} > \dfrac{kp+\eta}{k+1}$, we see that

$$\Re\{t(z)+c\} > 0 \quad (z \in \mathbb{U}). \tag{3.12}$$

Hence, applying Lemma 2.2 to (3.11), we can show that $p(z) \prec q_{p,k,\gamma}(z)$ so that $F_{c,p}(f) \in UK^{\lambda,\delta}_{p,\eta,\mu}(k;\gamma,\rho)$. □

Theorem 3.4. *Let $c(k+1)+kp+\eta \geq 0$. If $f \in UQ^{\lambda,\delta}_{p,\eta,\mu}(k;\gamma,\rho)$, then $F_{c,p}(f) \in UQ^{\lambda,\delta}_{p,\eta,\mu}(k;\gamma,\rho)$.*

Proof. Just as we derived Theorem 3.2 as consequence of Theorem 3.1, we easily deduce the integral-preserving property asserted by Theorem 3.4 by using Theorem 3.3. □

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