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WEIGHTED GDMP INVERSE OF OPERATORS BETWEEN HILBERT SPACES

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Abstract. We introduce new generalized inverses (named the WgDMP inverse and dual WgDMP inverse) for a bounded linear operator between two Hilbert spaces, using its Wg-Drazin inverse and its Moore-Penrose inverse. Some new properties of WgDMP inverse and dual WgDMP inverse are obtained and some known results are generalized.

1. Introduction

Let X and Y be arbitrary Hilbert spaces. Denote by $\mathcal{B}(X,Y)$ the set of all bounded linear operators from X to Y. Set $\mathcal{B}(X)=\mathcal{B}(X,X)$. For an operator $A\in\mathcal{B}(X,Y)$, the symbols $N(A)$, $R(A)$, $\sigma(A)$, respectively, will denote the null space, the range and the spectrum of A.

Let $W\in\mathcal{B}(Y,X)$ be a fixed nonzero operator. An operator $A\in\mathcal{B}(X,Y)$ is called Wg-Drazin invertible if there exists unique $B\in\mathcal{B}(X,Y)$ such that

$$AWB=BWA,\quad BWAWB=B \quad \text{and} \quad A-AWBWA \text{ is quasinilpotent.}$$

The Wg-Drazin inverse B of A is denoted by $A^{d,w}$ [4]. If $A-AWBWA$ is nilpotent in the above definition, then $A^{d,W}=A^{D,W}$ is the W-weighted Drazin inverse of A [3, 6, 7]. In the case that $X=Y$ and $W=I$, then $A^d=A^{d,W}$ is the generalized Drazin inverse (or the Koliha-Drazin inverse) of A [8] and $A^D=A^{D,W}$ is the Drazin inverse of A [5].

Let us recall that, if $A\in\mathcal{B}(X,Y)$ and $W\in\mathcal{B}(Y,X)\backslash\{0\}$, then the following conditions are equivalent [4]:

(1) A is Wg-Drazin invertible,
(2) AW is generalized Drazin invertible in $\mathcal{B}(Y)$ with $(AW)^d=A^{d,W}W$,
(3) WA is generalized Drazin invertible in $\mathcal{B}(X)$ with $(WA)^d=WA^{d,W}$.

Then, the Wg-Drazin inverse $A^{d,W}$ of A satisfies

$$A^{d,W}=((AW)^d)^2A=A((WA)^d)^2.$$

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The Moore-Penrose inverse of $A \in \mathcal{B}(X,Y)$ is the operator $B \in \mathcal{B}(Y,X)$ which satisfies the Penrose equations

$$ABA = A, \qquad BAB = B, \qquad (AB)^* = AB, \qquad (BA)^* = BA.$$

The Moore-Penrose inverse of A exists if and only if $R(A)$ is closed in Y. If the Moore-Penrose inverse of A exists, then it is unique and denoted by $A^\dagger$.

In [1], Baksalary and Trenkler introduced an inverse of a matrix named core inverse. Malik and Thome [9] generalized this definition and defined a new generalized inverse of a square matrix of an arbitrary index. They used the Drazin inverse and the Moore-Penrose inverse and therefore this new generalized inverse is called the DMP inverse. As an extension of the DMP inverse for a square matrix, the gDMP inverse for a Hilbert space operator was presented in [11]: let $A \in \mathcal{B}(X)$ be generalized Drazin invertible such that $R(A)$ is closed. The gDMP inverse of A is defined as

$$A^{d,\dagger} = A^d A A^\dagger.$$

The gDMP inverse of a Hilbert space operator can be seen as a generalization of the DMP inverse of a complex square matrix presented in [9] and as an extension of the core inverse [1]. Recall that, the MPgD inverse of the generalized Drazin invertible operator $A \in \mathcal{B}(X)$ such that $R(A)$ is closed, was defined as $A^{\dagger,d} = A^\dagger A A^d$ in [11].

Motivated by extension of the Drazin inverse to the W-weighted Drazin inverse, Meng [10] defined the W-weighted DMP inverse, generalizing the DMP inverse of a square matrix to a rectangular matrix.

Our aim is to define new generalized inverses named the WgDMP inverse and dual WgDMP inverse for a bounded linear operator between two Hilbert spaces using its Wg-Drazin inverse and its Moore-Penrose inverse. Thus, the WgDMP inverse of an operator is a generalization of the W-weighted DMP inverse of a rectangular matrix [10] and the gDMP inverse of a Hilbert space operator [11]. We give the matrix representations of the WgDMP inverse and dual WgDMP inverse of an operator. Some new properties of the WgDMP inverse and dual WgDMP inverse are investigated and their relation with the gDMP and MPgD inverse, respectively, of corresponding operators.

2. WgDMP inverse and dual WgDMP inverse

In this section, we define and study the WgDMP inverse and the dual WgDMP inverse of an operator between two Hilbert spaces. We start from a geometrical point of view.

Theorem 2.1. *Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible such that $R(A)$ is closed.*

(a) *The system of conditions*

$$AB = P_{R(AWA^{d,W}),N(A^{d,W}A^\dagger)} \quad \text{and} \quad R(B) \subseteq R(WAWA^{d,W}), \tag{1}$$

is consistent and it has the unique solution $B = WA^{d,W}WAA^\dagger$.

(b) *The system of conditions*

$$CA = P_{R(A^\dagger A^{d,W}),N(A^{d,W}WA)} \quad \textit{and} \quad R(C^*) \subseteq R((AWA^{d,W}W)^*) \tag{2}$$

is consistent and it has the unique solution $C = A^\dagger AWA^{d,W}W$.

Proof. (a) First, we check that $B = WA^{d,W}WAA^\dagger$ satisfies conditions (1). Obviously, $R(B) \subseteq R(WAWA^{d,W})$. From

$$\begin{aligned} BAB &= WA^{d,W}WAA^\dagger AB \\ &= WA^{d,W}WAWA^{d,W}WAA^\dagger \\ &= WA^{d,W}WAA^\dagger = B, \end{aligned} \tag{3}$$

we deduce that AB is a projector. Since $AB = AWA^{d,W}WAA^\dagger$ and

$$AWA^{d,W} = AWA^{d,W}WAA^\dagger AWA^{d,W} = ABAWA^{d,W},$$

we have that $R(AB) = R(AWA^{d,W})$. Also, by $AB = AWAWA^{d,W}A^\dagger$ and

$$\begin{aligned} A^{d,W}A^\dagger &= A^{d,W}WA^{d,W}WAA^\dagger \\ &= A^{d,W}WA^{d,W}WAWA^{d,W}WAA^\dagger \\ &= A^{d,W}WA^{d,W}WAB, \end{aligned}$$

notice that $N(AB) = N(A^{d,W}A^\dagger)$.

To prove that system (1) has unique solution, assume that two operators B_1 and B_2 satisfy (1). Then $A(B_1 - B_2) = P_{R(AWA^{d,W}),N(A^{d,W}A^\dagger)} - P_{R(AWA^{d,W}),N(A^{d,W}A^\dagger)} = 0$ implying $R(B_1 - B_2) \subseteq N(A) \subseteq N(WA^{d,W}WA)$. The conditions $R(B_1) \subseteq R(WAWA^{d,W})$ and $R(B_2) \subseteq R(WAWA^{d,W})$ give $R(B_1 - B_2) \subseteq R(WAWA^{d,W}) \cap N(WAWA^{d,W}) = \{0\}$. Thus, $B_1 = B_2$ and only one B satisfies (1).

(b) In the similar way, we prove that $C = A^\dagger AWA^{d,W}W$ is the unique solution of system (2). □

Definition 2.1. Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible such that $R(A)$ is closed.

(a) The WgDMP inverse of A is defined as

$$A^{d,W,\dagger} = WA^{d,W}WAA^\dagger.$$

(b) The dual WgDMP inverse of A is defined as

$$A^{\dagger,d,W} = A^\dagger AWA^{d,W}W.$$

If $A \in \mathcal{B}(X)$ and $W = I \in \mathcal{B}(X)$ in the above definition, then $A^{d,\dagger} = A^{d,W,\dagger}$ (or $A^{\dagger,d} = A^{\dagger,d}$) is the gDMP (MPgD) inverse of A [11].

Theorem 2.2. *Let* $W \in \mathcal{B}(Y,X)\backslash\{0\}$ *and let* $A \in \mathcal{B}(X,Y)$ *be* Wg*-Drazin invertible such that* $R(A)$ *is closed.*

(a) *The system of equations*

$$BAB = B \quad \text{and} \quad BA = WA^{d,W}WA \tag{4}$$

is consistent and $B = A^{d,W,\dagger}$ *is one of its solutions.*

(b) *The system of equations*

$$CAC = C \quad \text{and} \quad AC = AWA^{d,W}W \tag{5}$$

is consistent and $C = A^{\dagger,d,W}$ *is one of its solutions.*

Proof. (a) For $B = A^{d,W,\dagger}$, because $B = WA^{d,W}WAA^\dagger$, then $BA = WA^{d,W}WA$ and, by (3), $BAB = B$.

(b) Similarly as (a), we verify this part. □

Theorem 2.3. *Let* $W \in \mathcal{B}(Y,X)\backslash\{0\}$ *and let* $A \in \mathcal{B}(X,Y)$ *be* W*g-Drazin invertible such that* $R(A)$ *is closed.*

(a) *If* B *satisfies* (4)*, then* $AWA(B - A^\dagger)$ *and* $A - ABA$ *are quasinilpotent.*

(b) *If* C *satisfies* (5)*, then* $(A^\dagger - C)AWA$ *and* $A - ACA$ *are quasinilpotent.*

Proof. (a) Since $WAWA^{d,W}WA - WA = WA(WA)^dWA - WA$ is quasinilpotent, then $AWA(B - A^\dagger)$ is quasinilpotent by

$$\begin{aligned}\sigma(AWA(B - A^\dagger)) \cup \{0\} &= \sigma(WA(B - A^\dagger)A) \cup \{0\}\\ &= \sigma(WAWA^{d,W}WA - WA) \cup \{0\} = \{0\}.\end{aligned}$$

Evidently, $A - ABA = A - AWA^{d,W}WA$ is quasinilpotent.

The part (b) follows in the same manner. □

As a consequence of Theorem 2.1 and Theorem 2.2, we get the next result.

Corollary 2.1. *Let* $W \in \mathcal{B}(Y,X)\backslash\{0\}$ *and let* $A \in \mathcal{B}(X,Y)$ *be* W*g-Drazin invertible such that* $R(A)$ *is closed. The following statements hold*:

(a) $AA^{d,W,\dagger}$ *is a projector onto* $R(AWA^{d,W})$ *along* $N(A^{d,W}A^\dagger)$;

(b) $A^{d,W,\dagger}A = WA^{d,W}WA$ *is a projector onto* $R(WA^{d,W}WA)$ *along* $N(WA^{d,W}WA)$;

(c) $AA^{\dagger,d,W} = AWA^{d,W}W$ *is a projector onto* $R(AWA^{d,W}W)$ *along* $N(AWA^{d,W}W)$;

(d) $A^{\dagger,d,W}A$ *is a projector onto* $R(A^\dagger A^{d,W})$ *along* $N(A^{d,W}WA)$.

Now, we consider necessary and sufficient conditions for equalities between certain operators which appear in the previous corollary.

Lemma 2.1. *Let* $W \in \mathcal{B}(Y,X)\backslash\{0\}$ *and let* $A \in \mathcal{B}(X,Y)$ *be* W*g-Drazin invertible such that* $R(A)$ *is closed. Then*

(a) $AA^{d,W,\dagger}A = AWA^{d,W}$ *if and only if* $A^{d,W} = A^{d,W}WA$ *if and only if* $AA^{\dagger,d,W}A = AWA^{d,W}$;

(b) $AA^{d,W,\dagger} = A^{d,W}A^\dagger$ *if and only if* $A^{d,W} = AWA^{d,W}WA$ *if and only if* $A^{\dagger,d,W}A = A^\dagger A^{d,W}$;

(c) $AA^\dagger = AA^{d,W,\dagger}$ *if and only if* $A = AWA^{d,W}WA$ *if and only if* $A^\dagger A = A^{\dagger,d,W}A$;

(d) $A = AA^{d,W,\dagger}A$ *if and only if* $A = AWA^{d,W}WA$ *if and only if* $A = AA^{\dagger,d,W}A$.

Proof. The part (a) follows by

$$\begin{aligned} AA^{d,W,\dagger}A = AWA^{d,W} &\Leftrightarrow AWA^{d,W}WAA^\dagger A = AWA^{d,W} \\ &\Leftrightarrow A^{d,W}WA = A^{d,W} \\ &\Leftrightarrow AWA^{d,W} = A^{d,W} \\ &\Leftrightarrow AA^\dagger AWA^{d,W}WA = A^{d,W}WA \\ &\Leftrightarrow AA^{\dagger,d,W}A = AWA^{d,W}. \end{aligned}$$

Similarly, we check the rest. □

Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and let $A \in \mathcal{B}(X,Y)$ be such that $R(A)$ is closed. Observe that operators A and W have the following matrix representations with respect to the orthogonal sums $X = \overline{R(W)} \oplus N(W^*)$ and $Y = R(A) \oplus N(A^*)$:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix} \to \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \tag{6}$$

and

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix}, \tag{7}$$

where $D = A_1A_1^* + A_2A_2^*$ maps $R(A)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Notice that A_1, A_2 and D are linear bounded operators and

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix}. \tag{8}$$

Meng [10] presented the canonical form for the W-weighted DMP inverse of a rectangular matrix using the singular value decompositions which is a powerful tool to investigate various classes of complex matrices. To get the matrix representation for the WgDMP inverse, we use the matrix forms of a linear bounded operators (6) and (7), which are induced by some natural decompositions of Hilbert spaces.

Theorem 2.4. *Let* $W \in \mathcal{B}(Y,X)\backslash\{0\}$ *and let* $A \in \mathcal{B}(X,Y)$ *be* W*g-Drazin invertible such that* $R(A)$ *is closed. If* A *and* W *are written as in* (6) *and* (7), *respectively, then*

$$A^{d,W} = \begin{bmatrix} A_1^{d,W_1} & (A_1^{d,W_1}W_1)^2A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix} \to \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \tag{9}$$

and

$$A^{d,W,\dagger} = \begin{bmatrix} W_1A_1^{d,W_1}W_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix}. \tag{10}$$

Proof. Let

$$A^{d,W} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix} \to \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix}.$$

From $A^{d,W} = AWA^{d,W}WA^{d,W}$, we get $B_3 = B_4 = 0$. The equality $AWA^{d,W} = A^{d,W}WA$ gives $A_1W_1B_1 = B_1W_1A_1$ and $A_1W_1B_2 = B_1W_1A_2$. Further, by $A^{d,W} = A^{d,W}WAWA^{d,W}$, we obtain

$$B_1 = B_1W_1A_1W_1B_1$$

and

$$B_2 = B_1W_1(A_1W_1B_2) = B_1W_1B_1W_1A_2 = (B_1W_1)^2A_2.$$

Since

$$A - AWA^{d,W}WA = \begin{bmatrix} A_1 - A_1W_1B_1W_1A_1 & A_2 - A_1W_1B_1W_1A_2 \\ 0 & 0 \end{bmatrix}$$

is quasinilpotent, then $\sigma(A_1 - A_1W_1B_1W_1A_1) \subseteq \sigma(A - AWA^{d,W}WA) \cup \{0\} = \{0\}$ and so $A_1 - A_1W_1B_1W_1A_1$ is quasinilpotent. Therefore, $B_1 = A_1^{d,W_1}$ and

$$A^{d,W} = \begin{bmatrix} A_1^{d,W_1} & (A_1^{d,W_1}W_1)^2A_2 \\ 0 & 0 \end{bmatrix}.$$

Using (8), the WgDMP inverse of A is represented by

$$A^{d,W,\dagger} = WA^{d,W}WAA^\dagger = \begin{bmatrix} W_1A_1^{d,W_1}W_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix}.$$

□

Recall that $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and $A \in \mathcal{B}(X,Y)$ such that $R(A)$ is closed can be represented, with respect to the orthogonal sums $X = R(A^*) \oplus N(A)$ and $Y = \overline{R(W^*)} \oplus N(W)$, by:

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} R(A^*) \\ N(A) \end{bmatrix} \to \begin{bmatrix} \overline{R(W^*)} \\ N(W) \end{bmatrix},$$

and

$$W = \begin{bmatrix} W_1 & 0 \\ W_2 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W^*)} \\ N(W) \end{bmatrix} \to \begin{bmatrix} R(A^*) \\ N(A) \end{bmatrix},$$

where $E = A_1^*A_1 + A_2^*A_2$ maps $R(A^*)$ into itself and $E > 0$ (meaning $E \geq 0$ invertible). In addition, if A is Wg-Drazin invertible, we can prove that

$$A^{d,W} = \begin{bmatrix} A_1^{d,W_1} & 0 \\ A_2(W_1A_1^{d,W_1})^2 & 0 \end{bmatrix} : \begin{bmatrix} R(A^*) \\ N(A) \end{bmatrix} \to \begin{bmatrix} \overline{R(W^*)} \\ N(W) \end{bmatrix}$$

and

$$A^{\dagger,d,W} = \begin{bmatrix} W_1A_1^{d,W_1}W_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W^*)} \\ N(W) \end{bmatrix} \to \begin{bmatrix} R(A^*) \\ N(A) \end{bmatrix}.$$

By the previous matrix representations of $A^{d,W,\dagger}$ and $A^{\dagger,d,W}$, we have the next result.

Theorem 2.5. *Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible such that $R(A)$ is closed. Then*

(a) $A^{d,W,\dagger}AW = WAA^{d,W,\dagger}$ *if and only if* $WA^{d,W}W = A^{d,W,\dagger}$;

(b) $A^{\dagger,d,W}AW = WAA^{\dagger,d,W}$ *if and only if* $WA^{d,W}W = A^{\dagger,d,W}$;

(c) $AWAA^{\dagger}$ *is generalized Drazin invertible and* $A^{d,W,\dagger} = W(AWAA^{\dagger})^d$;

(d) $A^{\dagger}AWA$ *is generalized Drazin invertible and* $A^{\dagger,d,W} = (A^{\dagger}AWA)^d W$.

Proof. (a) Using (6), (7), (9) and (10), we get

$$A^{d,W,\dagger}AW = \begin{bmatrix} W_1A_1^{d,W_1}W_1A_1W_1 & W_1A_1^{d,W_1}W_1A_1W_2 \\ 0 & 0 \end{bmatrix},$$

$$WAA^{d,W,\dagger} = \begin{bmatrix} W_1A_1W_1A_1^{d,W_1}W_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and}$$

$$WA^{d,W}W = \begin{bmatrix} W_1A_1^{d,W_1}W_1 & W_1A_1^{d,W_1}W_2 \\ 0 & 0 \end{bmatrix}.$$

Hence,

$$\begin{aligned} A^{d,W,\dagger}AW = WAA^{d,W,\dagger} &\Leftrightarrow W_1A_1^{d,W_1}W_1A_1W_2 = 0 \\ &\Leftrightarrow W_1A_1^{d,W_1}W_2 = 0 \\ &\Leftrightarrow WA^{d,W}W = A^{d,W,\dagger}. \end{aligned}$$

(b) Analogously as part (a).

(c) Since A is Wg-Drazin invertible, by (6) and (7), $AW = \left[\begin{smallmatrix} A_1W_1 & A_1W_2 \\ 0 & 0 \end{smallmatrix}\right]$ is generalized Drazin invertible. Hence, A_1W_1 is generalized Drazin invertible which implies, applying (8), that $AWAA^{\dagger} = \left[\begin{smallmatrix} A_1W_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is generalized Drazin invertible. Now, we obtain

$$\begin{aligned} W(AWAA^{\dagger})^d &= \begin{bmatrix} W_1 & W_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (A_1W_1)^d & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} W_1A_1^{d,W_1}W_1 & 0 \\ 0 & 0 \end{bmatrix} = A^{d,W,\dagger}. \end{aligned}$$

(d) It follows as (c). □

Notice that part (c) and (d) of Theorem 2.5 give a method to obtain the WgDMP and dual WgDMP inverse by algebraic approach. The part (c) of Theorem 2.5 recovers [12, Theorem 3.3].

By corresponding definitions, we obtain the next connections between the WgDMP (or dual WgDMP) inverse of A and the gDMP (MPgD) inverse of WA (AW).

Lemma 2.2. *Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible such that $R(A)$ is closed.*

(a) *If $R(WA)$ is closed, then*
$$A^{d,W,\dagger} = (WA)^{d,\dagger} WAA^\dagger, (WA)^{d,\dagger} = A^{d,W,\dagger} A(WA)^\dagger$$
and $R(A^{d,W,\dagger}) = R((WA)^{d,\dagger})$.

(b) *If $R(AW)$ is closed, then*
$$A^{\dagger,d,W} = A^\dagger AW(AW)^{\dagger,d}, (AW)^{\dagger,d} = (AW)^\dagger AA^{\dagger,d,W}$$
and $N(A^{\dagger,d,W}) = N((AW)^{\dagger,d})$.

Notice that $A \in \mathcal{B}(X,Y)$ and $W \in \mathcal{B}(Y,X)\backslash\{0\}$ have the following matrix representations with respect to the orthogonal sums $X = \overline{R(W)} \oplus N(W^*)$ and $Y = \overline{R(A)} \oplus N(A^*)$:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(A)} \\ N(A^*) \end{bmatrix} \tag{11}$$

and

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(A)} \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix}, \tag{12}$$

where $D_1 = A_1A_1^* + A_2A_2^* \in \mathcal{B}(\overline{R(A)})$.

We now present the operator matrix form of the gDMP inverse of WA.

Theorem 2.6. *Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible such that $R(WA)$ is closed. If A and W are written as in* (11) *and* (12)*, respectively, then*

(13)
$$(WA)^{d,\dagger} = \begin{bmatrix} W_1A_1^{d,W_1}W_1D_1W_1^*(W_1D_1W_1^*)^\dagger & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix}.$$

Proof. The hypothesis A is Wg-Drazin invertible implies that
$$WA = \begin{bmatrix} W_1A_1 & W_1A_2 \\ 0 & 0 \end{bmatrix}$$
is generalized Drazin invertible. Therefore,
$$\begin{aligned}(WA)^\dagger = (WA)^*[WA(WA)^*]^\dagger &= \begin{bmatrix} A_1^*W_1^* & 0 \\ A_2^*W_1^* & 0 \end{bmatrix} \begin{bmatrix} (W_1D_1W_1^*)^\dagger & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_1^*W_1^*(W_1D_1W_1^*)^\dagger & 0 \\ A_2^*W_1^*(W_1D_1W_1^*)^\dagger & 0 \end{bmatrix}\end{aligned}$$
and, by [2, Theorem 2.3],
$$(WA)^d = \begin{bmatrix} (W_1A_1)^d & [(W_1A_1)^d]^2W_1A_2 \\ 0 & 0 \end{bmatrix}.$$
Now, we have
$$(WA)^{d,\dagger} = (WA)^dWA(WA)^\dagger = \begin{bmatrix} (W_1A_1)^dW_1DW_1^*(W_1DW_1^*)^\dagger & 0 \\ 0 & 0 \end{bmatrix}$$

which yields that (13) holds. □

In the case that $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and $A \in \mathcal{B}(X,Y)$ is W g-Drazin invertible such that $R(AW)$ is closed, using the following matrix representations with respect to the orthogonal sums $X = \overline{R(A^*)} \oplus N(A)$ and $Y = \overline{R(W^*)} \oplus N(W)$:

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(A^*)} \\ N(A) \end{bmatrix} \to \begin{bmatrix} \overline{R(W^*)} \\ N(W) \end{bmatrix},$$

and

$$W = \begin{bmatrix} W_1 & 0 \\ W_2 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W^*)} \\ N(W) \end{bmatrix} \to \begin{bmatrix} \overline{R(A^*)} \\ N(A) \end{bmatrix},$$

where $E_1 = A_1^*A_1 + A_2^*A_2 \in \mathcal{B}(R(A^*))$, we obtain

$$(AW)^{\dagger,d} = \begin{bmatrix} (W_1E_1W_1^*)^\dagger W_1E_1W_1^*A_1^{d,W_1}W_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W^*)} \\ N(W) \end{bmatrix} \to \begin{bmatrix} \overline{R(W^*)} \\ N(W) \end{bmatrix}.$$

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