

Maximum penalized likelihood estimation for a stress-strength reliability model using complete and incomplete data

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Abstract

The two parameter negative exponential distribution has many practical applications in queuing theory such as the service times of agents in system, the time it takes before your next telephone call, the time until a radioactive practical decays, the distance between mutations on a DNA strand, and the extreme values of annual snowfall or rainfall; consequently, has many applications in reliability systems. This paper considers an estimation problem of stress-strength model with two parameter negative parameter exponential distribution. We introduce a maximum penalized likelihood method, Bayes estimator using Lindley approximation to estimate stress-strength model and compare the proposed estimators with regular maximum likelihood estimator for complete data. We also introduce a maximum penalized likelihood method, Bayes estimator using a Markov chain Monte Carlo technique for incomplete data. A Monte Carlo simulation study is performed to compare stress-strength model estimates. Real data is used as a practical application of the proposed model.

Keywords: maximum penalized likelihood estimator, Lindley approximation, progressive type II censored data, Bayes estimator, simulation, Markov chain Monte Carlo technique, two parameter negative exponential distribution

1. Introduction

Let X be a two parameter negative exponential distribution with scale parameter λ measuring the failure rate and location parameter μ . Then the probability density function and cumulative distribution function of X are

$$f(x; \mu, \lambda) = \frac{1}{\lambda} \exp\left[-\frac{(x-\mu)}{\lambda}\right] I(x > \mu), \quad x > \mu, \mu \geq 0, \lambda > 0 \quad (1.1)$$

and

$$F(x; \mu, \lambda) = 1 - \exp\left[-\frac{(x-\mu)}{\lambda}\right] I(x > \mu), \quad x > \mu,$$

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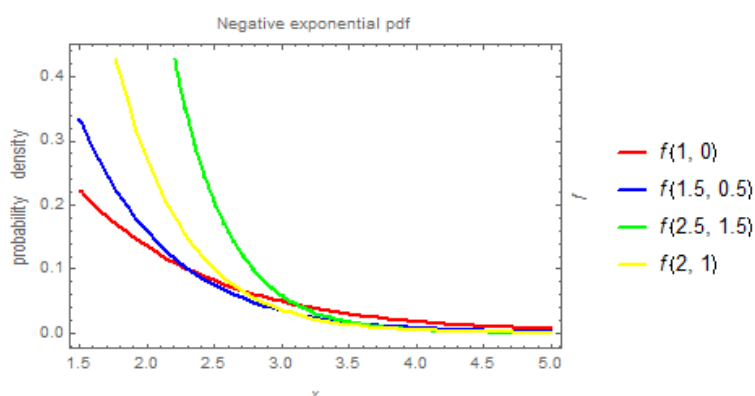


Figure 1: Two parameter negative exponential distribution.

where $I(\cdot)$ is the indicator function of (\cdot) that is $I(x > \mu) = 1$ for all $x > \mu$. Figure 1 shows the graph of two parameter negative exponential distribution for different values of μ and λ , we then observe that the two parameter negative exponential distribution is a positively skewed distribution.

The two parameter negative exponential distribution plays an important role in modeling in many fields such as queuing theory, physics, biology, hydrology, reliability, and reliability engineering. In this paper, we examine the problem of estimating the reliability parameter i.e., $R = P[Y < X]$. The reliability parameter is referred to the stress-strength model included in quality control, engineering statistics, and other fields. In reliability context, the stress-strength model describes the life of component which has a random strength variable X and is subjected to random variable stress Y . The system fails any time the stress is greater than the strength. The estimation of stress-strength model when X and Y are random variables having a specified distribution discussed by many authors starting from Birnbaum (1956), Basu (1964), Downtown (1973), Tong (1974, 1977), Beg (1980), Iwase (1987), McCool (1991). Recently, Ali *et al.* (2012), Greco and Ventura (2011), Rezaei *et al.* (2010), Wong (2012), Shahsanaei and Daneshkhah (2013), Hussian (2013), Al-Mutairi *et al.* (2013), Ghitany *et al.* (2015), Najrzadegan *et al.* (2016). The estimator of reliability parameter when X and Y are independent exponential random variables discussed by several authors such as Enis and Geisser (1971), Kelley *et al.* (1976), Sathe and Shah (1981). The pervious work of reliability parameter when X and Y are independent two parameter negative exponential random variables discussed by Pal *et al.* (2005) derived the distribution of UMVUE of the reliability parameter; subsequently, Krishnamoorthy *et al.* (2007) studied the problem of hypothesis testing and interval estimation of the reliability parameter.

In this paper, we introduce a maximum penalized likelihood estimator of a reliability parameter when X and Y are independent two parameter negative exponential random variables as well as introduce the Bayes estimator of it using Lindley approximation. This paper organized as: the stress-strength parameter when X and Y are independent two parameter negative exponential random variables with parameters (λ_1, μ_1) and (λ_2, μ_2) respectively is determined in Section 2. We present maximum penalized likelihood estimator of the reliability parameter and compare it by the maximum likelihood estimator also the Bayes estimator of the reliability parameter using Lindley approximation introduced for complete data in Section 3. We present the maximum penalized likelihood estimator of the reliability parameter and the Bayes estimator of the reliability parameter using a Markov chain Monte Carlo technique (MCMC) introduced for the incomplete data in Section 4. The performance of these estimator is assessed by simulation study in Section 5. A practical application is presented in Section 6. The conclusion of this paper introduced in Section 7.

2. Stress-strength model

Let X and Y be independent strength and stress random variables follow two parameter negative exponential distribution with parameters (λ_1, μ_1) and (λ_2, μ_2) respectively. Then the stress-strength model R define as

$$R = P[Y < X] = \int_{\min(\mu_1, \mu_2)}^{\infty} F_Y(x)f(x)dx.$$

For all cases of (μ_1, μ_2) we obtain the formula of R as:

- If $\mu_1 = \mu_2$, then

$$R = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \quad (2.1)$$

- If $\mu_1 \neq \mu_2$, then

$$R = \left[1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\frac{(\mu_2 - \mu_1)}{\lambda_2}} \right] I(\mu_1 > \mu_2) + \left[\frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\frac{(\mu_1 - \mu_2)}{\lambda_1}} \right] I(\mu_1 < \mu_2), \quad (2.2)$$

where $I(\cdot)$ is the indicator function.

In the next sections, we will obtain the different estimators of R .

3. Different estimators of R for complete data

In this section, the maximum penalized likelihood estimators of parameter R are obtained and compared by its regular maximum likelihood estimator for complete data and the Bayes estimator of reliability parameter using the Lindley approximation also introduced for complete data.

3.1. Maximum penalized likelihood estimator of R

Zheng (2013) introduced the maximum penalized likelihood estimator of two parameter negative exponential distribution parameters because the regular maximum likelihood estimator of distribution parameters were biased. In this subsection, we use the Zheng (2013) approach to find the maximum penalized estimator of R using the following algorithm.

Algorithm 1. Find maximum penalized likelihood estimator of R says R^* using the invariance property as follows

- Case 1: $\mu_1 = \mu_2$, then

$$\hat{R}^* = \frac{\hat{\lambda}_1^*}{\hat{\lambda}_1^* + \hat{\lambda}_2^*},$$

- Case 2: $\mu_1 \neq \mu_2$, then

$$\hat{R}^* = \left[1 - \frac{\hat{\lambda}_2^*}{\hat{\lambda}_1^* + \hat{\lambda}_2^*} e^{-\frac{\hat{\mu}_2^* - \hat{\mu}_1^*}{\hat{\lambda}_2^*}} \right] I(\hat{\mu}_1^* > \hat{\mu}_2^*) + \left[\frac{\hat{\lambda}_1^*}{\hat{\lambda}_1^* + \hat{\lambda}_2^*} e^{-\frac{\hat{\mu}_1^* - \hat{\mu}_2^*}{\hat{\lambda}_1^*}} \right] I(\hat{\mu}_1^* < \hat{\mu}_2^*),$$

where $\lambda_1^*, \mu_1^*, \lambda_2^*, \mu_2^*$ are the maximum penalized likelihood estimator of $\lambda_1, \mu_1, \lambda_2, \mu_2$ (Zheng, 2013).

Now, we will study the distribution of \hat{R}^* as follows case 1: $\mu_1 = \mu_2$, then

$$\hat{R}^* = \frac{\hat{\lambda}_1^*}{\hat{\lambda}_1^* + \hat{\lambda}_2^*} = \frac{1}{1 + \hat{\lambda}_2^*/\hat{\lambda}_1^*}.$$

Suppose $u = \hat{\lambda}_2^*/\hat{\lambda}_1^*$, then from convolution rule we get the probability density function of U is $f(u) = 1/(1 + u^2)$, $u > 0$ and the moments about zero of \hat{R}^* is given by

$$\begin{aligned} E[\hat{R}^*]^s &= E[1 + u]^{-s} \\ &= \frac{\pi 2^{-\frac{s}{2}}}{e^{-\frac{1}{4}i\pi s} + e^{\frac{i\pi s}{4}}} + \frac{\pi 2^{-\frac{s}{2}}}{\left(e^{-\frac{1}{4}i\pi s} + e^{\frac{i\pi s}{4}}\right)\left(e^{-\frac{1}{2}i\pi s} + e^{\frac{i\pi s}{2}}\right)} + \frac{{}_pF_q\left[\left\{\frac{1}{2}, 1, 1\right\}, \left\{1 - \frac{s}{2}, \frac{3}{2} - \frac{s}{2}\right\}, -1\right]}{s - 1}, \end{aligned}$$

and the mean square error (MSE) of \hat{R}^* is given by

$$\text{MSE}(\hat{R}^*) = E\left[(\hat{R}^* - R)^2\right] = \frac{\lambda_1^2 - (\pi - 2)\lambda_2\lambda_1 + \lambda_2^2}{2(\lambda_1 + \lambda_2)^2}.$$

But in case 2: $\mu_1 \neq \mu_2$, then

$$\hat{R}^* = \left[1 - \frac{\hat{\lambda}_2^*}{\hat{\lambda}_1^* + \hat{\lambda}_2^*} e^{\frac{\hat{\mu}_2^* - \hat{\mu}_1^*}{\hat{\lambda}_2^*}}\right] I(\hat{\mu}_1^* > \hat{\mu}_2^*) + \left[\frac{\hat{\lambda}_1^*}{\hat{\lambda}_1^* + \hat{\lambda}_2^*} e^{\frac{\hat{\mu}_1^* - \hat{\mu}_2^*}{\hat{\lambda}_1^*}}\right] I(\hat{\mu}_1^* < \hat{\mu}_2^*).$$

The exact distribution of \hat{R}^* does not exist. So, we find the asymptotic distribution of \hat{R}^* as: as $n \rightarrow \infty$ and $m \rightarrow \infty$ the \hat{R}^* has asymptotic normal with mean R and asymptotic variance $\text{Var}(R)$

$$\text{Var}(R) = \sum_{j=1}^4 \sum_{i=1}^4 \frac{\partial R}{\partial \Theta_i} \frac{\partial R}{\partial \Theta_j} I_{ij}^{-1},$$

where $\Theta = (\lambda_1, \mu_1, \lambda_2, \mu_2)$ and I_{ij}^{-1} is the $(i, j)^{th}$ element of the inverse of the Fisher information matrix I_{ij} which is given by

$$\begin{aligned} I_{ij} &= \begin{pmatrix} E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \lambda_1^2}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \lambda_1 \partial \mu_1}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \lambda_1 \partial \lambda_2}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \lambda_1 \partial \mu_2}\right] \\ E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \mu_1 \partial \lambda_1}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \mu_1^2}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \mu_1 \partial \lambda_2}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \mu_1 \partial \mu_2}\right] \\ E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \lambda_2 \partial \lambda_1}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \lambda_2 \partial \mu_1}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \lambda_2^2}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \lambda_2 \partial \mu_2}\right] \\ E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \mu_2 \partial \lambda_1}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \mu_2 \partial \mu_1}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \mu_2 \partial \lambda_2}\right] & E\left[\frac{-\partial^2 l^*(\Theta)}{\partial \mu_2^2}\right] \end{pmatrix}, \\ I_{ij}^{-1} &= \begin{pmatrix} \text{Var}(\lambda_1^*) & \text{Cov}(\lambda_1^*, \mu_1^*) & \text{Cov}(\lambda_1^*, \lambda_2^*) & \text{Cov}(\lambda_1^*, \mu_2^*) \\ \text{Cov}(\mu_1^*, \lambda_1^*) & \text{Var}(\mu_1^*) & \text{Cov}(\mu_1^*, \lambda_2^*) & \text{Cov}(\mu_1^*, \mu_2^*) \\ \text{Cov}(\lambda_2^*, \lambda_1^*) & \text{Cov}(\lambda_2^*, \mu_1^*) & \text{Var}(\lambda_2^*) & \text{Cov}(\lambda_2^*, \mu_2^*) \\ \text{Cov}(\mu_2^*, \lambda_1^*) & \text{Cov}(\mu_2^*, \mu_1^*) & \text{Cov}(\mu_2^*, \lambda_2^*) & \text{Var}(\mu_2^*) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \text{Var}(\lambda_1^*) &= \frac{\lambda_1^2}{n-1}, & \text{Var}(\lambda_2^*) &= \frac{\lambda_2^2}{m-1}, & \text{Var}(\mu_1^*) &= \frac{\lambda_1^2}{n(n-1)}, & \text{Var}(\mu_2^*) &= \frac{\lambda_2^2}{m(m-1)}, \\ \text{Cov}(\lambda_1^*, \mu_1^*) &= \text{Cov}(\mu_1^*, \lambda_1^*) = \frac{\lambda_1^2}{n}, & \text{Cov}(\lambda_1^*, \lambda_2^*) &= \text{Cov}(\lambda_2^*, \lambda_1^*) = 0, \\ \text{Cov}(\lambda_1^*, \mu_2^*) &= \text{Cov}(\mu_2^*, \lambda_1^*) = 0, & \text{Cov}(\mu_1^*, \lambda_2^*) &= \text{Cov}(\lambda_2^*, \mu_1^*) = 0, \\ \text{Cov}(\mu_1^*, \mu_2^*) &= \text{Cov}(\mu_2^*, \mu_1^*) = 0, & \text{Cov}(\lambda_2^*, \mu_2^*) &= \text{Cov}(\mu_2^*, \lambda_2^*) = \frac{\lambda_2^2}{m}, \\ \frac{\partial R}{\partial \lambda_1} &= -\frac{\lambda_1 e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{(\lambda_1 + \lambda_2)^2} + \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 + \lambda_2} - \frac{(\mu_1 - \mu_2) e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 (\lambda_1 + \lambda_2)} I(\mu_1 < \mu_2) + \frac{\lambda_2 e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{(\lambda_1 + \lambda_2)^2} I(\mu_1 > \mu_2), \\ \frac{\partial R}{\partial \lambda_2} &= -\frac{\lambda_1 e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{(\lambda_1 + \lambda_2)^2} I(\mu_1 < \mu_2) + \frac{\lambda_2 e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{(\lambda_1 + \lambda_2)^2} + \frac{(\mu_2 - \mu_1) e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_2 (\lambda_1 + \lambda_2)} - \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_1 + \lambda_2} I(\mu_1 > \mu_2), \\ \frac{\partial R}{\partial \mu_1} &= \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 + \lambda_2} I(\mu_1 < \mu_2) + \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_1 + \lambda_2} I(\mu_1 > \mu_2), \\ \frac{\partial R}{\partial \mu_2} &= -\frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 + \lambda_2} I(\mu_1 < \mu_2) - \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_1 + \lambda_2} I(\mu_1 > \mu_2). \end{aligned}$$

Hence an asymptotic $100(1 - \alpha)\%$ confidence interval for R is given by

$$R \in \left(\hat{R}^* \pm \frac{Z_{\alpha/2}}{\sqrt{n+m}} \sqrt{\widehat{\text{Var}}(R^*)} \right), \tag{3.1}$$

where $Z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ quantile of standard normal distribution and $\widehat{\text{Var}}(R^*)$ is the value of $\text{Var}(R^*)$ at maximum penalized likelihood estimators of parameters $\Theta = (\lambda_1, \mu_1, \lambda_2, \mu_2)$.

3.2. Bayes estimator of R

Lindley (1980) proposed on approximation technique to find the Bayes estimator of the stress-strength parameter under squared error loss function and under Jeffrey prior (ρ) as follows.

Algorithm 2.

1. Find the penalized maximum likelihood estimator of $\Theta = (\lambda_1, \mu_1, \lambda_2, \mu_2)$ and R says $\Theta^* = (\lambda_1^*, \mu_1^*, \lambda_2^*, \mu_2^*)$ and \hat{R}^* .
2. Compute the Bayes estimator of R says R^B as

$$R^B = \hat{R}^* + \frac{1}{2} \left(\sum_{i=1}^4 \sum_{j=1}^4 (\hat{R}_{ij}^* + 2\hat{R}_i^* \rho_j) \sigma_{ij} \right) + \frac{1}{2} \left(\sum_{i=1}^4 \sum_{j=1}^4 \sum_{k=1}^4 \sum_{l=1}^4 L_{ijkl} \sigma_{ij} \sigma_{kl} \hat{R}_l^* \right).$$

Note: all terms and derivatives are calculated by replacing $\Theta = (\lambda_1, \mu_1, \lambda_2, \mu_2)$ and R with their

maximum penalized likelihood estimator. Where

$$\begin{aligned}
 R_{11} &= \frac{\partial^2 \hat{R}^*}{\partial \lambda_1^2} = -\frac{2\lambda_2 e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{(\lambda_1 + \lambda_2)^3} I[\mu_1 > \mu_2] \\
 &\quad + \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}} \left(-2\lambda_1^3 (\lambda_2 - \mu_1 + \mu_2) + \lambda_1^2 (\mu_1 - \mu_2) (2\lambda_2 + \mu_1 - \mu_2) + 2\lambda_2 \lambda_1 (\mu_1 - \mu_2)^2 + \lambda_2^2 (\mu_1 - \mu_2)^2 \right)}{\lambda_1^3 (\lambda_1 + \lambda_2)^3} I[\mu_1 < \mu_2], \\
 R_{12} = R_{21} &= \frac{\partial^2 \hat{R}^*}{\partial \lambda_1 \partial \mu_1} = -\frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{(\lambda_1 + \lambda_2)^2} I[\mu_1 > \mu_2] - \frac{(\mu_1 - \mu_2) e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1^2 (\lambda_1 + \lambda_2)} - \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{(\lambda_1 + \lambda_2)^2} I[\mu_1 < \mu_2], \\
 R_{13} = R_{31} &= \frac{\partial^2 \hat{R}^*}{\partial \lambda_1 \partial \lambda_2} = \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}} (\lambda_1 (\lambda_2 + \mu_1 - \mu_2) - \lambda_2 (\lambda_2 - \mu_1 + \mu_2))}{\lambda_2 (\lambda_1 + \lambda_2)^3} I[\mu_1 > \mu_2] \\
 &\quad + \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}} \left(-\lambda_1 (\lambda_2 - \mu_1 + \mu_2) + \lambda_2 (\mu_1 - \mu_2) + \lambda_1^2 \right)}{\lambda_1 (\lambda_1 + \lambda_2)^3} I[\mu_1 < \mu_2], \\
 R_{14} = R_{41} &= \frac{\partial^2 \hat{R}^*}{\partial \lambda_1 \partial \mu_2} = \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{(\lambda_1 + \lambda_2)^2} I[\mu_1 > \mu_2] + \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}} (\lambda_1 \mu_1 + \lambda_2 \mu_1 - (\lambda_1 + \lambda_2) \mu_2 + \lambda_1^2)}{\lambda_1^2 (\lambda_1 + \lambda_2)^2} I[\mu_1 < \mu_2], \\
 R_{22} &= \frac{\partial^2 \hat{R}^*}{\partial \mu_1^2} = -\frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_2 (\lambda_1 + \lambda_2)} I[\mu_1 > \mu_2] - \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 (\lambda_1 + \lambda_2)} I[\mu_1 < \mu_2], \\
 R_{23} = R_{32} &= \frac{\partial^2 \hat{R}^*}{\partial \lambda_2 \partial \mu_1} = \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}} (\lambda_1 (\mu_1 - \mu_2) - \lambda_2 (\lambda_2 - \mu_1 + \mu_2))}{\lambda_2^2 (\lambda_1 + \lambda_2)^2} I[\mu_1 > \mu_2] - \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{(\lambda_1 + \lambda_2)^2} I[\mu_1 < \mu_2], \\
 R_{24} = R_{42} &= \frac{\partial^2 \hat{R}^*}{\partial \mu_2 \partial \mu_1} = \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_2 (\lambda_1 + \lambda_2)} I[\mu_1 > \mu_2] - \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 (\lambda_1 + \lambda_2)} I[\mu_1 < \mu_2], \\
 R_{33} &= \frac{\partial^2 \hat{R}^*}{\partial \lambda_2^2} = \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}} \left(\lambda_2^2 (\mu_1 - \mu_2) (2\lambda_2 - \mu_1 + \mu_2) + 2\lambda_1 \lambda_2 (\lambda_2 (\mu_1 - \mu_2) + \lambda_2^2 - (\mu_1 - \mu_2)^2) - \lambda_1^2 (\mu_1 - \mu_2)^2 \right)}{\lambda_2^3 (\lambda_1 + \lambda_2)^3} I[\mu_1 > \mu_2] \\
 &\quad + \frac{2\lambda_1 e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{(\lambda_1 + \lambda_2)^3} I[\mu_1 < \mu_2], \\
 R_{34} = R_{43} &= \frac{\partial^2 \hat{R}^*}{\partial \lambda_2 \partial \mu_2} \\
 &= \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}} (\lambda_1 (\mu_2 - \mu_1) + \lambda_2 (\lambda_2 - \mu_1 + \mu_2))}{\lambda_2^2 (\lambda_1 + \lambda_2)^2} I[\mu_1 > \mu_2] + \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}} (\lambda_1 \mu_1 + \lambda_2 \mu_1 - (\lambda_1 + \lambda_2) \mu_2 + \lambda_1^2)}{\lambda_1^2 (\lambda_1 + \lambda_2)^2} I[\mu_1 < \mu_2], \\
 R_{44} &= \frac{\partial^2 \hat{R}^*}{\partial \mu_2^2} = -\frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_2 (\lambda_1 + \lambda_2)} I[\mu_1 > \mu_2] + \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 (\lambda_1 + \lambda_2)} I[\mu_1 < \mu_2], \\
 \sigma_{ij} &= I_{ij}^{-1}, \\
 \rho_1 &= \frac{\partial \rho}{\partial \lambda_1} = \frac{\lambda_1 + 2}{\lambda_1^2 + 4\lambda_1 + 2} - \frac{1}{\lambda_1} - \frac{1}{1 + \lambda_1}, \\
 \rho_2 &= \frac{\partial \rho}{\partial \mu_1} = \frac{\mu_1 + 2}{\mu_1^2 + 4\mu_1 + 2} - \frac{1}{\mu_1} - \frac{1}{1 + \mu_1}, \\
 \rho_3 &= \frac{\partial \rho}{\partial \lambda_2} = \frac{\lambda_2 + 2}{\lambda_2^2 + 4\lambda_2 + 2} - \frac{1}{\lambda_2} - \frac{1}{1 + \lambda_2},
 \end{aligned}$$

$$\begin{aligned} \rho_4 &= \frac{\partial \rho}{\partial \mu_2} = \frac{\mu_2 + 2}{\mu_2^2 + 4\mu_2 + 2} - \frac{1}{\mu_2} - \frac{1}{1 + \mu_2}, \\ R_1 &= \frac{\partial R}{\partial \lambda_1} = -\frac{\lambda_1 e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{(\lambda_1 + \lambda_2)^2} + \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 + \lambda_2} - \frac{(\mu_1 - \mu_2) e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 (\lambda_1 + \lambda_2)} I(\mu_1 < \mu_2) + \frac{\lambda_2 e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{(\lambda_1 + \lambda_2)^2} I(\mu_1 > \mu_2), \\ R_2 &= \frac{\partial R}{\partial \mu_1} = \frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 + \lambda_2} I(\mu_1 < \mu_2) + \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_1 + \lambda_2} I(\mu_1 > \mu_2), \\ R_3 &= \frac{\partial R}{\partial \lambda_2} = -\frac{\lambda_1 e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{(\lambda_1 + \lambda_2)^2} I(\mu_1 < \mu_2) + \frac{\lambda_2 e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{(\lambda_1 + \lambda_2)^2} + \frac{(\mu_2 - \mu_1) e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_2 (\lambda_1 + \lambda_2)} - \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_1 + \lambda_2} I(\mu_1 > \mu_2), \\ R_4 &= \frac{\partial R}{\partial \mu_2} = -\frac{e^{\frac{\mu_1 - \mu_2}{\lambda_1}}}{\lambda_1 + \lambda_2} I(\mu_1 < \mu_2) - \frac{e^{\frac{\mu_2 - \mu_1}{\lambda_2}}}{\lambda_1 + \lambda_2} I(\mu_1 > \mu_2), \\ L_{111} &= \frac{\partial^3 l^*(\Theta)}{\partial \lambda_1^3} = \frac{2(n-1)}{\lambda_1^3}, \\ L_{333} &= \frac{\partial^3 l^*(\Theta)}{\partial \lambda_2^3} = \frac{2(m-1)}{\lambda_2^3}, \\ L_{121} &= L_{211} = L_{112} = \frac{\partial^3 l^*(\Theta)}{\partial \lambda_1^2 \partial \mu_1} = \frac{2n}{\lambda_1^3}, \\ L_{433} &= L_{334} = L_{343} = \frac{\partial^3 l^*(\Theta)}{\partial \lambda_2^2 \partial \mu_2} = \frac{2m}{\lambda_2^3}, \\ L_{221} &= L_{122} = L_{212} = \frac{\partial^3 l^*(\Theta)}{\partial \lambda_1 \partial \mu_1^2} = \frac{2n(n-1)}{\lambda_1^3}, \\ L_{443} &= L_{344} = L_{434} = \frac{\partial^3 l^*(\Theta)}{\partial \lambda_2 \partial \mu_2^2} = \frac{2m(m-1)}{\lambda_2^3}. \end{aligned}$$

Note: other $L_{ijk} = 0$.

4. Different estimators of R for incomplete data

It is very difficult to get a complete data and the data are often incomplete. In medical, industrial or others fields researchers might stop their experiment before all items fail because they often do not have time to observe the lifetime of all items in the study or items fail due to random causes. Consequently, researchers use censored data. There are several types of censored data (Type I and II) including right, left, interval, and truncation censoring. Saracoglu *et al.* (2012) combine type II censoring and a progressive censoring scheme called progressive type II censoring. This scheme allows the researcher to remove active units during the experiment and is defined as: Given $m < n$, and P_1, \dots, P_m non-negative integers such that $P_1 + \dots + P_m = n - m$. Where n stands for the items are on the life test at the same time. At the time of the first failure, one chooses randomly P_1 items from the rest of the active $n - 1$ and then discards. In a similar way, at time of the second failure, one selects P_2 out of $n - P_1 - 2$ remaining items at random and consequently remove it. Eventually, at the time of the m^{th} failure, all the remaining active items are removed (see Saracoglu *et al.* (2012) and references for the advantages of this censoring scheme).

4.1. Maximum penalized likelihood estimator of R for incomplete data

In this subsection, maximum penalized likelihood estimator of R is obtained for progressive type II censored data as follows:

Algorithm 3.

1. Suppose two progressive censoring schemes $(n_1, m_1, P_1, \dots, P_{m_1})$ and $(n_2, m_2, Q_1, \dots, Q_{m_2})$ for two independent random variables X and Y with two parameter negative exponential distribution $\text{NED}(\lambda_1, \mu_1)$ and $\text{NED}(\lambda_2, \mu_2)$ respectively.
2. Suppose two progressively censored samples as $X = (X_{1:m_1:n_1}, \dots, X_{m_1:m_1:n_1})$ from strength variable X and $Y = (Y_{1:m_2:n_2}, \dots, Y_{m_2:m_2:n_2})$ from stress variable Y .
3. Find the joint density probability functions of $X = (X_{1:m_1:n_1}, \dots, X_{m_1:m_1:n_1})$ and $Y = (Y_{1:m_2:n_2}, \dots, Y_{m_2:m_2:n_2})$ as:

$$f_{(X_{1:m_1:n_1}, \dots, X_{m_1:m_1:n_1})}(x_1, \dots, x_{m_1}) = c_1 \prod_{i=1}^{m_1} f(x_i)(1 - F(x_i))^{P_i},$$

$$f_{(Y_{1:m_2:n_2}, \dots, Y_{m_2:m_2:n_2})}(y_1, \dots, y_{m_2}) = c_2 \prod_{j=1}^{m_2} f(y_j)(1 - F(y_j))^{Q_j},$$

where

$$c_1 = n_1(n_1 - P_1 - 1) \cdots (n_1 - m_1 - P_1 - \cdots - P_{m_1-1} + 1),$$

$$c_2 = n_2(n_2 - Q_2 - 1) \cdots (n_2 - m_2 - Q_2 - \cdots - Q_{m_2-1} + 1)$$

are normalizing constants see Balakrishnan and Aggarwala (2000).

4. Find the penalized likelihood function

$$L^{Pr}(\Theta) = c_1 c_2 (x_{1:m_1} - \mu_1)(y_{1:m_2} - \mu_2) \prod_{i=1}^{m_1} f(x_i)(1 - F(x_i))^{P_i} \prod_{j=1}^{m_2} f(y_j)(1 - F(y_j))^{Q_j}.$$

5. Taking the logarithm of penalized likelihood function

$$l^{Pr}(\Theta) = \text{Log}c_1 + \text{Log}c_2 + \text{Log}(x_{1:m_1} - \mu_1) + \text{Log}(y_{1:m_2} - \mu_2) - m_1 \text{Log}\lambda_1 - m_2 \text{Log}\lambda_2$$

$$- \frac{1}{\lambda_1} \sum_{i=1}^{m_1} (P_i + 1)(x_i - \mu_1) - \frac{1}{\lambda_2} \sum_{j=1}^{m_2} (Q_j + 1)(y_j - \mu_2).$$

6. Find maximum penalized likelihood estimators of $\Theta = (\lambda_1, \lambda_2, \mu_1, \mu_2)$ as

$$\lambda_1^{Pr} = \frac{1}{m_1} \left[\sum_{i=1}^{m_1} (P_i + 1)x_i - x_{1:m_1} \sum_{i=1}^{m_1} (P_i + 1) \right],$$

$$\lambda_2^{Pr} = \frac{1}{m_2} \left[\sum_{j=1}^{m_2} (Q_j + 1)y_j - y_{2:m_2} \sum_{j=1}^{m_2} (Q_j + 1) \right],$$

$$\mu_1^{pr} = x_{1:m_1} - \frac{\sum_{i=1}^{m_1} (P_i + 1)x_i - x_{1:m_1} \sum_{i=1}^{m_1} (P_i + 1)}{(m_1 - 1) \sum_{i=1}^{m_1} (P_i + 1)},$$

$$\mu_2^{pr} = y_{1:m_2} - \frac{\sum_{j=1}^{m_2} (Q_j + 1)y_j - y_{1:m_2} \sum_{j=1}^{m_2} (Q_j + 1)}{(m_2 - 1) \sum_{j=1}^{m_2} (Q_j + 1)}.$$

7. Find maximum penalized likelihood estimator of R says R^{pr} using the invariance property as:

- Case 1: $\mu_1 = \mu_2$, then

$$R^{pr} = \frac{\lambda_1^{pr}}{\lambda_1^{pr} + \lambda_2^{pr}},$$

- Case 2: $\mu_1 \neq \mu_2$, then

$$R^{pr} = \left[1 - \frac{\lambda_2^{pr}}{\lambda_1^{pr} + \lambda_2^{pr}} e^{\frac{\mu_2^{pr} - \mu_1^{pr}}{\lambda_2^{pr}}} \right] I(\mu_1^{pr} > \mu_2^{pr}) + \frac{\lambda_1^{pr}}{\lambda_1^{pr} + \lambda_2^{pr}} e^{\frac{(\mu_1^{pr} - \mu_2^{pr})}{\lambda_2^{pr}}} I(\mu_1^{pr} < \mu_2^{pr}).$$

Now to find the asymptotic $100(1 - \alpha)\%$ confidence interval for R is given by

$$R \in \left(R^{pr} \pm \frac{Z_{\alpha/2}}{\sqrt{n+m}} \sqrt{\widehat{\text{Var}}(R^{pr})} \right), \tag{4.1}$$

where $Z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ quantile of standard normal distribution and $\widehat{\text{Var}}(R^{pr})$ is the value of $\text{Var}(R^{pr})$ at the penalized maximum likelihood estimators of parameters $\Theta = (\lambda_1, \mu_1, \lambda_2, \mu_2)$, where

$$\text{Var}(R^{pr}) = \sum_{j=1}^4 \sum_{i=1}^4 \frac{\partial R}{\partial \Theta_i} \frac{\partial R}{\partial \Theta_j} (I_{ij}^{-1})^{pr},$$

where $\Theta = (\lambda_1, \mu_1, \lambda_2, \mu_2)$ and I_{ij}^{-1} is the $(i, j)^{th}$ element of the inverse of the Fisher information matrix I_{ij} given by

$$I_{ij}^{pr} = \begin{pmatrix} E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \lambda_1^2} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \lambda_1 \partial \mu_1} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \lambda_1 \partial \lambda_2} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \lambda_1 \partial \mu_2} \right] \\ E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \mu_1 \partial \lambda_1} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \mu_1^2} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \mu_1 \partial \lambda_2} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \mu_1 \partial \mu_2} \right] \\ E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \lambda_2 \partial \lambda_1} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \lambda_2 \partial \mu_1} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \lambda_2^2} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \lambda_2 \partial \mu_2} \right] \\ E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \mu_2 \partial \lambda_1} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \mu_2 \partial \mu_1} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \mu_2 \partial \lambda_2} \right] & E \left[\frac{-\partial^2 l^{pr}(\Theta)}{\partial \mu_2^2} \right] \end{pmatrix},$$

$$(I_{ij}^{-1})^{pr} = \begin{pmatrix} \text{Var}(\lambda_1^{pr}) & \text{Cov}(\lambda_1^{pr}, \mu_1^{pr}) & \text{Cov}(\lambda_1^{pr}, \lambda_2^{pr}) & \text{Cov}(\lambda_1^{pr}, \mu_2^{pr}) \\ \text{Cov}(\mu_1^{pr}, \lambda_1^{pr}) & \text{Var}(\mu_1^{pr}) & \text{Cov}(\mu_1^{pr}, \lambda_2^{pr}) & \text{Cov}(\mu_1^{pr}, \mu_2^{pr}) \\ \text{Cov}(\lambda_2^{pr}, \lambda_1^{pr}) & \text{Cov}(\lambda_2^{pr}, \mu_1^{pr}) & \text{Var}(\lambda_2^{pr}) & \text{Cov}(\lambda_2^{pr}, \mu_2^{pr}) \\ \text{Cov}(\mu_2^{pr}, \lambda_1^{pr}) & \text{Cov}(\mu_2^{pr}, \mu_1^{pr}) & \text{Cov}(\mu_2^{pr}, \lambda_2^{pr}) & \text{Var}(\mu_2^{pr}) \end{pmatrix},$$

and

$$\begin{aligned}\text{Var}(\lambda_1^{pr}) &= \frac{2(m_1 - 1) \sum_{i=1}^{m_1} (1 + P_i) - m_1^2}{\lambda_1^2 m_1^2}, & \text{Var}(\lambda_2^{pr}) &= \frac{2(m_2 - 1) \sum_{j=1}^{m_2} (1 + Q_j) - m_2^2}{\lambda_2^2 m_2^2}, \\ \text{Var}(\mu_1^{pr}) &= \frac{\lambda_1^2}{n(n-1)}, & \text{Var}(\mu_2^{pr}) &= \frac{\lambda_2^2}{m(m-1)}, \\ \text{Cov}(\lambda_1^{pr}, \mu_1^{pr}) &= \text{Cov}(\mu_1^{pr}, \lambda_1^{pr}) = \frac{\lambda_1^2}{\sum_{i=1}^{m_1} (1 + P_i)}, & \text{Cov}(\lambda_1^{pr}, \lambda_2^{pr}) &= \text{Cov}(\lambda_2^{pr}, \lambda_1^{pr}) = 0, \\ \text{Cov}(\lambda_1^{pr}, \mu_2^{pr}) &= \text{Cov}(\mu_2^{pr}, \lambda_1^{pr}) = 0, & \text{Cov}(\mu_1^{pr}, \lambda_2^{pr}) &= \text{Cov}(\lambda_2^{pr}, \mu_1^{pr}) = 0, \\ \text{Cov}(\mu_1^{pr}, \mu_2^{pr}) &= \text{Cov}(\mu_2^{pr}, \mu_1^{pr}) = 0, & \text{Cov}(\lambda_2^{pr}, \mu_2^{pr}) &= \text{Cov}(\mu_2^{pr}, \lambda_2^{pr}) = \frac{\lambda_2^2}{\sum_{j=1}^{m_2} (1 + Q_j)}.\end{aligned}$$

4.2. Bayes estimation and credible interval R for incomplete data

To find the Bayes estimator of R as:

Assume all parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$ are random variables having gamma prior distributions with parameters (a_i, b_i) , $i = 1, 2, 3, 4$ respectively. Where

$$\begin{aligned}\pi(\lambda_1) &= \frac{b_1^{a_1}}{\Gamma(a_1)} \lambda_1^{a_1-1} e^{-\lambda_1 b_1}, & \lambda_1, a_1, b_1 &> 0, \\ \pi(\lambda_2) &= \frac{b_2^{a_2}}{\Gamma(a_2)} \lambda_2^{a_2-1} e^{-\lambda_2 b_2}, & \lambda_2, a_2, b_2 &> 0, \\ \pi(\mu_1) &= \frac{b_3^{a_3}}{\Gamma(a_3)} \mu_1^{a_3-1} e^{-\mu_1 b_3}, & \mu_1, a_3, b_3 &> 0, \\ \pi(\mu_2) &= \frac{b_4^{a_4}}{\Gamma(a_4)} \mu_2^{a_4-1} e^{-\mu_2 b_4}, & \mu_2, a_4, b_4 &> 0.\end{aligned}$$

Then the joint posterior density function of $\lambda_1, \lambda_2, \mu_1, \mu_2$ is given by

$$L(\lambda_1, \lambda_2, \mu_1, \mu_2 | \text{data}) = \frac{L(\text{data} | \lambda_1, \lambda_2, \mu_1, \mu_2) \pi(\lambda_1) \pi(\lambda_2) \pi(\mu_1) \pi(\mu_2)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty L(\text{data} | \lambda_1, \lambda_2, \mu_1, \mu_2) \pi(\lambda_1) \pi(\lambda_2) \pi(\mu_1) \pi(\mu_2) d\lambda_1 d\lambda_2 d\mu_1 d\mu_2}.$$

But the joint posterior density function can not compute analytically. So we use MCMC technique to compute Bayes estimator of R and construct the corresponding credible interval.

Now compute the Bayes estimator using MCMC technique when $\mu_1 = \mu_2$ and under the assumption that the parameters $\lambda_1, \lambda_2, \mu$ have independent inverse gamma (IG) priors with the probability density functions as:

$$\begin{aligned}\pi_1(\lambda_1 | a_1, b_1) &= \frac{b_1^{a_1}}{\Gamma(a_1)} \lambda_1^{-a_1-1} e^{-\frac{b_1}{\lambda_1}}, & \lambda_1 &> 0, \\ \pi_2(\lambda_2 | a_2, b_2) &= \frac{b_2^{a_2}}{\Gamma(a_2)} \lambda_2^{-a_2-1} e^{-\frac{b_2}{\lambda_2}}, & \lambda_2 &> 0, \\ \pi_3(\mu | a_3, b_3) &= \frac{b_3^{a_3}}{\Gamma(a_3)} \mu^{-a_3-1} e^{-\frac{b_3}{\mu}}, & \mu &> 0,\end{aligned}$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are chosen to reflect prior knowledge about $\lambda_1, \lambda_2, \mu$. The posterior probability density functions of λ_1, λ_2 are

$$\lambda_1 | \mu, \lambda_2, \text{data} \sim \text{IG} \left(m_1 + a_1, b_1 + \sum_{i=1}^{m_1} (P_i + 1)(x_i - \mu) \right),$$

$$\lambda_2 | \mu, \lambda_1, \text{data} \sim \text{IG} \left(m_2 + a_2, b_2 + \sum_{j=1}^{m_2} (Q_j + 1)(y_j - \mu) \right)$$

and

$$f(\mu | \lambda_1, \lambda_2, \text{data}) \propto (x_{1:m_1} - \mu)(y_{1:m_2} - \mu) \left(\frac{1}{\mu} \right)^{a_3 - 1} \exp \left[-\frac{b_3}{\mu} - \frac{1}{\lambda_1} \sum_{i=1}^{m_1} (P_i + 1)(x_i - \mu) - \frac{1}{\lambda_2} \sum_{j=1}^{m_2} (Q_j + 1)(y_j - \mu) \right].$$

The posterior probability density function of μ is unknown, but we assume it is a normal distribution and use Metropolis-Hastings algorithm to find the Bayes estimator as:

Algorithm 4.

1. Start with an initial guess $\lambda_1^{(0)}, \lambda_2^{(0)}, \mu^{(0)}$.
2. Set $t = 1$.
3. Using Metropolis-Hastings, generate $\mu^{(t)}$ from $f(\mu | \lambda_1, \lambda_2, \text{data})$ using Metropolis-Hastings algorithm with the proposal distribution $N(\mu^{t-1}, 1)$.
4. Generate $\lambda_1^{(t)}$ from $\text{IG}(m_1 + a_1, b_1 + \sum_{i=1}^{m_1} (P_i + 1)(x_i - \mu^{(t-1)}))$.
5. Generate $\lambda_2^{(t)}$ from $\text{IG}(m_2 + a_2, b_2 + \sum_{j=1}^{m_2} (Q_j + 1)(y_j - \mu^{(t-1)}))$.
6. Compute $R^{(t)}$.
7. Set $t = t + 1$.
8. Repeat steps 3–7, T times.

The Bayes estimator of R says R_{pr}^B compute under square error loss function is given by

$$R_{pr}^B = \frac{1}{T - M} \sum_{t=M+1}^{T-M} R^t,$$

where M is burn-in period. Also the highest probability density (HPD) $100(1 - \gamma)\%$ credible interval is obtained by the method of Chen and Shao (1999).

5. Simulation study

In this section a simulation study is performed to compare the performance of estimates of R obtained by using different methods; all computations are obtained using Mathematica 10.

Table 1: Results of simulation study for complete data

| (n, m) | $(\lambda_1^*, \lambda_2^*, \mu_1^*, \mu_2^*)$ | Maximum penalized likelihood estimator | | | Bayes estimator | | | ACI |
|---|--|--|---------|--------|-----------------|---------|--------|------------------|
| | | \hat{R}^* | Bias | MSE | R^B | Bias | MSE | |
| $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1, 1, 1.5, 1)$ and $R = 0.6967, \mu_1 > \mu_2$ | | | | | | | | |
| (30, 30) | (0.0738, 0.0520, 2.4090, 1.9489) | 0.9999 | 0.0050 | 0.0015 | 0.9997 | -0.0001 | 0.0000 | (0.6200, 1.0000) |
| (30, 50) | (0.0597, 0.0677, 2.4450, 1.9320) | 0.9997 | 0.0037 | 0.0011 | 0.9920 | -0.0004 | 0.0000 | (0.6101, 0.9999) |
| (30, 100) | (0.4850, 0.0772, 2.4505, 1.9222) | 0.9993 | 0.0023 | 0.0007 | 0.9989 | 0.0003 | 0.0000 | (0.6011, 0.9995) |
| (50, 50) | (0.1026, 0.0653, 2.4016, 1.9354) | 0.9996 | 0.0030 | 0.0009 | 0.9991 | -0.0004 | 0.0000 | (0.6412, 1.0000) |
| (50, 100) | (0.0567, 0.0682, 2.4444, 1.9356) | 0.9996 | 0.0020 | 0.0006 | 0.9993 | -0.0002 | 0.0000 | (0.6454, 1.0000) |
| (100, 100) | (0.0689, 0.0677, 2.4232, 1.9323) | 0.9996 | 0.0015 | 0.0004 | 0.9993 | -0.0002 | 0.0000 | (0.6600, 1.0000) |
| $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1.5, 1, 1.5, 1.5)$ and $R = 0.6, \mu_1 = \mu_2$ | | | | | | | | |
| (30, 30) | (0.1064, 0.0659, 2.9110, 2.4360) | 0.6192 | 0.0003 | 0.0000 | 0.6182 | -0.0006 | 0.0000 | (0.5999, 0.6400) |
| (30, 50) | (0.0922, 0.0764, 2.9090, 2.4234) | 0.5468 | -0.0006 | 0.0000 | 0.5457 | -0.0001 | 0.0000 | (0.5124, 0.7000) |
| (30, 100) | (0.1572, 0.0612, 2.8360, 2.4398) | 0.7197 | 0.0009 | 0.0001 | 0.7195 | -0.0001 | 0.0000 | (0.5300, 0.7500) |
| (50, 50) | (0.1417, 0.0751, 2.8666, 2.4295) | 0.6536 | 0.0005 | 0.0000 | 0.6524 | -0.0011 | 0.0000 | (0.5504, 0.7700) |
| (50, 100) | (0.1039, 0.0565, 2.8953, 2.4490) | 0.6474 | 0.0003 | 0.0000 | 0.6473 | 0.0003 | 0.0000 | (0.5965, 0.7584) |
| (100, 100) | (0.0885, 0.1064, 2.9112, 2.3955) | 0.4539 | -0.0007 | 0.0001 | 0.4519 | -0.0007 | 0.0000 | (0.4210, 0.7145) |
| $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1, 1.5, 1.5, 2)$ and $R = 0.2420, \mu_1 < \mu_2$ | | | | | | | | |
| (100, 100) | (0.0838, 0.1059, 2.4260, 3.3815) | 0.0001 | 0.2419 | 0.4419 | 0.0016 | 0.1622 | 0.4386 | (0.0001, 0.2500) |

MSE = mean square error; ACI = asymptotic confidence interval.

5.1. Simulation study for complete data

1. For given values of $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ compute the true value of R from (2.1) or (2.2).
2. For given (n, m) generate sample from size n from (1.1) with given parameters (λ_1, μ_1) and generate sample from size m from (1.1) with given parameters (λ_2, μ_2) .
3. Evaluate maximum penalized likelihood estimators of $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ and R using Algorithm 1.
4. Repeat step 2–3 $N = 10^4$ times and calculate the biases and MSEs.
5. Calculate the asymptotic confidence interval (ACI) of R according to (3.1).
6. Using 2–4 to compute R^B using Algorithm 2.
7. Calculate the biases and MSEs for the Bayes estimator.

Table 1 shows the simulation study results for the complete data. We observe that for large samples size the estimator values of \hat{R}^* and R^B became constant, when $\mu_1 = \mu_2$ the estimator values of R nearest of its true vales. R has bad estimator when $\mu_1 < \mu_2$, also biases and MSE's decrease when sample sizes increase.

5.2. Simulation study for incomplete data

In this subsection we compare the maximum penalized likelihood estimator and Bayes estimator using the MCMC technique of R in bias, MSE as well as given the ACI and HPD credible intervals for progressive type II censored data as:

1. For different parameter values $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1, 1, 1.5, 0.5), (1, 1, 1.5, 1.5),$ and $(1, 1, 2.5, 2.5)$.

Table 2: Results of simulation study for incomplete data (biases and MSE's) of maximum penalized likelihood estimator and Bayes estimator of R

| CS (P, Q) | Maximum penalized likelihood estimator | | | Bayes estimator | | | | | | | | |
|---|--|---------|--------|-----------------|---------|--------|------------|---------|--------|------------|---------|--------|
| | | | | Prior 1 | | | Prior 2 | | | Prior 3 | | |
| | R_{pr}^B | Bias | MSE | R_{pr}^B | Bias | MSE | R_{pr}^B | Bias | MSE | R_{pr}^B | Bias | MSE |
| $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1, 1, 0.5, 0.5)$ | | | | | | | | | | | | |
| (I, I) | 0.5520 | 0.0026 | 0.0001 | 0.5001 | -0.0001 | 0.0000 | 0.5108 | -0.0108 | 0.0001 | 0.5027 | -0.0027 | 0.0000 |
| (I, II) | 0.3480 | -0.0075 | 0.0011 | 0.4874 | 0.0126 | 0.0001 | 0.4840 | 0.0160 | 0.0002 | 0.5149 | -0.0149 | 0.0002 |
| (I, III) | 0.5338 | 0.0016 | 0.0000 | 0.4951 | 0.0049 | 0.0000 | 0.5058 | -0.0058 | 0.0000 | 0.4988 | 0.0012 | 0.0000 |
| (II, II) | 0.3125 | -0.0093 | 0.0017 | 0.4977 | 0.0023 | 0.0000 | 0.4922 | 0.0078 | 0.0000 | 0.4872 | 0.0128 | 0.0001 |
| (II, III) | 0.5245 | 0.0012 | 0.0000 | 0.4942 | 0.0058 | 0.0000 | 0.5135 | -0.0135 | 0.0001 | 0.4816 | 0.0184 | 0.0003 |
| (III, III) | 0.4017 | -0.0049 | 0.0004 | 0.5132 | -0.0132 | 0.0001 | 0.4822 | 0.0178 | 0.0003 | 0.4992 | 0.0008 | 0.0000 |
| $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1, 1, 1.5, 1.5)$ | | | | | | | | | | | | |
| (I, I) | 0.5753 | 0.0037 | 0.0002 | 0.4790 | 0.0209 | 0.0004 | 0.5064 | -0.0064 | 0.0000 | 0.5023 | -0.0023 | 0.0000 |
| (I, II) | 0.7423 | 0.0121 | 0.0029 | 0.4919 | 0.0082 | 0.0000 | 0.4788 | 0.0211 | 0.0004 | 0.4917 | 0.0082 | 0.0006 |
| (I, III) | 0.5731 | 0.0036 | 0.0029 | 0.5065 | -0.0065 | 0.0000 | 0.5280 | -0.0280 | 0.0007 | 0.4847 | 0.0152 | 0.0002 |
| (II, II) | 0.3524 | -0.0073 | 0.0010 | 0.5108 | -0.0106 | 0.0001 | 0.5013 | -0.0013 | 0.0000 | 0.4968 | 0.0031 | 0.0000 |
| (II, III) | 0.3085 | -0.0057 | 0.0006 | 0.4850 | 0.0144 | 0.0003 | 0.4904 | 0.0095 | 0.0000 | 0.5106 | 0.0106 | 0.0001 |
| (III, III) | 0.3461 | -0.0076 | 0.0011 | 0.5185 | -0.0185 | 0.0003 | 0.5220 | -0.0220 | 0.0004 | 0.5152 | -0.0192 | 0.0002 |
| $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1, 1, 2.5, 2.5)$ | | | | | | | | | | | | |
| (I, I) | 0.6344 | 0.0067 | 0.0009 | 0.4820 | 0.0179 | 0.0003 | 0.4780 | 0.0219 | 0.0004 | 0.4958 | 0.0041 | 0.0000 |
| (I, II) | 0.5930 | 0.0046 | 0.0004 | 0.5116 | -0.0116 | 0.0001 | 0.4886 | 0.0113 | 0.0001 | 0.5162 | -0.0162 | 0.0002 |
| (I, III) | 0.5433 | 0.0021 | 0.0000 | 0.4999 | 0.0001 | 0.0000 | 0.5046 | -0.0046 | 0.0000 | 0.4990 | 0.0609 | 0.0000 |
| (II, II) | 0.3009 | -0.0099 | 0.0019 | 0.5094 | -0.0094 | 0.0000 | 0.5023 | -0.0023 | 0.0000 | 0.4933 | 0.0066 | 0.0000 |
| (II, III) | 0.4359 | -0.0032 | 0.0002 | 0.4696 | 0.0303 | 0.0009 | 0.5295 | -0.0295 | 0.0008 | 0.4812 | 0.0187 | 0.0003 |
| (III, III) | 0.5191 | 0.0009 | 0.0000 | 0.5089 | -0.0089 | 0.0000 | 0.5016 | -0.0016 | 0.0000 | 0.4939 | 0.0060 | 0.0000 |

MSE = mean square error; CS = censoring schemes.

Also use censoring schemes (CS) as

$$I = (10, 30) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 20),$$

$$II = (10, 30) = (20, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$

$$III = (10, 30) = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2).$$

2. Find R_{pr}^B using Algorithm 3.
3. Compute bias and MSE.
4. Compute ACI of (4.1).
5. Find the R_{pr}^B and compute HPD credible interval using Algorithm 4 and use censoring schemes in 1 and the following priors

Prior 1. $a_i = 0, b_i = 0, \quad i = 1, 2, 3.$

Prior 2. $a_i = 1, b_i = 2, \quad i = 1, 2, 3.$

Prior 3. $a_i = 2, b_i = 3, \quad i = 1, 2, 3.$

Tables 2 and 3 show results of simulation study of R for incomplete data.

6. Data analysis

In this section, we present a real example to sure our proposed model can apply in practice. Badar and Priest (1982) reported the data represent the strength measured in GPA for single carbon fibers, and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 20

Table 3: Comparison between ACI and HPD credible interval of R

| CS | ACI | HPD credible interval of R | | |
|---|------------------|------------------------------|------------------|------------------|
| | | Prior 1 | Prior2 | Prior 3 |
| $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1, 1, 0.5, 0.5)$ | | | | |
| (I, I) | (0.4832, 0.6207) | (0.4510, 0.5491) | (0.4607, 0.5608) | (0.4534, 0.5519) |
| (I, II) | (0.3978, 0.6021) | (0.4397, 0.5352) | (0.4366, 0.5314) | (0.4644, 0.5638) |
| (I, III) | (0.3490, 0.6509) | (0.4466, 0.5436) | (0.4562, 0.5554) | (0.4499, 0.5476) |
| (II, II) | (0.3002, 0.6997) | (0.4490, 0.5465) | (0.4440, 0.5405) | (0.4395, 0.5350) |
| (II, III) | (0.4436, 0.5563) | (0.4458, 0.5426) | (0.4631, 0.5638) | (0.4344, 0.5288) |
| (III, III) | (0.3803, 0.6196) | (0.4629, 0.5635) | (0.4349, 0.5294) | (0.4503, 0.5481) |
| $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1, 1, 1.5, 1.5)$ | | | | |
| (I, I) | (0.3894, 0.6105) | (0.4320, 0.5259) | (0.4568, 0.5561) | (0.4531, 0.5515) |
| (I, II) | (0.3839, 0.6160) | (0.4435, 0.5398) | (0.4319, 0.5278) | (0.4436, 0.5399) |
| (I, III) | (0.3899, 0.6105) | (0.4568, 0.5561) | (0.4762, 0.5797) | (0.4372, 0.5322) |
| (II, II) | (0.3685, 0.6314) | (0.4608, 0.5609) | (0.4522, 0.5504) | (0.4481, 0.5455) |
| (II, III) | (0.3606, 0.6393) | (0.4850, 0.5326) | (0.4424, 0.5385) | (0.4606, 0.5607) |
| (III, III) | (0.3997, 0.6002) | (0.4677, 0.5193) | (0.4709, 0.5732) | (0.4647, 0.5657) |
| $(\lambda_1, \lambda_2, \mu_1, \mu_2) = (1, 1, 2.5, 2.5)$ | | | | |
| (I, I) | (0.4190, 0.5809) | (0.4348, 0.5293) | (0.4311, 0.5248) | (0.4472, 0.5443) |
| (I, II) | (0.2338, 0.7661) | (0.4615, 0.5618) | (0.4407, 0.5365) | (0.4656, 0.5666) |
| (I, III) | (0.4190, 0.5809) | (0.4509, 0.5489) | (0.4551, 0.5541) | (0.4501, 0.5479) |
| (II, II) | (0.3724, 0.6275) | (0.4595, 0.5593) | (0.4531, 0.5515) | (0.4449, 0.5416) |
| (II, III) | (0.4389, 0.5610) | (0.4236, 0.5156) | (0.4776, 0.5814) | (0.4340, 0.5283) |
| (III, III) | (0.4501, 0.5498) | (0.4590, 0.5588) | (0.4524, 0.5507) | (0.4455, 0.5423) |

CS = censoring schemes; ACI = asymptotic confidence interval; HPD = highest probability density.

mm (data set 1) and 10 mm (data set 2). These data have been used previously by Raqab and Kundu (2005), Kundu and Gupta (2006), Kundu and Raqab (2009), and Asgharzadeh *et al.* (2011).

- Data set I: (gauge lengths of 20 mm).

1.312 1.314 1.479 1.552 1.700 1.803 1.861 1.865 1.944 1.958 1.966 1.997 2.006 2.021 2.027 2.055
 2.063 2.098 2.140 2.179 2.224 2.240 2.253 2.270 2.272 2.274 2.301 2.301 2.359 2.382 2.382 2.426
 2.434 2.435 2.478 2.490 2.511 2.514 2.535 2.554 2.566 2.570 2.586 2.629 2.633 2.642 2.648 2.684
 2.697 2.726 2.770 2.773 2.800 2.809 2.818 2.821 2.848 2.880 2.954 3.012 3.067 3.084 3.090 3.096
 3.128 3.233 3.433 3.585 3.585

- Data set II (gauge lengths of 10 mm).

1.901 2.132 2.203 2.228 2.257 2.350 2.361 2.396 2.397 2.445 2.454 2.474 2.518 2.522 2.525 2.532
 2.575 2.614 2.616 2.618 2.624 2.659 2.675 2.738 2.740 2.856 2.917 2.928 2.937 2.937 2.977 2.996
 3.030 3.125 3.139 3.145 3.220 3.223 3.235 3.243 3.264 3.272 3.294 3.332 3.346 3.377 3.408 3.435
 3.493 3.501 3.537 3.554 3.562 3.628 3.852 3.871 3.886 3.971 4.024 4.027 4.225 4.395 5.020

We apply our model as:

1. Check the validity of two parameter negative exponential for given data sets see Figure 2.
2. Compute maximum penalized likelihood estimator and Bayes estimator of R for complete data. Table 4 shows the results.
3. Compute the maximum penalized likelihood and Bayes estimator for incomplete data and corresponding confidence interval using two different progressively censored samples schemes (Table 5) and data analysis results for incomplete data (Table 6).

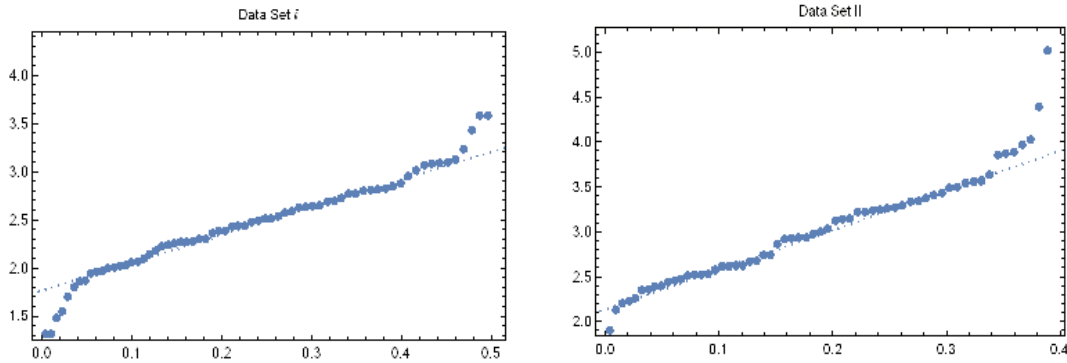


Figure 2: Fitted two parameter negative exponential distribution for data set I and data set II.

Table 4: Data analysis results for complete data

| | | | |
|--|-------------|--------|------------------|
| $(\lambda_1^*, \lambda_2^*, \mu_1^*, \mu_2^*)$ | \hat{R}^* | R^B | ACI |
| (1.3707, 1.214, 1.8792, 1.2944) | 0.70986 | 0.6807 | (0.6985, 0.7098) |

ACI = asymptotic confidence interval.

Table 5: Data and the corresponding censored schemes

| | | | | | | | | | | |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| x_i | 1.312 | 1.479 | 1.552 | 1.803 | 1.944 | 1.858 | 1.966 | 2.027 | 2.055 | 2.098 |
| P_i | 1 | 0 | 1 | 2 | 0 | 0 | 3 | 0 | 1 | 50 |
| y_j | 1.901 | 2.132 | 2.257 | 2.361 | 2.396 | 2.445 | 2.373 | 2.523 | 2.532 | 2.570 |
| Q_j | 0 | 2 | 1 | 0 | 1 | 1 | 2 | 0 | 0 | 44 |

Table 6: Data analysis results for incomplete data

| | | |
|--|------------------|---------------------|
| $(\lambda_1^{pp}, \lambda_2^{pp}, \mu_1^{pp}, \mu_2^{pp}) = (2.8849, 2.2840, 1.257, 1.2576)$ | | |
| Method | Estimator of R | Confidence interval |
| Maximum penalized likelihood | 0.5580 | (0.5000, 0.5800) |
| Bayes estimator (prior 1) | 0.5313 | (0.4793, 0.5834) |
| Bayes estimator (prior 2) | 0.4756 | (0.4289, 0.5222) |
| Bayes estimator (prior 3) | 0.5138 | (0.4634, 0.5641) |

7. Conclusions

In this paper, we apply a maximum penalized likelihood method to estimate a stress-strength parameter based on two parameter negative exponential distribution and compare it with the regular maximum likelihood method. Subsequently, it was decided the maximum penalized likelihood method is better than the regular maximum likelihood method because it has a smaller variance. We apply the maximum penalized likelihood method for complete and incomplete data. We also introduce a Bayes estimator using Lindley approximation for complete data and MCMC technique for incomplete data. A simulation study is performed to compare between different estimators in biases and MSEs for complete and incomplete data. The Bayes estimate of R , and the corresponding credible interval can be obtained using the Gibbs sampling technique. In addition, we apply our model on real data and recommend the use of two parameter negative exponential distribution as a reliability model for complete and incomplete data.

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