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THE PROPERTIES OF RIEMANNIAN FOLIATIONS ADMITTING TRANSVERSAL CONFORMAL FIELDS

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Abstract. Let $(M, \mathcal{F})$ be a closed, oriented Riemannian manifold of a foliation $\mathcal{F}$ with a nonisometric transversal conformal field. Then $(M, \mathcal{F})$ is transversally isometric to the sphere under some transversal concircular curvature conditions.

1. Introduction

Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Then Lee and Richardson [8] generalized the Lichnerowicz inequality, which states that if the transversal Ricci curvature Ric^Q satisfies that $\mathrm{Ric}^Q(X) \geq (q-1)c^2 X$ for some c and for every normal vector field X, then the smallest nonzero eigenvalue λ_B of the basic Laplacian satisfies $\lambda_B \geq qc^2$. In addition, the equality holds if and only if $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where G is the discrete subgroup of $O(q)$ acting by isometries on the last q coordinates of the standard q-sphere $S^q(1/c)$ of radius $\frac{1}{c}$ in Euclidean space $\mathbb{R}^{q+1}$. A Riemannian foliation $(M, \mathcal{F})$ is *transversally isometric* to (W, G), where G is a discrete group acting by isometries on a Riemannian manifold (W, g_W), if there exists a homeomorphism $\eta : W/G \to M/\mathcal{F}$ that is locally covered by isometries [8]. In particular, if $\mathcal{F}$ is transversally Einsteinian with constant transversal scalar curvature σ^Q and the basic mean curvature form is coclosed, then the following conditions are equivalent to each other:

$(F1)$ $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where G is a discrete subgroup of $O(q)$.

$(F2)$ M admits a transversal nonisometric conformal field.

$(F3)$ M admits a non constant basic function f such that $\Delta_B f = qc^2 f$.

$(F4)$ M admits a non constant basic function f such that $\nabla_X \nabla f = -c^2 f X$ for any normal vector field X.

Precisely, see [8] for $(F1) \Leftrightarrow (F3)$, [4] for $(F1) \Leftrightarrow (F4)$, [2] for $(F2) \Rightarrow (F1)$ and [9] for $(F4) \Rightarrow (F2)$.

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Actually, the equivalence between $(F1)$ and $(F4)$ is called the *generalized Obata theorem* for foliations [4]. By using the generalized Obata theorem, S. D. Jung [3] characterized the Riemannian foliation admitting the transversal conformal fields under some conditions of the tensor E^Q (see Section 4 for details), which is defined by

$$E^Q(s) = \mathrm{Ric}^Q(s) - \frac{\sigma^Q}{q}s \tag{1.1}$$

for any vector $s \in Q$, where $Q = TM/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$. Note that if E^Q vanishes, then $\mathcal{F}$ is transversally Einsteinian. In fact,

Theorem 1.1 ([3]). *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a minimal foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M. Assume that the transversal scalar curvature $\sigma^Q(\neq 0)$ is constant. If M admits a transversal nonisometric conformal field $\bar{Y}$ such that*

$$\theta(Y)|E^Q|^2 = 0, \tag{1.2}$$

then $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^Q}{q(q-1)}$.

Remark. Theorem 1.1 yields $(F2) \Rightarrow (F1)$ when $\mathcal{F}$ is transversally Einsteinian (cf. [2]).

Now we define the transversal concircular curvature tensor Z^Q by

$$Z^Q(X, Y) = R^Q(X, Y) - R^Q_\sigma(X, Y) \tag{1.3}$$

for any $X, Y \in T\mathcal{F}^\perp$, where R^Q is the transversal curvature tensor and

$$R^Q_\sigma(X, Y)s = \frac{\sigma^Q}{q(q-1)}\{g_Q(\pi(Y), s)\pi(X) - g_Q(\pi(X), s)\pi(Y)\}$$

for any $X, Y \in TM$ and $s \in Q$. Here $\pi : TM \to Q$ is a natural projection and g_Q is a holonomy invariant metric on Q. Trivially, if $Z^Q = 0$, then $\mathcal{F}$ is a foliation of transversally constant sectional curvature. In a Riemannian geometry, the concircular curvature tensor is invariant under a concirclar transformation [14]. A concircular transformation is a conformal transformation preserving geodesic circles. We observe immediately that Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor estimates a measure of the failure of a Riemannian manifold to be of constant curvature.

In this paper, we study the Riemannian foliations admitting the transversal nonisometric conformal fields under some conditions of the transversal concircular curvature tensor Z^Q. Namely,

Theorem 1.2 (Cf. Corollary 4.6). *Let $(M, g_M, \mathcal{F})$ be as in Theorem 1.1. Assume that the transversal scalar curvature $\sigma^Q(\neq 0)$ is constant. If M admits a transversal nonisometric conformal field $\bar{Y} = \pi(Y)$ such that*

$$\theta(Y)|Z^Q|^2 = 0 \ (\textit{or } \theta(Y)|R^Q|^2 = 0),$$

then $(M,\mathcal{F})$ *is transversally isometric to* $(S^q(1/c),G)$, *where* $c^2=\frac{\sigma^Q}{q(q-1)}$.

When $\mathcal{F}$ is a point foliation, Theorem 1.2 was found in [16] by K. Yano for an ordinary manifold.

2. The basic facts on Riemannian foliations

Let $(M,g_M,\mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$ [16]. Let TM be the tangent bundle of M, $T\mathcal{F}$ its integrable subbundle given by $\mathcal{F}$, and $Q=TM/T\mathcal{F}$ the corresponding normal bundle. Then there exists an exact sequence of vector bundles

$$0\longrightarrow T\mathcal{F}\longrightarrow TM\underset{\sigma}{\overset{\pi}{\rightleftarrows}}Q\longrightarrow 0,$$

where $\sigma:Q\to T\mathcal{F}^\perp$ is a bundle map satisfying $\pi\circ\sigma=\mathrm{id}$. Let g_Q be the holonomy invariant metric on Q induced by $g_M=g_{T\mathcal{F}}+g_{T\mathcal{F}^\perp}$. This means that $\theta(X)g_Q=0$ for any $X\in T\mathcal{F}$, where $\theta(X)$ is the transversal Lie derivative, which is defined by $\theta(X)s=\pi[X,\sigma(s)]$ for any $s\in\Gamma Q$. Let ∇ be the transverse Levi-Civita connection in Q, which is defined [5] by

$$\nabla_X s=\begin{cases}\pi([X,\sigma(s)]) & \forall X\in T\mathcal{F},\\ \pi(\nabla^M_X\sigma(s)) & \forall X\in T\mathcal{F}^\perp\end{cases}$$

for any $s\in\Gamma Q$, where ∇^M is the Levi-Civita connection of g_M. The transversal curvature tensor R^Q of ∇ is defined by $R^Q(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ for any vector fields $X,Y\in\Gamma TM$. Let Ric^Q and σ^Q be the transversal Ricci operator and the transversal scalar curvature of $\mathcal{F}$, respectively. The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if $\mathrm{Ric}^Q=\frac{1}{q}\sigma^Q\cdot\mathrm{id}$ with constant transversal scalar curvature σ^Q. The foliation $\mathcal{F}$ is said to be *minimal* if the mean curvature vector field τ vanishes. Here the mean curvature vector field τ is defined by $\tau=\sum_i\pi(\nabla^M_{E_i}E_i)$, where $\{E_i\}(i=1,\dots,p)$ is a local orthonormal frame field on $T\mathcal{F}$.

Let $V(\mathcal{F})$ be the space of all infinitesimal automorphisms Y of $(M,\mathcal{F})$, that is, $[Y,Z]\in\Gamma T\mathcal{F}$ for all $Z\in\Gamma T\mathcal{F}$ [13]. Let $\bar V(\mathcal{F})=\{\bar Y=\pi(Y)\,|\,Y\in V(\mathcal{F})\}\subset Q$. It is trivial that an element s of $\bar V(\mathcal{F})$ satisfies $\nabla_X s=0$ for all $X\in T\mathcal{F}$ [6]. For the later use, we recall the transversal divergence theorem [17] on a foliated Riemannian manifold.

Theorem 2.1 ([17]). *Let* $(M,g_M,\mathcal{F})$ *be a closed, oriented Riemannian manifold with a transversally oriented foliation* $\mathcal{F}$ *and a bundle-like metric* g_M *with respect to* $\mathcal{F}$. *Then*

$$\int_M \mathrm{div}_\nabla(s)=\int_M g_Q(s,\tau)$$

for all $s\in\Gamma Q$, *where* $\mathrm{div}_\nabla s$ *denotes the transversal divergence of* s *with respect to the connection* ∇.

A differential form $\omega \in \Omega^r(M)$ is *basic* if $i(X)\omega = 0$ and $i(X)d\omega = 0$ for all $X \in T\mathcal{F}$. Let $\Omega_B^r(\mathcal{F})$ be the set of all basic r-forms on M. Then $\Omega^*(M) = \Omega_B^*(\mathcal{F}) \oplus \Omega_B^*(\mathcal{F})^\perp$ [1] and $\Omega_B^1(\mathcal{F}) \cong \bar{V}(\mathcal{F})$. Let κ be the mean curvature form of $\mathcal{F}$, which is given by $\kappa(s) = g_Q(\tau, s)$ for any $s \in Q$. Then the basic part κ_B of the mean curvature form is closed, i.e., $d\kappa_B = 0$ [1]. Let d_B be the restriction of d on $\Omega_B(\mathcal{F})$ and δ_B its formal adjoint operator of d_B with respect to the global inner product $\ll \cdot, \cdot \gg$, which is given by

$$\ll \phi, \psi \gg = \int_M \phi \wedge \bar{*}\psi \wedge \chi_{\mathcal{F}},$$

where $\bar{*}$ is the star operator on $\Omega_B^*(\mathcal{F})$ and $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ [12]. The *basic Laplacian* Δ_B acting on $\Omega_B^*(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B.$$

Now we define the connection ∇ on $\Omega_B^*(\mathcal{F})$, which is induced from the connection ∇ on Q and Riemannian connection ∇^M of g_M. This connection ∇ extends the partial Bott connection, which satisfies $\nabla_X\omega = \theta(X)\omega$ for any $X \in T\mathcal{F}$ [7].

Lastly, we recall the generalized Obata theorem for foliations for later use.

Theorem 2.2 ([4]). *Let $(M, g_M, \mathcal{F})$ be a connected, complete Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M, and let c be a positive real number. Then the following are equivalent*:

(1) *There exists a non constant basic function f such that $\nabla_X\nabla f = -c^2 fX$ for all vectors $X \in T\mathcal{F}^\perp$.*

(2) *$(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where G is the discrete subgroup of $O(q)$ acting by isometries on the last q coordinates of the q-sphere $S^q(1/c)$ of radius $1/c$ in Euclidean space $\mathbb{R}^{q+1}$.*

3. Transversal concircular curvature tensor

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 2f_Y g_Q$ for a basic function f_Y depending on Y, then $\bar{Y}$ is said to be a *transversal conformal field* of $\mathcal{F}$ [2, 10, 11]; in this case, we have

$$f_Y = \frac{1}{q}\,\mathrm{div}_\nabla(\bar{Y}). \tag{3.1}$$

And a transversal conformal field with $f_Y = 0$ is called a *transversal Killing field.* Let $\{E_a\}$ $(a = 1, \ldots, q)$ be the local orthonormal frame on $T\mathcal{F}^\perp$.

Lemma 3.1 ([2]). *If $\bar{Y}$ is a transversal conformal field, i.e., $\theta(Y)g_Q = 2f_Y g_Q$, then*

$$g_Q((\theta(Y)R^Q)(E_a, E_b)E_c, E_d) = \delta_b^d\nabla_a f_c - \delta_b^c\nabla_a f_d - \delta_a^d\nabla_b f_c + \delta_a^c\nabla_b f_d, \tag{3.2}$$

$$\theta(Y)\sigma^Q = 2(q-1)(\Delta_B f_Y - \kappa_B^\sharp(f_Y)) - 2f_Y\sigma^Q, \tag{3.3}$$

where $\nabla_a = \nabla_{E_a}$ and $f_a = \nabla_a f_Y$.

Now, we recall two tensors E^Q and Z^Q from (1.1) and (1.3): for any $s \in \Gamma Q$,

$$E^Q(s) = \mathrm{Ric}^Q(s) - \frac{\sigma^Q}{q}s, \tag{3.4}$$

$$Z^Q(X,Y)s = R^Q(X,Y)s - R^Q_\sigma(X,Y)s \tag{3.5}$$

for any vector fields X and Y. It is well-known that for any $s \in \Gamma Q$,

$$\sum_a Z^Q(\sigma(s), E_a)E_a = E^Q(s). \tag{3.6}$$

Then the following identities hold [3];

$$\mathrm{tr}_Q E^Q = 0, \quad \mathrm{div}_\nabla(E^Q) = \frac{q-2}{2q}\nabla\sigma^Q, \quad |E^Q|^2 = |\mathrm{Ric}^Q|^2 - \frac{(\sigma^Q)^2}{q}, \tag{3.7}$$

where $\mathrm{tr}_Q E^Q = \sum_{a=1}^q g_Q(E^Q(E_a), E_a)$. If σ^∇ is constant, then $\mathrm{div}_\nabla(E^Q) = 0$. For more properties of E^Q, see [3] precisely.

Lemma 3.2. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M. Then*

$$|Z^Q|^2 = |R^Q|^2 - \frac{2(\sigma^Q)^2}{q(q-1)}.$$

Proof. From (3.5), a direct calculation gives

$$\begin{aligned}
|Z^Q|^2 &= \sum_{a,b,c} g_Q(Z^Q(E_a,E_b)E_c, Z^Q(E_a,E_b)E_c) \\
&= |R^Q|^2 - \frac{4\sigma^Q}{q(q-1)}\sum_{a,b}(g_Q(R^Q(E_a,E_b)E_b, E_a) \\
&\quad + \frac{2(\sigma^Q)^2}{q^2(q-1)^2}\sum_{a,b}(\delta^a_a\delta^b_b - \delta^b_a\delta^b_a) \\
&= |R^Q|^2 - \frac{2(\sigma^Q)^2}{q(q-1)}.
\end{aligned}$$

□

Lemma 3.3. *Let $(M, g_M, \mathcal{F})$ be as in Lemma 3.2. If $\bar{Y}$ is a transversal conformal field with $\theta(Y)g_Q = 2f_Y g_Q$, then*

$$\theta(Y)|Z^Q|^2 = -8g_Q(\nabla\nabla f_Y, E^Q) - 4f_Y|Z^Q|^2.$$

Proof. From (3.5), we have

$$\begin{aligned}
(\theta(Y)Z^Q)(E_a,E_b)E_c = (\theta(Y)R^Q)(E_a,E_b)E_c - \frac{1}{q(q-1)}(\theta(Y)\sigma^Q)(\delta^c_b E_a - \delta^c_a E_b) \\
- \frac{2f_Y\sigma^Q}{q(q-1)}(\delta^c_b E_a - \delta^c_a E_b).
\end{aligned}$$

Hence from Lemma 3.1, we have

$$g_Q((\theta(Y)Z^Q)(E_a,E_b)E_c, E_d) = \delta^d_b\nabla_a f_c - \delta^c_b\nabla_a f_d - \delta^d_a\nabla_b f_c + \delta^c_a\nabla_b f_d \tag{3.8}$$

$$-\frac{2}{q}(\Delta_B f_Y - \kappa_B^\sharp(f_Y))(\delta_a^d\delta_b^c - \delta_b^d\delta_a^c).$$

On the other hand, a direct calculation with $\theta(Y)g_Q = 2f_Y g_Q$ gives

$$\begin{aligned}
&\sum_{a,b,c} g_Q(Z^Q(\theta(Y)E_a, E_b)E_c, Z^Q(E_a,E_b)E_c)\\
=&\sum_{a,b,c,d} g_Q(\theta(Y)E_a, E_d)g_Q(Z^Q(E_d,E_b)E_c, Z^Q(E_a,E_b)E_c)\\
=&-\sum_{a,b,c,d}\{(\theta(Y)g_Q)(E_a,E_d)+g_Q(E_a,\theta(Y)E_d)\}g_Q(Z^Q(E_d,E_b)E_c, Z^Q(E_a,E_b)E_c)\\
=&-2f_Y|Z^Q|^2 - \sum_{a,b,c} g_Q(Z^Q(\theta(Y)E_a,E_b)E_c, Z^Q(E_a,E_b)E_c).
\end{aligned}$$

Hence we have

$$\sum_{a,b,c} g_Q(Z^Q(\theta(Y)E_a,E_b)E_c, Z^Q(E_a,E_b)E_c) = -f_Y|Z^Q|^2. \tag{3.9}$$

Similarly, we have

$$\sum_{a,b,c} g_Q(Z^Q(E_a,E_b)\theta(Y)E_c, Z^Q(E_a,E_b)E_c) = -f_Y|Z^Q|^2. \tag{3.10}$$

From (3.6) and $\mathrm{tr}_Q E^Q = 0$, we have

$$\begin{aligned}
&\sum_{a,b,c} g_Q((\theta(Y)Z^Q)(E_a,E_b)E_c, Z^Q(E_a,E_b)E_c)\\
=&-4\sum_{a,b,c}(\nabla_a f_c)g_Q(Z^Q(E_a,E_b)E_b,E_c)\\
&-\frac{4}{q}\Big(\Delta_B f_Y - \kappa_B^\sharp(f_Y)\Big)\sum_{a,b} g_Q(Z^Q(E_a,E_b)E_b,E_a)\\
=&-4\sum_{a,b}(\nabla_a f_b)g_Q(E^Q(E_a),E_b) - \frac{4}{q}\Big(\Delta_B f_Y - \kappa_B^\sharp(f_Y)\Big)\mathrm{tr}_Q E^Q\\
=&-4g_Q(\nabla\nabla f_Y, E^Q). \qquad (3.11)
\end{aligned}$$

Therefore, from (3.9), (3.10) and (3.11), we have

$$\begin{aligned}
\theta(Y)|Z^Q|^2 &= \sum_{a,b,c}\theta(Y)g_Q(Z^Q(E_a,E_b)E_c, Z^Q(E_a,E_b)E_c)\\
&= -8g_Q(\nabla\nabla f_Y, E^Q) - 4f_Y|Z^Q|^2,
\end{aligned}$$

which completes the proof. □

Proposition 3.4. *Let $(M, g_M, \mathcal{F})$ be a closed, oriented Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M. Assume* *that the transversal scalar curvature σ^Q is constant. If $\bar{Y}$ is a transversal conformal field with $\theta(Y)g_Q = 2f_Y g_Q$, $f_Y \neq 0$, then*

$$\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) = \frac{1}{2}\int_M \{{f_Y}^2|Z^Q|^2 + \frac{1}{4}f_Y\theta(Y)|Z^Q|^2\} + \int_M g_Q(\mathrm{Ric}^Q(f_Y\nabla f_Y), \kappa_B^\sharp).$$

Proof. By a direct calculation, we have

$$\begin{aligned} \mathrm{div}_\nabla(E^Q(f_Y\nabla f_Y)) &= g_Q(\mathrm{div}_\nabla(E^Q), f_Y\nabla f_Y) + g_Q(E^Q(\nabla f_Y), \nabla f_Y) \\ &\quad + g_Q(f_Y E^Q, \nabla\nabla f_Y) \\ &= g_Q(E^Q(\nabla f_Y), \nabla f_Y) + g_Q(f_Y E^Q, \nabla\nabla f_Y). \end{aligned} \tag{3.12}$$

Since σ^Q is constant, $\mathrm{div}_\nabla(E^Q) = 0$ from (3.7). Hence the last equality of (3.12) holds. By integrating (3.12) and using the transversal divergence theorem (Theorem 2.1), we have

$$\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) = \int_M g_Q(E^Q(f_Y\nabla f_Y), \kappa_B^\sharp) - \int_M g_Q(f_Y E^Q, \nabla\nabla f_Y). \tag{3.13}$$

Now, we calculate the first term of the right hand side in (3.13). By definition of E^Q,

$$g_Q(E^Q(f_Y\nabla f_Y), \kappa_B^\sharp) = g_Q(\mathrm{Ric}^Q(f_Y\nabla f_Y), \kappa_B^\sharp) - \frac{\sigma^Q}{q}g_Q(f_Y\nabla f_Y, \kappa_B^\sharp).$$

Since $\delta_B\kappa_B = 0$, we have $\int_M g_Q(\kappa_B^\sharp, f_Y\nabla f_Y) = \frac{1}{2}\int_M \kappa_B^\sharp(f_Y^2) = 0$. Hence

$$\int_M g_Q(E^Q(f_Y\nabla f_Y), \kappa_B^\sharp) = \int_M g_Q(\mathrm{Ric}^Q(f_Y\nabla f_Y), \kappa_B^\sharp). \tag{3.14}$$

From Lemma 3.3 and (3.14), the proof is completed. □

4. Riemannian foliations admitting transversal conformal fields

In this section, we characterize the Riemannian foliations admitting transversal nonisometric conformal fields. First, we review the known facts.

Theorem 4.1 ([2]). *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M such that $\delta_B\kappa_B = 0$. Assume that the transversal scalar curvature σ^Q is constant. If $\bar{Y}$ is a transversal conformal field with $\theta(Y)g_Q = 2f_Y g_Q$, $f_Y \neq 0$, then*

$$\int_M |\nabla f_Y|^2 = \frac{\sigma^Q}{q-1}\int_M f_Y^2$$

and the scalar curvature σ^Q is non-negative.

Theorem 4.2 ([3])**.** *Let $(M, g_M, \mathcal{F})$ be as in Theorem 4.1. Assume that the transversal scalar curvature σ^Q is constant. If $\bar{Y}$ is a transversal conformal field with $\theta(Y)g_Q = 2f_Y g_Q$, $f_Y \neq 0$, then*

$$\int_M \{g_Q(E^Q(\nabla f_Y), \nabla f_Y) + |\nabla\nabla f_Y + \frac{\sigma^Q}{q(q-1)} f_Y g_Q|^2\} = 0.$$

Theorem 4.3 ([3])**.** *Let $(M, g_M, \mathcal{F})$ be as in Theorem 4.1. Assume that the transversal scalar curvature σ^Q is non-zero constant. If M admits a transversal conformal field $\bar{Y}$ with $\theta(Y)g_Q = 2f_Y g_Q$, $f_Y \neq 0$, such that*

$$\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) \geq 0,$$

then $(M, \mathcal{F})$ is transversally isometric to the sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^Q}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

Remark. Theorem 4.3 yields the result in [2] when $\mathcal{F}$ is transversally Einsteinian. Also, Riemannian version of Theorem 4.3 can be found in [15].

Theorem 4.4. *Let $(M, g_M, \mathcal{F})$ be as in Theorem 4.1. Assume that the transversal scalar curvature σ^Q is non-zero constant. If $\bar{Y}$ is a transversal conformal field with $\theta(Y)g_Q = 2f_Y g_Q$, $f_Y \neq 0$, then*

$$q(q-1)\int_M g_Q(\mathrm{Ric}^Q(\nabla f_Y), \nabla f_Y) \leq (\sigma^Q)^2 \int_M f_Y^2. \tag{4.1}$$

Equality holds if and only if $(M, \mathcal{F})$ is transversally isometric to the sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^Q}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

Proof. Let $\bar{Y}$ be a transversal conformal field such that $\theta(Y)g_Q = 2f_Y g_Q$ ($f_Y \neq 0$). Then from Theorem 4.2,

$$\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) \leq 0. \tag{4.2}$$

Equivalently, from (3.5) and (4.2) we have

$$\int_M g_Q(\mathrm{Ric}^Q(\nabla f_Y), \nabla f_Y) - \frac{\sigma^Q}{q}\int_M |\nabla f_Y|^2 \leq 0. \tag{4.3}$$

From (4.3) and Theorem 4.1, the proof of (4.1) follows. Equality holds if and only if $\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) = 0$. From Theorem 4.3, the proof is completed. □

Theorem 4.5. *Let $(M, g_M, \mathcal{F})$ be as in Theorem 4.1, and suppose that $\mathcal{F}$ is minimal. Assume that the transversal scalar curvature σ^Q is constant. If M admits a transversal conformal field $\bar{Y}$ with $\theta(Y)g_Q = 2f_Y g_Q$, $f_Y \neq 0$ such that*

$$\theta(Y)|Z^Q|^2 = \lambda f_Y |Z^Q|^2 \ (\lambda \geq -4), \tag{4.4}$$

then $(M, \mathcal{F})$ is transversally isometric to the sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^Q}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

Proof. From Proposition 3.4, if we use (4.4) and the minimality of $\mathcal{F}$, then

$$\int_M g_Q(E^Q(\nabla f_Y), \nabla f_Y) = \frac{4+\lambda}{8}\int_M {f_Y}^2|Z^Q|^2.$$

Since $\lambda \geq -4$, the proof follows from Theorem 4.3. □

From Lemma 3.2, if the transversal scalar curvature is constant, then

$$\theta(Y)|Z^Q|^2 = \theta(Y)|R^Q|^2.$$

Hence we have the following corollary.

Corollary 4.6. *Let $(M, g_M, \mathcal{F})$ be as in Theorem 4.1, and suppose that $\mathcal{F}$ is minimal. Assume that the transversal scalar curvature σ^Q is constant. If M admits a transversal nonisometric conformal field $\bar{Y}$ such that*

$$\theta(Y)|Z^Q|^2 = 0 \text{ (or } \theta(Y)|R^Q|^2 = 0),$$

then $(M, \mathcal{F})$ is transversally isometric to the sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^Q}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

Corollary 4.7. *Let $(M, g_M, \mathcal{F})$ be as in Theorem 4.1. If $|R^Q|$ or $|Z^Q|$ is constant, then $(M, \mathcal{F})$ is transversally isometric to the sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^Q}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.*

Lastly, we give some property of the Riemannian foliation on complete Riemannian manifolds admitting the transversal conformal field.

Theorem 4.8. *Let $(M, g_M, \mathcal{F})$ be a complete Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric g_M. Assume that the transversal scalar curvature σ^Q is positive constant. If $\bar{Y}$ is a transversal conformal field with $\theta(Y)g_Q = 2f_Y g_Q$, $f_Y \neq 0$, then*

$$|\nabla\nabla f_Y|^2 \geq \frac{(\sigma^Q)^2}{q(q-1)^2}{f_Y}^2.$$

Equality holds if and only if $(M, \mathcal{F})$ is transversally isometric to the sphere $(S^q(1/c), G)$, where $c^2 = \frac{\sigma^Q}{q(q-1)}$ and G is a discrete subgroup of $O(q)$.

Proof. Since σ^Q is constant, from (3.3) we have

$$\sum_a \nabla_a\nabla_a f_Y = -(\Delta_B - \kappa_B^\sharp)f_Y = -\frac{\sigma^Q}{q-1}f_Y. \tag{4.5}$$

Hence, we have from (3.4)

$$\begin{aligned} 0 &\leq |\nabla\nabla f_Y + \frac{\sigma^Q}{q(q-1)}f_Y g_Q|^2 \\ &= |\nabla\nabla f_Y|^2 + \frac{2\sigma^Q}{q(q-1)}f_Y\sum_a \nabla_a\nabla_a f_Y + \frac{(\sigma^Q)^2}{q(q-1)^2}{f_Y}^2 \end{aligned}$$

$$= |\nabla\nabla f_Y|^2 - \frac{(\sigma^Q)^2}{q(q-1)^2} {f_Y}^2,$$

which complete the proof. Equality holds if and only if

$$\nabla\nabla f_Y = -\frac{\sigma^Q}{q(q-1)} f_Y g_Q.$$

Hence if we put $c^2 = \frac{\sigma^Q}{q(q-1)}$, then by the generalized Obata theorem (Theorem 2.2) [4], the proof is completed. □

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