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Sharp Coefficient Bounds for the Quotient of Analytic Functions

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Abstract. We derive sharp upper bound on the initial coefficients and Hankel determinants for normalized analytic functions belonging to a class, introduced by Silverman, defined in terms of ratio of analytic representations of convex and starlike functions. A conjecture related to the coefficients for functions in this class is posed and verified for the first five coefficients.

1. Introduction

Let $\mathcal{S}$ be the class of univalent analytic functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \tag{1.1}$$

defined in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. It is well-known that the coefficient of the functions in the class $\mathcal{S}$ satisfy $|a_n| \leq n$. This result was put before, as a conjecture, by Bieberbach in 1916, and it took around 68 year to prove and was finally affirmatively settled by de Branges. In those 68 years many researchers tried to prove or disprove it which lead to explore many subclasses of the class $\mathcal{S}$. The class $\mathcal{S}^*$ of starlike functions is a collection of functions $f \in \mathcal{S}$ for which $\mathrm{Re}(zf'(z)/f(z)) > 0$ for all $z \in \mathbb{D}$. However, the class $\mathcal{K}$ of convex functions is a collection of all those functions $f \in \mathcal{S}$ for which $\mathrm{Re}(1 + zf''(z)/f'(z)) > 0$ for all $z \in \mathbb{D}$. These subclasses are among the most studied subclasses of $\mathcal{S}$. In 1997, Silverman [18] investigated a class of normalised analytic functions involving an expression of the quotient of the analytic representations of convex and starlike

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functions. For $0 < \mu \leq 1$, he defined the class $\mathcal{G}_\mu$ as follows:

$$\mathcal{G}_\mu := \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < \mu \right\}$$

and proved that the function in the class $\mathcal{G}_\mu$ are starlike of order $2/(1+\sqrt{1+8b})$. He also investigated many sufficient conditions for functions to be starlike and convex of positive order. Further, this result was improved by Obradovič and Tuneski [12]. In 2003, Tuneski [19] investigated the condition for functions in the class $\mathcal{G}_\mu$ to be Janowski starlike. For further related results reader can refer [1, 7, 11] and the references cited therein.

Recall that for analytic functions f and g, we say that f is subordinate to g, denoted by $f \prec g$, if there is a Schwarz function w with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. Further, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In view of this definition, we can write

$$\mathcal{G}_\mu := \left\{ f \in \mathcal{A} : \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \mu z \right\}.$$

The sharp bound on the functional $|a_2a_4 - a_3^2|$ for starlike and convex functions were obtained by Janteng [5]. He proved that for starlike and convex functions, this quantity is bounded above by 1 and 1/8, respectively. This functional is related to the Hankel determinants. Recall that for given natural numbers n, q, the Hankel determinant $H_{q,n}(f)$ of a function $f \in \mathcal{A}$ is defined by means of the following determinant

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix},$$

with $a_1 = 1$. The quantity $H_{2,1}(f) = a_3 - a_2^2$ is the well-known Fekete-Szegö functional. The second Hankel determinant is given by the expression $H_{2,2}(f) := a_2a_4 - a_3^2$. Further, the quantity $H_{3,\,1}(f) := a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$ is called the *third Hankel determinant*. The Hankel determinant $H_{q,n}(f)$ for the class of univalent functions was investigated by Pommerenke [14] and Hayman [4]. For a chronological development in this direction till 2013 reader may refer [8].

The results related to the Hankel determinants usually are derived by relating the functions in the class under consideration to the Carathéodory functions. For this purpose we recall the definition of this class. Let $\mathcal{P}$ denote the class of Carathéodory [2, 3] functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathbb{D}). \tag{1.2}$$

The following results shall be used as tools:

Lemma 1.1.([9, 10, Libera and Zlotkiewicz]) *If* $p \in \mathcal{P}$ *has the form given by* (1.2) *with* $p_1 \geq 0$, *then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{1.3}$$

and

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)y \tag{1.4}$$

for some x *and* y *such that* $|x| \leq 1$ *and* $|y| \leq 1$.

Lemma 1.2.([17, Ravichandran and Verma]) *Let* $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ *and* $\hat{a}$ *satisfy the inequalities* $0 < \hat{\alpha} < 1$, $0 < \hat{a} < 1$ *and*

$$8a(1-a)[(\hat{\alpha}\hat{\beta} - 2\hat{\gamma})^2 + (\hat{\alpha}(\hat{a} + \hat{\alpha}) - \hat{\beta})^2] + \hat{\alpha}(1 - \hat{\alpha})(\hat{\beta} - 2\hat{a}\hat{\alpha})^2 \leq 4\hat{a}\hat{\alpha}^2(1 - \hat{\alpha})^2(1 - \hat{a}).$$

If $p \in \mathcal{P}$ *has the form given by* (1.2), *then*

$$|\hat{\gamma}p_1^4 + \hat{a}p_2^2 + 2\hat{\alpha}p_1p_3 - (3/2)\hat{\beta}p_1^2p_2 - p_4| \leq 2.$$

The following result is due to Prokhorov and Szynal [15]. Since the result is lengthy, so we are quoting here some specific part of their result which we need in our further investigation.

Let $\mathcal{B}$ be the class of analytic functions $w(z) = \sum_{n=1}^{\infty} c_n z^n$ $(z \in \mathbb{D})$ and satisfying the condition $|w(z)| < 1$ for $z \in \mathbb{D}$. Consider a functional $\Psi(w) = |c_3 + \alpha c_1 c_2 + \beta c_1^3|$ for $w \in \mathcal{B}$ and α, $\beta \in \mathbb{R}$. Define the sets $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ and Ω_4 by

$$\Omega_1 := \left\{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \leq 1/2, \;\; -1 \leq \beta \leq 1\right\},$$

$$\Omega_2 := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \frac{1}{2} \leq |\alpha| \leq 2, \;\; \frac{4}{27}(|\alpha| + 1)^3 - (|\alpha| + 1) \leq \beta \leq 1\right\},$$

$$\Omega_3 := \left\{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \leq 2, \;\; \beta \geq 1\right\},$$

$$\Omega_4 := \left\{(\alpha, \beta) \in \mathbb{R}^2 : 2 \leq |\alpha| \leq 4, \;\; \beta \geq \frac{1}{12}(\alpha^2 + 8)\right\},$$

and

$$\Omega_5 := \left\{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \geq 4, \; \beta \geq \frac{2}{3}(|\alpha| - 1)\right\}.$$

Lemma 1.3.([15, Lemma 2, p. 128]) *If* $w \in \mathcal{B}$, *then for any real numbers* α *and* β *the sharp estimate* $\Psi(w) \leq \Phi(\alpha, \beta)$ *holds, where*

$$\Phi(\alpha, \beta) = \begin{cases} 1, & \text{if } (\alpha, \beta) \in \Omega_1 \cup \Omega_2, \\ |\beta|, & \text{if } (\alpha, \beta) \in \Omega_3 \cup \Omega_4 \cup \Omega_5. \end{cases}$$

Lemma 1.4.([13, Ohno and Sugawa]) *For any real real numbers a, b and c, let the quantity $Y(a,b,c)$ be given by*

$$Y(a,b,c) = \max_{z\in\overline{\mathbb{D}}} \left\{|a+bz+cz^2| + 1 - |z|^2\right\},$$

where $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$. If $ac \ge 0$, then

$$Y(a,b,c) = \begin{cases} |a|+|b|+|c|, & \text{if } |b| \ge 2(1-|c|), \\ 1+|a|+\frac{b^2}{4(1-|c|)}, & \text{if } |b| < 2(1-|c|). \end{cases}$$

Further, if $ac < 0$, then

$$Y(a,b,c) = \begin{cases} 1-|a|+\frac{b^2}{4(1-|c|)}, & \text{if } -4ac(c^{-2}-1) \le b^2 \text{ and } |b| < 2(1-|c|), \\ 1+|a|+\frac{b^2}{4(1+|c|)}, & \text{if } b^2 < \min\{4(1+|c|)^2, -4ac(c^{-2}-1)\}, \\ R(a,b,c), & \text{otherwise}, \end{cases}$$

where

$$R(a,b,c) = \begin{cases} |a|+|b|-|c|, & \text{if } |c|(|b|+4|a|) \le |ab|, \\ -|a|+|b|+|c|, & \text{if } |ab| \le |c|(|b|-4|a|), \\ (|c|+|a|)\sqrt{1-\frac{b^2}{4ac}}, & \text{otherwise}. \end{cases}$$

2. Coefficient Bounds

The following theorem gives the sharp upper bound for the initial coefficients for functions in the class $\mathcal{G}_\mu$.

Theorem 2.1. *Let $f \in \mathcal{G}_\mu$. Assume that $\mu_0 \approx 0.335$ be the smallest positive root of*

$$\begin{aligned} &236196 - 2932686\mu^2 - 563472\mu^3 + 8764817\mu^4 + 6932820\mu^5 \\ &\qquad - 15654024\mu^6 - 13969152\mu^7 + 22902912\mu^8 = 0. \end{aligned} \tag{2.1}$$

Then the following sharp inequalities hold:

(1) $|a_2| \le \mu$,

(2) $|a_3| \le \begin{cases} \frac{\mu}{4}, & 0 < \mu \le 1/4; \\ \mu^2, & 1/4 \le \mu \le 1, \end{cases}$

(3) $|a_4| \le \begin{cases} \frac{\mu}{9}, & 0 < \mu \le 1/3; \\ \mu^3, & 1/3 \le \mu \le 1, \end{cases}$

(4) $|a_5| \le \frac{\mu}{16}$ $(0 < \mu \le \mu_0)$,

(5) $|a_3 - \nu a_2^2| \le \frac{\mu}{4} \max\{1; 4\mu|\nu-1|\}$, $\nu \in \mathbb{C}$.

Proof. Since $f \in \mathcal{G}_\mu$, it follows that there exists a Schwarz function $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathcal{B}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)}(1 + \mu w(z)). \tag{2.2}$$

To prove the result we use the relation $w(z) = (p(z) - 1)/(p(z) + 1)$ between the Schwarz function w and the Carathéodory function $p(z) = 1+p_1 z+p_2 z^2+p_3 z^3+\cdots \in \mathcal{P}$. On comparing the coefficients of like power terms in (2.2), we get

$$a_2 = \frac{\mu}{2}p_1, \quad a_3 = \frac{1}{16}\mu[(4\mu - 1){p_1}^2 + 2p_2], \tag{2.3}$$

and

$$a_4 = \frac{\mu}{288}[(4 - 21\mu + 36\mu^2){p_1}^3 - 2(8 - 21\mu)p_1 p_2 + 16p_3] \tag{2.4}$$

(1) & (2) Using the well-known facts (see [6, 16]) $|p_n| \leq 2$ and for any complex number ν, $|p_2 - \nu p_1^2| \leq \max 2\{1, |2\nu - 1|\}$, the upper bound on $|a_2|$ and $|a_3|$ follow immediately.

(3) To find the estimate on the fourth coefficient we shall write the coefficients (a_i) in terms of Schwarz's coefficients (c_i) by equating the coefficients of similar power terms in (2.2) as follows:

$$a_2 = \mu c_1, \quad a_3 = \frac{1}{4}\mu\left(c_2 + 4\mu {c_1}^2\right), \quad a_4 = \frac{1}{3}\mu\left(3\mu^2 {c_1}^3 + \frac{7}{4}\mu c_1 c_2 + \frac{1}{3}c_3\right) \tag{2.5}$$

On setting $\alpha = 21\mu/4, \quad \beta = 9\mu^2$, we can write

$$|a_4| = \frac{\mu}{9}|c_3 + \alpha c_1 c_2 + \beta {c_1}^3| =: \frac{\mu}{9}H(\alpha, \beta). \tag{2.6}$$

To get the desired estimate, we now consider the following cases. For this we first assume that the symbols $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ and Ω_4 are as defined in Lemma 1.3 with the choice $\alpha = 21\mu/4$ and $\beta = 9\mu^2$.

(i) Let $0 < \mu \leq 2/21$. It is a simple matter to verify that $|\alpha| = \alpha \leq 1/2, \ -1 \leq \beta \leq 1$ hold for all $\mu \in (0, 2/21)$ and so $(\alpha, \beta) \in \Omega_1$.

(ii) Let $2/21 \leq \mu \leq 1/3$. Here, in this case, we see that the conditions

$$\frac{1}{2} \leq |\alpha| \leq 2 \ \text{ and } \ \frac{4}{27}(|\alpha| + 1)^3 - (|\alpha| + 1) \leq \beta \leq 1$$

are equivalent to

$$\frac{2}{21} \leq \mu \leq \frac{8}{21} \ \text{ and } \ \frac{1}{432}(21\mu + 4)\left(441\mu^2 + 168\mu - 92\right) \leq 9\mu^2 \leq 1$$

which hold for all $\mu \in (2/21, 1/3)$ and so $(\alpha, \beta) \in \Omega_2$.

(iii) Let $1/3 \leq \mu \leq 8/21$. In this case, it is seen that the inequalities $|\alpha| \leq 2$ and $\beta \geq 1$ hold good. Therefore $(\alpha, \beta) \in \Omega_3$.

(iv) Let $8/21 \leq \mu \leq 16/21$. Now we see that $2 \leq |\alpha| \leq 4$ and $12\beta \geq \alpha^2 + 8$ hold for all such values of μ and hence $(\alpha, \beta) \in \Omega_4$.

(v) Let $16/21 \leq \mu \leq 1$. We can easily verify that $|\alpha| \geq 4$ and $3\beta \geq 2(|\alpha| - 1)$ hold for all such values of μ and hence $(\alpha, \beta) \in \Omega_5$.

Now in view of Lemma 1.3 and the cases (i) and (ii), we conclude that if $0 < \mu \leq 1/3$, then $H(\alpha, \beta) \leq 1$. Further, the cases (iii)–(v) and Lemma 1.3 give $H(\alpha, \beta) \leq \beta$, for $1/3 \leq \mu \leq 1$. Thus, the result follows from (2.6).

(4) We now find the estimate on the fifth coefficient. Comparing the coefficients of z^5 on both sides of (2.2), we have

$$a_5 = \frac{\mu}{4608}[(-18 + 107\mu - 276\mu^2 + 288\mu^3){p_1}^4 + 4(27 - 107\mu + 138\mu^2){p_1}^2 p_2 \\ + 16(20\mu - 9)p_1 p_3 + 36((3\mu - 2){p_2}^2 + 4p_4)] \tag{2.7}$$

Let us denote

$$\hat{a} := \frac{1}{4}(2 - 3\mu), \quad \hat{\alpha} := \frac{9 - 20\mu}{18}, \quad \hat{\beta} := \frac{1}{54}(27 - 107\mu + 138\mu^2)$$

and

$$\hat{\gamma} := \frac{1}{144}(18 - 107\mu + 276\mu^2 - 288\mu^3).$$

Using the above notations, (2.7) can be re-written as

$$|a_5| = \frac{\mu}{32}\left|\hat{\gamma}{p_1}^4 - \frac{3}{2}\hat{\beta}{p_1}^2 p_2 + 2\hat{\alpha}p_1 p_3 + \hat{a}{p_2}^2 - p_4\right|. \tag{2.8}$$

Here it is a simple matter to verify that the inequalities $0 < \hat{a} < 1$ and $0 < \hat{\alpha} < 1$ hold for $0 < \mu < 9/20$. Now a computation shows that

$$\begin{aligned} &8\hat{a}(1 - \hat{a})[(\hat{\alpha}\hat{\beta} - 2\hat{\gamma})^2) + (\hat{\alpha}(\hat{a} + \hat{\alpha}) - \hat{\beta})^2] \\ &\quad + \hat{\alpha}(1 - \hat{\alpha})(\hat{\beta} - 2\hat{a}\hat{\alpha})^2 - 4\hat{a}\hat{\alpha}^2(1 - \hat{\alpha})^2(1 - \hat{a}) \\ &= \frac{1}{3779136}(-236196 + 2932686\mu^2 + 563472\mu^3 - 8764817\mu^4 \\ &\quad - 6932820\mu^5 + 15654024\mu^6 + 13969152\mu^7 - 22902912\mu^8). \end{aligned} \tag{2.9}$$

It is easy to check that the right hand side of (2.9) is negative if $0 < \mu \leq \mu_0 \approx 0.335$. Here μ_0 is the smallest positive roots of the Equation (2.1). Thus all the conditions of Lemma 1.2 are satisfied for $0 < \mu \leq \mu_0$ and from (2.8), we deduce that $|a_5| \leq \mu/16$.

(5) From (2.3), for any complex number ν, we have

$$\begin{aligned} |a_3 - \nu {a_2}^2| &= \frac{\mu}{8}\left[p_2 - \frac{4\mu(\nu - 1) + 1}{2}p_1^2\right] \\ &= \frac{\mu}{4}\max\{1, 4\mu|\nu - 1|\}. \end{aligned} \tag{2.10}$$

This is the desired estimate. The equality holds in case of the function f defined by (2.2) with choice of the function $w(z) = z$.

For $n = 2, 3, 4$ and 5, the equality of estimates on $|a_n|$ in the case $0 < \mu \leq (n-1)^{-2/(n-2)}$ holds for the function f defined by (2.2) with choice of the function $w(z) = z^{n-1}$, whereas in the case $(n-1)^{-2/(n-2)} \leq \mu \leq 1$ equality holds for the choice of $w(z) = z$. This ends the proof. □

Theorem 2.2. *Let $f \in \mathcal{G}_\mu$. Then the following sharp inequalities hold:*

(1) $|a_2a_4 - {a_3}^2| \leq \begin{cases} \frac{\mu^2}{16}, & 0 < \mu \leq \frac{1}{6}; \\ \frac{\mu^2(4+3\mu)^2}{36(7+12\mu)}, & \frac{1}{6} < \mu \leq 1. \end{cases}$

(2) $|a_2a_3 - a_4| \leq \frac{\mu}{9}$.

Proof. **(1)** Proceeding as in the proof of Theorem 2.1, and using (2.3) and (2.4), we have

$$\begin{aligned} a_2a_4 - {a_3}^2 &= \frac{(7-12\mu)\mu^2}{2304}{p_1}^4 + \frac{4(6\mu-7)\mu^2}{2304}{p_1}^2p_2 - \frac{36\mu^2}{2304}{p_2}^2 + \frac{64\mu^2}{2304}p_1p_3 \\ &= \frac{\mu^2}{2304}\left[(7-12\mu){p_1}^4 + 4(6\mu-7){p_1}^2p_2 - 36{p_2}^2 + 64p_1p_3\right]. \end{aligned} \tag{2.11}$$

We substitute equivalent expressions for p_2 and p_3 in terms of p_1 from (1.3) and (1.4) in (2.11). Thus we have

$$\begin{aligned} a_2a_4 - {a_3}^2 = \frac{\mu^2}{2304}\big[12\mu{p_1}^2(4-{p_1}^2)x - (7{p_1}^2+36)(4-{p_1}^2)x^2 \\ +32p_1(4-{p_1}^2)(1-|x|^2)y.\big] \end{aligned} \tag{2.12}$$

Since $p \in \mathcal{P}$, without loss of any generality, we can assume that $p_1 = |p_1| =: s \in [0, 2]$. Further since $|x| \leq 1$ and $|y| \leq 1$ for some $x, y \in \mathbb{C}$, using this facts and the triangle inequality in (2.12) we can write

$$|a_2a_4 - {a_3}^2| \leq \frac{\mu^2}{72}s(4-s^2)\left[\left|\frac{3\mu s}{8}x - \frac{7s^2+36}{32s}x^2\right| + 1 - |x|^2\right]. \tag{2.13}$$

We note that for $s = p_1 = 0$, and $s = p_1 = 2$ from (2.12), we have $|a_2a_4 - {a_3}^2| \leq \mu^2/16$ and $|a_2a_4 - {a_3}^2| = 0$, respectively.

Now we assume that $s \in (0, 2)$. Then form (2.13) we obtain

$$|a_2a_4 - {a_3}^2| \leq \frac{\mu^2}{72}s(4-s^2)F(a,b,c), \tag{2.14}$$

where

$$F(a,b,c) := |a + bx + cx^2| + 1 - |x|^2,$$

with

$$a := 0, \quad b := \frac{3\mu s}{8} \text{ and } c := -\frac{7s^2+36}{32s}.$$

Here it is easily seen that $ac = 0$ and $|b| \geq 2(1-|c|)$. Therefore

$$\begin{aligned} |a_2a_4 - {a_3}^2| &\leq \frac{\mu^2}{72} s(4-s^2)F(a,b,c) \\ &= \frac{\mu^2}{72}\left(\frac{3\mu s^2(4-s^2)}{8} + \frac{(7s^2+36)(4-s^2)}{32}\right) \\ &= \frac{\mu^2}{72} g(s), \end{aligned}$$

where the function $g : (0,2) \to \mathbb{R}$ is defined by

$$g(s) = \frac{3\mu s^2(4-s^2)}{8} + \frac{(7s^2+36)(4-s^2)}{32}.$$

To find the maximum of g, we shall consider two cases namely, (i) $0 < \mu \leq 1/6$ and (ii) $1/6 < \mu \leq 1$. It is easy to verify in the first case when $0 < \mu \leq 1/6$, the function g has no critical point in $(0,2)$ and so $|a_2a_4 - {a_3}^2| \leq \mu^2/16$. Further in the second case when $1/6 < \mu \leq 1$, it can be easily verified that $g'(s) = 0$ holds for $s = s_0 = 2\sqrt{6\mu-1}/\sqrt{7+12\mu}$ and the second derivative of g is negative at s_0. So by the second derivative test, it is clear that g has its maximum at s_0 and

$$|a_2a_4 - {a_3}^2| \leq \frac{\mu^2}{72} g(s_0) = \frac{\mu^2(4+3\mu)^2}{36(7+12\mu)}.$$

For the case when $1/6 < \mu \leq 1$, the equality occurs for the function f defined by

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)}\left(1 + \mu\frac{p(z)-1}{p(z)+1}\right) \quad \text{with} \quad p(z) = \frac{1-z^2}{1-s_0z+z^2}.$$

For the function defined above, on comparing the coefficient of like power terms, we have

$$a_2 = \frac{\mu\sqrt{6\mu-1}}{\sqrt{12\mu+7}}, \quad a_3 = \frac{\mu\left(12\mu^2-5\mu-4\right)}{2(12\mu+7)}$$

and

$$a_4 = \frac{\mu\left(108\mu^3 - 81\mu^2 - 96\mu - 16\right)\sqrt{6\mu-1}}{18(12\mu+7)^{3/2}}.$$

A computation gives

$$|a_2a_4 - {a_3}^2| = \frac{\mu^2(3\mu+4)^2}{36(12\mu+7)}.$$

On the other hand when $0 < \mu \leq 1/6$, the equality holds in case of the function 2.2 for the choice of the function $w(z) = z^2$.

(2) We shall now find the estimate on $|a_2a_3 - a_4|$. For this using (2.3) and (2.4), we can write

$$a_2a_3 - a_4 = \frac{\mu}{72}\left\{(3\mu-1){p_1}^3 + (2-3\mu)p_1(2p_2) - 4p_3\right\}. \tag{2.15}$$

Substituting expressions for p_2 and p_3 in terms of p_1 from (1.3) and (1.4) in (2.15) and using the facts that $|x| \leq 1$ and $|y| \leq 1$ for some $x, y \in \mathbb{C}$, we can write

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{\mu}{72}(4-s^2)\left[\left|-3\mu s x + s x^2\right| + 2(1-|x|^2)\right] \\ &= \frac{\mu}{36}(4-s^2)\left[\left|-\frac{3\mu s}{2}x + \frac{s}{2}x^2\right| + 1 - |x|^2\right] \\ &= \frac{\mu}{36}(4-s^2)G(A,B,C), \end{aligned} \tag{2.16}$$

where $p_1 = |p_1| =: s \in [0,2]$, $A := 0, B := -3\mu s/2$ and $C := s/2$ and the function $G : [0,2] \to \mathbb{R}$ is defined by

$$G(A,B,C) := \left|A + Bx + Cx^2\right| + 1 - |x|^2.$$

Now from (2.16), we see that if $s = 0$, then $|a_2a_3 - a_4| \leq \mu/9$ and if $s = 2$, then $|a_2a_3 - a_4| = 0$.

We now consider the case $s \in (0,2)$. Here it is easy to verify that $AB = 0$ for all $s \in (0,2)$. Here we have two cases now:

(i) Let $s \in (0, 4/(2+3\mu))$. Then $|B| < 2(1-|C|)$ and therefore by Lemma 1.4, we have

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{\mu}{36}(4-s^2)G(A,B,C) \\ &= \frac{\mu}{36}(4-s^2)\left(1 + |A| + \frac{B^2}{4(1-|C|)}\right) \\ &= \frac{\mu}{288}h(s), \end{aligned}$$

where $h : (0, 4/(2+3\mu)) \to \mathbb{R}$ is function defined by $h(s) = (2+s)(16 - 8s + 9s^2\mu^2)$. Since $h'(s) = 0$ occurs only at $s = t_0 := 4(4-9\mu^2)/27\mu^2 \in (0, 4/(2+3\mu))$ for $2\left(\sqrt{3}-1\right)/3 < \mu < 2/3$ and $h''(t_0) = 16 - 36\mu^2 > 0$ whenever $2\left(\sqrt{3}-1\right)/3 < \mu < 2/3$. Therefore h has a maxima at s_0.

(ii) Let $s \in [4/(2+3\mu), 2)$. Then $|B| \geq 2(1-|C|)$. Therefore, using Lemma 1.4, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &= \frac{\mu}{36}(4-s^2)G(A,B,C) \\ &= \frac{\mu}{36}(4-s^2)(|A| + |B| + |C|) \\ &= \frac{\mu}{72}k(s), \end{aligned}$$

where $k : [4/(2+3\mu), 2) \to \mathbb{R}$ is function defined by $k(s) = s(4-s^2)(3\mu+1)$. A computation shows that the function k has its maximum at $s = r_0 := 2/\sqrt{3}$ and therefore

$$|a_2a_3 - a_4| \leq \frac{\mu}{72}k(r_0) = \frac{2\mu(1+3\mu)}{27\sqrt{3}}.$$

Therefore, as discussed above in the cases (i) and (ii), for all $s \in (0,2)$, we have

$$|a_2a_3 - a_4| \leq \frac{2\mu(1+3\mu)}{27\sqrt{3}}.$$

Therefore, for all $s \in [0,2]$, we conclude that

$$|a_2a_3 - a_4| \leq \max\left\{\frac{2\mu(1+3\mu)}{27\sqrt{3}}, \frac{\mu}{9}\right\} = \frac{\mu}{9}.$$

This gives the desired estimate. The equality holds in case of the function f defined by (2.2) with the choice $w(z) = z^3$. Hence the theorem. □

Using the above results, we deduce the following estimates on the third Hankel determinant:

Corollary 2.3. *Let $f \in \mathcal{G}_\mu$. Then the following holds:*

$$|H_{3,\,1}(f)| \leq \begin{cases} \frac{\mu^2(81\mu+145)}{5184}, & 0 < \mu \leq 1/6; \\ \frac{\mu^2\left(324\mu^3+864\mu^2+2316\mu+1015\right)}{5184(12\mu+7)}, & 1/6 < \mu \leq 1/4; \\ \frac{\mu^2\left(1296\mu^4+3456\mu^3+2304\mu^2+1740\mu+1015\right)}{5184(12\mu+7)}, & 1/4 < \mu \leq 1/3. \end{cases}$$

Remark 2.4. It should be noted that all the estimates derived so far in this paper are sharp except the bound on the third Hankel determinant $|H_{3,\,1}(f)|$ mentioned in Corollary 2.3. Finding the sharp bound on the third Hankel determinants for many classes of analytic functions, including the class under consideration here, are still unsettled . However, here we have a partial solution for sharp bound on the third Hankel determinant under certain conditions.

It is known that a function $f \in \mathcal{A}$ is said to be n-fold symmetric if $f(\epsilon z) = \epsilon f(z)$ holds for all $z \in \mathbb{D}$, where $\epsilon = e^{2\pi i/n}$. The set of all n-fold symmetric functions is denoted by $\mathcal{A}^{(n)}$ and functions in this class are of the form $g_n(z) = z + a_{n+1}z^{n+1} + a_{2n+1}z^{2n+1} + \cdots$. In particular, any function g in the class $\mathcal{A}^{(3)}$ has the form $g_3(z) = z + b_4z^4 + b_7z^7 + \cdots$. Thus it is clear that $|H_{3,\,1}(f)| = |b_4^2|$. Since the functions in the class $\mathcal{G}_\mu$ are starlike, it follows that $f \in \mathcal{G}_\mu$ if and only if $g_3(z) = \sqrt[3]{f(z^3)} = z + (a_2/3)z^4 + \cdots \in \mathcal{G}_\mu$. Thus $b_4 = a_2/3$ and $|H_{3,\,1}(f)| = |b_4^2| = |a_2|^2/9$. Now from the first part of Theorem 2.1, we have $|H_{3,\,1}(f)| = |a_2|^2/9 \leq \mu^2/9$. The result is sharp in case of the function f_0 defined by

$$1 + \frac{zf_0''(z)}{f_0'(z)} = \frac{zf_0'(z)}{f_0(z)}(1 + \mu z^3).$$

Remark 2.5. A close observation on the results in Theorem 2.1 and the extremal functions reveals the following expected results, which have already been verified for $n = 2, 3, 4$ in Theorem 2.1. Further in the same theorem this conjecture is also verified for $n = 5$ for the range $0 < \mu \leq \mu_0 \approx 0.335$.

Conjecture 2.6. Let $f \in \mathcal{G}_\mu$. Then, for any natural number $n \geq 5$, the following sharp inequalities hold:

$$|a_n| \leq \begin{cases} \frac{\mu}{(n-1)^2}, & 0 < \mu \leq (n-1)^{-2/(n-2)}; \\ \mu^{n-1}, & (n-1)^{-2/(n-2)} \leq \mu \leq 1. \end{cases}$$

The extremal function, in both the cases $0 < \mu \leq (n-1)^{-2/(n-2)}$ and $(n-1)^{-2/(n-2)} \leq \mu \leq 1$, is given by (2.2) with choice of the function $w(z) = z^{n-1}$ and $w(z) = z$, respectively.

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