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GENERATING SAMPLE PATHS AND THEIR CONVERGENCE OF THE GEOMETRIC FRACTIONAL BROWNIAN MOTION

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Abstract. We derive discrete time model of the geometric fractional Brownian motion. It provides numerical pricing scheme of financial derivatives when the market is driven by geometric fractional Brownian motion. With the convergence analysis, we guarantee the convergence of Monte Carlo simulations. The strong convergence rate of our scheme has order H which is Hurst parameter. To obtain our model we need to convert Wick product term of stochastic differential equation into Wick free discrete equation through Malliavin calculus but ours does not include Malliavin derivative term. Finally, we include several numerical experiments for the option pricing.

1. Introduction

Most models of financial processes are assuming Brownian motion of the asset prices. Gaussian and Markovian properties of Brownian motion facilitate greatly pricing of many financial derivatives. But it is found that empirical asset returns may have memory about past history. Empirical asset prices are affected not only by their most current values, but also by their history (see Mandelbrot [18] and Shiryaev [25]). It implies that market is not exactly Markovian, too. Therefore most classical models need to be modified so that they may describe the dynamics of markets more accurately. One alternative is to use the fractional Brownian motion (fBm) instead of the Brownian motion. Then the classical Black-Scholes model can be extended to the fractional Black-Scholes model. However, Björk and Hult [8] pointed out the difficulties of interpretation based on Wick calculus. Also the pricing models based on geometric fBm can give explicit arbitrage strategies in [25].

However, to exclude arbitrage several modifications of the fractional market setting have been suggested (Hu and Økesendal [15], Elliott and van der

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Hoek [12] and Rostek [23]). In case that market prices move at least slightly faster than any market participant can react, arbitrage can be excluded. Renouncement of continuous trability can obtain a reasonable financial model where no arbitrage occurs (Rostek [22]). In the more realistic market models which includes transaction costs, the ideal continuous time trading strategies turn out to be of bounded variation. In this case Guasoni [13], Guasoni et al., [14] show that geometric fBm models can be economically meaningful. In this respect, Valkeila [28] studied the approximation of geometric fBm in the sense of weak convergence.

The fractional Brownian motion can be represented as a limit of a random walk (see Sottinen [26]). There are many methods of simulation for fractional Brownian motion like Hosking method, Cholesky method, Davies and Harte method, FFT method as is reviewed in [16]. The stochastic differential equation describing fractional Brownian market includes stochastic terms consisting of Wick product of fractional Brownian motion. In the continuous time stochastic differential equation, even when such a solution can be found, it may be only in implicit form or too complicated to visualize and evaluate numerically. The time discrete approximation or discretisation of stochastic differential equation is the method that generate values $\widehat{S}_{\Delta t}, \widehat{S}_{2\Delta t}, \widehat{S}_{3\Delta t}, \ldots, \widehat{S}_{n\Delta t}, \ldots$ at given discretization times $0 < \Delta t < 2\Delta t < \cdots < n\Delta t < \cdots$. The main difficulty of sampling geometric fractional Brownian motion is that Wick product cannot be evaluated pathwise, but depends on all possible other paths as is stated in Bender [5]. We suggest a recursive method for generating sample paths of geometric fractional Brownian motion in the context of Wick-Itô integral. The main idea is to replace Wick products by ordinary products plus expectations of possible other paths. Our scheme has convergence rate H which is Hurst parameter.

Theorem 1.1. *Suppose $S_t, t \in [0,T]$ is geometric fractional Brownian motion and $\widehat{S}_{k\Delta t}, k \in \{0, 1, \ldots, \frac{T}{\Delta t}\}$ is discrete geometric fractional Brownian motion. There is a constant C depending only on H such that*

$$E[|S_T - \widehat{S}_T|] \le C(T+1)e^{T^{2H}} \Delta t^H.$$

Through numerical experiments we compare sample paths from our algorithm with exact paths of the geometric fractional Brownian motion. We also evaluate European option price when underlying asset is governed by geometric fractional Brownian motion by Monte Carlo method.

2. Discrete model of the geometric fractional Brownian motion

2.1. Fractional Brownian motion

The fractional Brownian motion B_t^H is Gaussian process which satisfies the following condition:

$$E(B_t^H) = 0 \quad \forall t \in \mathbb{R},$$

$$E(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}),$$

where H is the Hurst parameter and in case $H = \frac{1}{2}$, B_t^H is the classical Brownian motion. For the case $0 < H < \frac{1}{2}$, the covariance of the increments is negative, i.e., the process is mean-reverting and the increments of the process are negatively correlated. So the processes are called anti-persistent. For the case $\frac{1}{2} < H < 1$, the covariance of the increments is positive, i.e., increments of the process are positively correlated. So the processes are called persistent and have long-range dependency property. The Case $H > \frac{1}{2}$ makes the fractional Brownian motion a plausible model in mathematical finance. Several empirical studies of financial time series say that the log-returns have long-range dependence. In this paper, we consider the case $H > \frac{1}{2}$ in our financial model.

The fractional Brownian increment

$$\Delta B_{t,s}^H = B_t^H - B_s^H$$

has the following moment properties;

$$E(\Delta B_{t,s}^H) = E(B_t^H - B_s^H) = E(B_t^H) - E(B_s^H) = 0, \forall t, s \in \mathbb{R},$$

$$E((\Delta B_{t,s}^H)^2) = E((B_t^H - B_s^H)(B_t^H - B_s^H)) = |t-s|^{2H}, \forall t, s \in \mathbb{R}.$$

The covariance of two non-overlapping increments is

$$\begin{aligned} E(\Delta B_{t,s}^H \Delta B_{s,0}^H) &= E((B_t^H - B_s^H)(B_s^H - B_0^H)) \\ &= E(B_t^H B_s^H) - E(B_t^H B_0^H) - E((B_s^H)^2) + E(B_s^H B_0^H) \\ &= \frac{1}{2}[t^{2H} + s^{2H} - (t-s)^{2H}] - s^{2H} \\ &= \frac{1}{2}[t^{2H} - s^{2H} - (t-s)^{2H}]. \end{aligned} \tag{1}$$

Therefore the increments of fBm are correlated except for $H = \frac{1}{2}$.

Consider partitions $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of the interval $[0, T]$ such that $|\pi_n| \longrightarrow 0$ for $n \longrightarrow \infty$. Then we obtain the following:

Lemma 2.1. *For $p \geq 1$,*

$$\lim_{|\pi| \to 0} \sum_{k=1}^{n} \left| B_{t_k}^H - B_{t_{k-1}}^H \right|^p = \begin{cases} \infty & \text{if } pH < 1, \\ E|B_T^H|^p & \text{if } pH = 1, \\ 0 & \text{if } pH > 1. \end{cases} \tag{2}$$

Proof can be referred to Rogers [21]. We can conclude quadratic variation of fractional Brownian motion is zero in case $H > \frac{1}{2}$. We can obtained the following corollary:

Corollary 2.2. *For the fBm quadratic variation is zero if $H > \frac{1}{2}$ and does not exist if $H < \frac{1}{2}$. Moreover B^H has unbounded variation $\mathbb{P}$-a.s.*

2.2. Wick-Itô integration

The stochastic integration with respect to fBm is based on a renormalization operator Wick product introduced by Duncan, Hu and Pasik-Duncan [11] and Hu and Øksendal [15].

Let $H \in (\frac{1}{2}, 1)$. We define the fractional kernel $\phi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\phi(s,t) = H(2H-1)|t-s|^{2H-2}.$$

We endow the space of Borel measurable functions $f, g : [0,T] \to \mathbb{R}$ with the norm $||\cdot||_\phi^2$ and inner product $\langle \cdot, \cdot \rangle_\phi$:

$$||f||_\phi^2 := \int_0^T \int_0^T f(s)f(t)\phi(s,t)dsdt,$$

$$\langle f, g\rangle_\phi := \int_0^T \int_0^T f(s)g(t)\phi(s,t)dsdt$$

and we define

$$\mathbb{L}_\phi^2([0,T]) = \{f : [0,T] \to \mathbb{R} : f \text{ is Borel measurable, and } ||f||_\phi^2 < +\infty\}.$$

For a deterministic function $f \in \mathbb{L}_\phi^2([0,T])$ we define its Wick integral in the following way. Let $\pi_n = \{0 = t_0 < \cdots < t_n = T\}$ be a sequence of partitions of $[0,T]$ such that $|\pi_n| \to 0$, and f_n be the step functions approximating f :

$$f_n(t) = \sum_i a_i^n 1_{[t_i, t_{i+1})}(t) \longrightarrow f(t) \quad \text{in} \quad \mathbb{L}_\phi^2([0,T]).$$

Then we define

$$\int_0^T f_n(t)dB_t^H := \sum_i a_i^n (B_{t_{i+1}}^H - B_{t_i}^H)$$

and

$$\int_0^T f(t)dB_t^H := \lim_{n\to\infty} \int_0^T f_n(t)dB_t^H.$$

For a more detailed discussion of stochastic integral of fractional Brownian motion, we refer [7].

Then the following property can be obtained [11]:

$$\begin{aligned} &E[\int_0^T f dB^H] = 0, \\ &E[\int_0^T f dB^H \int_0^T g dB^H] = \langle f, g\rangle_\phi, \\ &E[\int_0^T f dB^H]^2 = ||f||_\phi^2 \quad \text{(Wick-Itô isometry)}. \end{aligned} \tag{3}$$

We denote by $I(f) := \int_0^T f(s)dB_s^H$ for every $f \in \mathbb{L}_\phi^2([0,T])$. Random variables of $L^p(\Omega, \mathfrak{F}, \mathbb{P})$ can be approximated with arbitrary exactness by linear combinations of so-called Wick exponentials $\exp^\diamond(I(f))$ that are defined by fractional integrals with deterministic integrands f's:

$$\exp^{\diamond}(I(f)) := \exp(I(f) - \frac{1}{2}||f||_{\phi}^2)$$

(cf. [11], [20]). Duncan et al. [11] defined the Wick product implicitly on these Wick exponentials by

$$\exp^{\diamond}(I(f)) \diamond \exp^{\diamond}(I(g)) = \exp^{\diamond}(I(f+g)).$$

Furthermore, $X \diamond Y$, that is, the Wick product can be extended to random variables X, Y in L^p. For an explicit definition of the Wick product based on a representation using Hermite polynomials, we may refer to Hu and Øksendal [15]. The following equation holds:

$$\exp^{\diamond}(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^{\diamond k}. \tag{4}$$

Let Φ be the following functional

$$(\Phi g)(t) = \int_0^T \phi(t,u) g(u) du.$$

We define ϕ-derivative of X in the direction Φg as

$$D_{\Phi g} X(\omega)(t) = \lim_{\delta \to 0} \frac{X(\omega + \delta \int_0^t (\Phi g)(v) dv) - X(\omega)}{\delta}.$$

The following properties are obtained (see Definition 3.1 and Equations (3.6)-(3.8) in [11]);

$$D_{\Phi g} \int_0^T f(s) dB_s^H = \langle f, g \rangle_{\phi}, \tag{5}$$

$$D_{\Phi g} \exp^{\diamond}(I(f)) = \exp^{\diamond}(I(f)) \langle f, g \rangle_{\phi}, \tag{6}$$

$$X \diamond \int_0^T g(s) dB_s^H = X \int_0^T g(s) dB_s^H - D_{\Phi g} X.$$

We now define the fractional Wick-Itô integral for random integrand. Let $X_t \in L^2(\Omega, \mathfrak{F}_t, \mathbb{P})$ for filtration $\mathfrak{F}_t, t \in [0,T]$ and $\pi = \{0 = t_0, \ldots, t_n = T\}$ be a partition of $[0,T]$. Then we define the Riemann sum $S_{\diamond}(X, \pi)$ of Wick type with respect to a partition π as

$$S_{\diamond}(X, \pi) := \sum_{i=0}^{n-1} X_{t_i} \diamond (B_{t_{i+1}}^H - B_{t_i}^H),$$

and one defines the fractional Wick-Itô integral as the limit of the sequence of Riemann sums

$$\int_0^T X_s d^{\diamond} B_s^H = \lim_{|\pi| \to 0} S_{\diamond}(X, \pi)$$

if the Wick product and the limit exists in $L^2(\Omega, \mathfrak{F}, \mathbb{P})$.

Duncan et al. [11] introduced fractional Itô formula.

Theorem 2.3. *Let $F \in C^2(\mathbb{R})$ with bounded second derivative. Then*

$$F(B_t^H) - F(B_s^H) = \int_s^t F'(B_u^H) d^\diamond B_u^H + H \int_s^t F''(B_u^H) u^{2H-1} du. \tag{7}$$

Theorem 2.4. *Let $F \in C^{1,2}([0,T] \times \mathbb{R})$ with bounded derivative and let $Y_t = \int_0^t X_u d^\diamond B_u^H$. Assume that there is an $\alpha > 1 - H$ such that*

$$E|X_u - X_v|^2 \le C|u - v|^{2\alpha},$$

where $|u - v| \le \delta$ for some $\delta > 0$ and

$$\lim_{0 \le u, v \le t, |u-v| \to 0} E|D_u^\phi (X_u - X_v)|^2 = 0.$$

Then for $0 \le t \le T$,

$$\begin{aligned} F(t, Y_t) = F(0,0) &+ \int_0^t \frac{\partial F}{\partial s}(s, Y_s) ds + \int_0^t \frac{\partial F}{\partial x}(s, Y_s) X_s d^\diamond B_s^H \\ &+ \int_0^t \frac{\partial^2 F}{\partial x^2}(s, Y_s) X_s D_s^\phi Y_s ds \end{aligned} \tag{8}$$

(cf. Theorem 4.3 in [11]).

2.3. Fractional Black-Scholes model

The fractional Black-Scholes model consists of two assets, riskless asset A and a risky asset S. The dynamics of the asset A are deterministic

$$dA_t = r_t A_t dt, \tag{9}$$

where r_t is a deterministic interest rate. The dynamics of the asset S are

$$dS_t = \mu S_t dt + \sigma S_t d^\diamond B_t^H, \ S_0 = s_0, \tag{10}$$

where $\diamond$ is Wick product and $H > \frac{1}{2}$. The wealth portfolio V_t is defined by $V_t = \alpha_t A_t + \beta_t S_t$ for weights α_t and β_t.

Why is it Wick product not ordinary product? If the dynamics of risky asset is governed by

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H \tag{11}$$

in the pathwise sense, the corresponding random stochastic integral does not have zero expectation which already suggests the possibility of riskless gains. Obvious drawbacks and examples of arbitrage in (11) are given by Shiryaev [24], Dasgupta and Kallianpur [10] and Bender [2]. The solution of (11) is

$$S_t = S_0 e^{\mu t + \sigma B_t^H}. \tag{12}$$

See Shiryaev [24].

The Wick-based integral in (10) have zero expectation. Duncan and Hu and Pasik-Duncan [11] provided a stochastic integration calculus with respect to fractional Brownian motion that is based on the Wick product. However, it turns out that the unrestricted fractional market settings allows arbitrage

(Rogers [21], Sottinen [27], Cheridito [9], Bender [4,6]). But again, to exclude arbitrage several modifications of the fractional market setting have been suggested (Hu and Økesendal [15], Elliott and van der Hoek [12] and Rostek [23]). In *a* case that market prices move at least slightly faster than any market participant can react, arbitrage can be excluded. Renouncement of continuous trability can obtain a reasonable financial model where no arbitrage occurs (Rostek [22]).

The solution of the stochastic differential equation (10) is

$$S_t = S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t^{2H} + \sigma B_t^H) \tag{13}$$

through fractional Wick-Itô integral. See Hu and Øksendal [15] or fractional Itô formula (7) and (8).

2.4. Discretization of the geometric fractional Brownian motion

As is motivated by classical geometric Brownian motion, one may introduce Euler-Maruyama scheme

$$S_n = S_{n-1} \diamond (1 + \mu\Delta t + \sigma\Delta B_n^H) \tag{14}$$

from Wick-Itô integral $\int_0^T dS_t = \int_0^T \mu S_t dt + \int_0^T \sigma S_t d^\diamond B_t^H$. In fact Bender ([4,5]) constructed random path by binomial process and conducted iteration to get discrete geometric fractional Brownian motion. He proved weak convergence of discrete version of his model. A drawback of his scheme is exponential number of iterations to get a path.

To simplify (10), let $\mu = 0$, $\sigma = 1$, $s_0 = 1$. Then (10) become fractional Doléans-Dade equation,

$$dS_t = S_t d^\diamond B_t^H,\ S_0 = 1 \tag{15}$$

and its integral form is $\int_0^T dS_t = \int_0^T S_t d^\diamond B_t^H$. The solution is $S_T = \exp^\diamond(B_T^H) = \exp(B_T^H - \frac{1}{2}T^{2H})$ by the fractional Itô formula. Let us discretize time into $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0, T]$ and $t_i - t_{i-1} = \Delta t$ for all index i. Then we obtain a recursive equation of (15)

$$\widehat{S}_n = \widehat{S}_{n-1} \diamond (1 + \Delta B_n^H), \tag{16}$$

where $\widehat{S}_n$ and ΔB_n^H denote $\widehat{S}_{t_n}$ and $B_{t_n}^H - B_{t_{n-1}}^H$, respectively. As we can see in Bender ([4,5]), it is costly to generate sample path because of Wick product term and Malliavin derivative.

Therefore we suggest memoryless discrete recursion scheme which avoids calculating discrete Malliavin derivative;

$$\begin{aligned} \widehat{S}_n &= \widehat{S}_{n-1}(1 + \Delta B_n^H - E(B_{n-1}^H \Delta B_n^H)) \\ &= \widehat{S}_{n-1}(1 + \Delta B_n^H - \frac{1}{2}(t_n^{2H} - t_{n-1}^{2H} - (t_n - t_{n-1})^{2H})) \end{aligned} \tag{17}$$

starting $\widehat{S}_0 = S_0 = 1$.

The motivation of Equation (17) is based on the following rough arguments assuming very regular condition. By property of Wick product, (16) becomes

$$S_n = S_{n-1} + S_{n-1} \diamond \Delta B_n^H$$

and Wick product term is

$$S_{n-1} \diamond \Delta B_n^H = S_{n-1} \diamond \int_0^T f_n(t)\, dB_t^H,$$

where $f_n(t) = \chi_{[t_{n-1},t_n]}$ and χ is characteristic function. By (7)

$$\widehat{S}_{n-1} \diamond \int_0^T f_n(t)\, dB_t^H = \widehat{S}_{n-1} \int_0^T f_n(t)\, dB_t^H - D_{\Phi f_n} S_{n-1} \tag{18}$$

and from (16)

$$\widehat{S}_n = S_0 \diamond (1 + \Delta B_1^H) \diamond (1 + \Delta B_2^H) \diamond \cdots \diamond (1 + \Delta B_n^H) \tag{19}$$

and

$$\begin{aligned} \exp^{\diamond}(\Delta B_k^H) &= \exp(\Delta B_k^H - \frac{1}{2}(\Delta t)^{2H}) \\ &= 1 + (\Delta B_k^H - \frac{1}{2}(\Delta t)^{2H}) + \frac{1}{2}(\Delta B_k^H - \frac{1}{2}(\Delta t)^{2H})^2 + \cdots \\ &\approx 1 + \Delta B_k^H \text{ in } L^2 \end{aligned}$$

for small Δt by (4) and Corollary 2.2.

Then (19) is roughly

$$\begin{aligned} \widehat{S}_{n-1} &\approx S_0(\exp^{\diamond}(\Delta B_1^H)) \diamond (\exp^{\diamond}(\Delta B_2^H)) \diamond \cdots \diamond (\exp^{\diamond}(\Delta B_{n-1}^H)) \\ &= S_0 \exp^{\diamond}(B_{n-1}^H) \end{aligned}$$

for small Δt and by (6)

$$\begin{aligned} D_{\Phi f_n} S_0 \exp^{\diamond}(B_{n-1}^H) &= S_0 \exp^{\diamond}(B_{n-1}^H) \langle f_1 + f_2 + \cdots + f_{n-1}, f_n \rangle_\phi \\ &\approx \widehat{S}_{n-1} \langle f_1 + f_2 + \cdots + f_{n-1}, f_n \rangle_\phi \end{aligned}$$

for small Δt.

Therefore (18) becomes

$$\widehat{S}_{n-1} \diamond \int_0^T f_n(t)\, dB_t^H \approx \widehat{S}_{n-1}(\Delta B_n^H - \langle f_1 + f_2 + \cdots + f_{n-1}, f_n \rangle_\phi)$$

and thus (16) becomes

$$\begin{aligned} \widehat{S}_n &\approx S_{n-1}(1 + \Delta B_n^H - \langle f_1 + f_2 + \cdots + f_{n-1}, f_n \rangle_\phi) \\ &= \widehat{S}_{n-1}(1 + \Delta B_n^H - E(B_{n-1}^H \Delta B_n^H)) \\ &= \widehat{S}_{n-1}(1 + \Delta B_n^H - \frac{1}{2}(t_n^{2H} - t_{n-1}^{2H} - (t_n - t_{n-1})^{2H})) \end{aligned}$$

by (3) and(1) for small Δt.

Now we prove that the recursion scheme (17) has convergence rate H. First, a lemma for kurtosis type estimate of ΔB_t^H is necessary. The following lemma is related to Hörmander theorem for the fractional Brownian motion [1]. In this paper, the lemma is proved differently through the simple properties of the increments ΔB_i^H.

Lemma 2.5. *We have*

$$E[\Big|\sum_i^n(|\Delta B_i^H|^2 - |\Delta t|^{2H})\Big|^2] \le CT\Delta t^{4H-1} \quad \textit{if} \quad \frac{1}{2} < H < \frac{3}{4},$$

$$E[\Big|\sum_i^n(|\Delta B_i^H|^2 - |\Delta t|^{2H})\Big|^2] \le C(T+\log(T+1))\Delta t^2|\log \Delta t| \quad \textit{if} \quad H = \frac{3}{4},$$

$$E[\Big|\sum_i^n(|\Delta B_i^H|^2 - |\Delta t|^{2H})\Big|^2] \le CT^{4H-2}\Delta t^2 \quad \textit{if} \quad \frac{3}{4} < H < 1$$

for a constant C depending only on H.

Proof. We know that

$$E[|\Delta B_i^H|^2] = \Delta t^{2H}$$

and it follows that

$$E[\Big|\sum_i^n(|\Delta B_i^H|^2 - |\Delta t|^{2H})\Big|^2] = E[\Big|\sum_i^n |\Delta B_i^H|^2\Big|^2] - (n\Delta t^{2H})^2,$$

where $\Delta t = \frac{T}{n}$. To compute $E[\big|\sum_i^n |\Delta B_i^H|^2\big|^2]$ we need bivariate normal distribution of ΔB_i and ΔB_j. If we assume $i > j$, the covariance of ΔB_i and ΔB_j is given by

$$E[\Delta B_i \Delta B_j] = \frac{1}{2}((t_{i-1} - t_j)^{2H} + (t_i - t_{j-1})^{2H} - 2(t_i - t_j)^{2H}).$$

Letting $\tau = t_i - t_j$ we have

$$(t_{i-1} - t_j)^{2H} + (t_i - t_{j-1})^{2H} - 2(t_i - t_j)^{2H} = (\tau - \Delta t)^{2H} + (\tau + \Delta t)^{2H} - 2\tau^{2H}.$$

By mean value theorem, we have

$$(\tau - \Delta t)^{2H} + (\tau + \Delta t)^{2H} - 2\tau^{2H} \le C\tau^{2H-2}\Delta t^2 \tag{20}$$

for $\tau \ge \Delta t$. Hence we get

$$|E[\Delta B_i \Delta B_j]| \le C(H)|t_i - t_j|^{2H-2}\Delta t^2 \le C(H)\Delta t^{2H}$$

for a $C(H)$ depending only on H.

Because ΔB_i^H has normal distribution, the bivariate distribution of $x = \Delta B_i^H$ and $y = \Delta B_j^H$ is

$$p(x,y) = \frac{1}{\sqrt{1-\rho^2}2\pi\sigma_x\sigma_y}\exp(\frac{-1}{2(1-\rho^2)}(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2\rho\frac{x}{\sigma_x}\frac{y}{\sigma_y})),$$

where $\sigma_x = \sigma_y = \Delta t^H$ and the correlation

$$\rho = \frac{E[xy]}{\sigma_x \sigma_y} = \frac{1}{2\Delta t^{2H}}((t_{i-1} - t_j)^{2H} + (t_i - t_{j-1})^{2H} - 2(t_i - t_j)^{2H}).$$

Since $E[x^2] = \sigma_x^2$, we have

$$E[|x^2 - \sigma_x^2|^2] = \int_{R^2} (x^2 - \sigma_x^2)^2 p(x, y) dydx = \int_R (x^2 - \sigma_x^2)^2 \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}} dx$$
$$= C\sigma_x^4 = C\Delta t^{4H}.$$

Also the covariance $E[(x^2 - \sigma_x^2)(y^2 - \sigma_y^2)]$ is

$$\begin{aligned} &\int_{R^2} (x^2 - \sigma_x^2)(y^2 - \sigma_y^2) p(x, y) dydx \\ &= 2\rho^2 \sigma_x^2 \sigma_y^2 \\ &= \frac{1}{2}\big((t_{i-1} - t_j)^{2H} + (t_i - t_{j-1})^{2H} - 2(t_i - t_j)^{2H}\big)^2. \end{aligned}$$

By the estimate (20), we have

$$|E[(x^2 - \sigma_x^2)(y^2 - \sigma_y^2)]| \le C(t_i - t_j)^{4H-4} \Delta t^4.$$

Therefore we have

$$\begin{aligned} E[\big|\sum_i^n (|\Delta B_i^H|^2 - |\Delta t|^{2H})\big|^2] &= \sum_{i=1}^n \sum_{j=1}^n E[(|\Delta B_i^H|^2 - \Delta t^{2H})(|\Delta B_j^H|^2 - \Delta t^{2H})] \\ &\le C \sum_{i=1}^n \sum_{j=1}^n |t_i - t_j|^{4H-4} \Delta t^4 \\ &= C\Delta t^{4H} \sum_{i=1}^n \sum_{j=1}^n |i - j|^{4H-4}. \end{aligned}$$

If $\frac{1}{2} < H < \frac{3}{4}$, we have

$$\sum_{i=1}^n \sum_{j=1}^n |i - j|^{4H-4} \le C(H)n = C(H)T\Delta t^{-1}$$

and

$$E[\big|\sum_i^n (|\Delta B_i^H|^2 - |\Delta t|^{2H})\big|^2] \le C(H)T\Delta t^{4H-1}.$$

If $\frac{3}{4} < H < 1$, we have

$$\sum_{i=1}^n \sum_{j=1}^n |i - j|^{4H-4} \le C(H)n^{4H-2} = C(H)T^{4H-2}\Delta t^{2-4H}$$

and

$$E[\big|\sum_i^n (|\Delta B_i^H|^2 - |\Delta t|^{2H})\big|^2] \le C(H)T^{4H-2}\Delta t^2.$$

Also when $H = \frac{3}{4}$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|i-j|^{4H-4} \leq Cn\log(n) = -CT(1+\log(T+1))\Delta t^{-1}|\log \Delta t|$$

and

$$E[\big|\sum_{i}^{n}(|\Delta B_i^H|^2 - |\Delta t|^{2H})\big|^2] \leq C(H)T(1+\log(T+1))\Delta t^2|\log \Delta t|.$$

□

Lemma 2.6. *Let*

$$S_T = \exp(B_T^H - \frac{1}{2}T^{2H})$$

and

$$\widehat{S}_T = \prod_{i=1}^{n}(1+\Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H})),$$

and A be the event such that $1 + \Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H}) < 0$ for some i. Then the expectation

$$E[I_A|S_T - \widehat{S}_T|] \leq C(\Delta t)^s \text{ for any } s > 1, \tag{21}$$

where I_A is indicator function of event A.

Proof. Let A_i be the event such that $1+\Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H}) < 0$. Then $P(A) \leq \sum_{i=1}^{n} P(A_i)$ and

$$P(A_i) = \int_{-\infty}^{-1+\frac{1}{2}(t_i^{2H}-t_{i-1}^{2H}-(t_i-t_{i-1})^{2H})} \frac{1}{\sqrt{2\pi}(\Delta t)^H}\exp(-\frac{x^2}{2(\Delta t)^{2H}})dx.$$

By changing variable $y = \frac{x}{(\Delta t)^H}$,

$$P(A_i) = \int_{-\infty}^{-k}\frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})dy = \int_{k}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})dy,$$

where $k = (1 - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H}))/(\Delta t)^H$. By following inequality

$$\int_k^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})dy \leq \frac{1}{k}\int_k^{\infty}\frac{y}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})dy = \frac{\exp(-\frac{k^2}{2})}{k\sqrt{2\pi}},$$

$$P(A_i) \leq C(\Delta t)^H \exp(-\frac{(\Delta t)^{-2H}}{2}). \tag{22}$$

Therefore

$$P(A) \leq CT(\Delta t)^{H-1}\exp(-\frac{(\Delta t)^{-2H}}{2}) \tag{23}$$

and

$$E[I_A|S_T - \widehat{S}_T|] \leq E[I_A|S_T|] + E[I_A|\widehat{S}_T|].$$

By Hölder's inequality and equation (23), we have

$$E[I_A|S_T|] = \int_A |S_T|(\omega)dP(\omega) \leq \sqrt{P(A)}\sqrt{\int_A |S_T|^2 dP(\omega)}$$
$$\leq \sqrt{P(A)}\sqrt{\int_\Omega |S_T|^2 dP(\omega)} \leq \widetilde{C}\sqrt{(\Delta t)^{H-1}\exp(-\frac{(\Delta t)^{-2H}}{2})}.$$

The last inequality follows from $E|S_T|^2 = \int_\Omega |S_T|^2 dP(\omega) < \infty$. Similarly,

$$E[I_A|\widehat{S}_T|] = \int_A |\widehat{S}_T|(\omega)dP(\omega) = \int_A |\prod_{i=1}^n X_i(\omega)|dP(\omega),$$

where $X_i = 1+\Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H})$. Then by Hölder's inequality,

(24)
$$\int_A |\prod_{i=1}^n X_i(\omega)|dP(\omega) \leq \Big(\int_A |X_1|^n dP\Big)^{\frac{1}{n}}\Big(\int_A |X_2|^n dP\Big)^{\frac{1}{n}} \cdots \Big(\int_A |X_n|^n dP\Big)^{\frac{1}{n}}$$
$$\leq \Big(\sqrt{P(A)}\sqrt{\int_A |X_1|^{2n} dP}\Big)^{\frac{1}{n}}\Big(\sqrt{P(A)}\sqrt{\int_A |X_2|^{2n} dP}\Big)^{\frac{1}{n}}$$
$$\cdots \Big(\sqrt{P(A)}\sqrt{\int_A |X_n|^{2n} dP}\Big)^{\frac{1}{n}}$$
$$\leq \sqrt{P(A)}\Big(\int_\Omega |X_1|^{2n} dP\Big)^{\frac{1}{2}}$$

since X_i have same distribution on sample space Ω. Also last term

$$\int_\Omega |X_1|^{2n} dP \leq \int_\Omega |1 + \Delta B_1^H|^{2n} dP.$$

By the Minkowski inequality,

$$\int_\Omega |1 + \Delta B_1^H|^{2n} dP \leq (1 + (\int_\Omega |\Delta B_1^H|^{2n} dP)^{\frac{1}{2n}})^{2n}$$
$$= (1 + ((\Delta t)^{2nH}(2n-1)!!)^{\frac{1}{2n}})^{2n}$$
$$= (1 + (\Delta t)^H((2n-1)!!)^{\frac{1}{2n}})^{2n}$$

since $\Delta B_1^H \sim N(0, (\Delta t)^{2H})$, $E[(\Delta B_1^H)^{2n}] = (\Delta t)^{2nH}(2n-1)!!$.

By Stirling's approximation, $(2n-1)!! \leq Cn^n(2/e)^n$ and thus $1+(\Delta t)^H((2n-1)!!)^{\frac{1}{2n}} = 1 + \widetilde{C}(T/n)^H\sqrt{n} = 1 + \widehat{C}n^{\frac{1}{2}-H}$ for some constants $\widetilde{C}$ and $\widehat{C}$. For $\frac{1}{2} < H < 1$, we have $\frac{1}{2} - H < 0$ and it follows that

$$(\int_\Omega |X_1|^{2n} dP)^{\frac{1}{2}} \leq (1 + \widehat{C}n^{\frac{1}{2}-H})^n \leq \exp(\widehat{C}n^{\frac{3}{2}-H}).$$

Therefore from this inequality and (23), the equation (24)

$$\sqrt{P(A)}(\int_\Omega |X_1|^{2n}dP)^{\frac{1}{2}} \leq \sqrt{CT}(\Delta t)^{(H-1)/2}\exp(-\frac{(\Delta t)^{-2H}}{4})\exp(\widehat{C}n^{\frac{3}{2}-H})$$
$$= C_1 n^{\frac{1-H}{2}}\exp(-C_2 n^{2H} + C_3 n^{\frac{3}{2}-H})$$

for some positive constant C_1, C_2, C_3 depending on T, H. For our case $\frac{1}{2} < H < 1$, $2H > \frac{3}{2} - H$ and so $E[I_A|\widehat{S}_T|]$ goes to zero faster than polynomials of any degree for large n.

Therefore $E[I_A|S_T - \widehat{S}_T|]$ goes to zero faster than polynomials of any degree for small Δt where A is the event such that $1 + \Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H}) < 0$ for some i. □

By Lemma 2.6, we can neglect the event A in the following main theorem.

Theorem 2.7. *There is a constant C depending only on H such that*

$$E[|S_T - \widehat{S}_T|] \leq C(T+1)e^{T^{2H}}\Delta t^H.$$

Proof. By Equation (17)

$$\widehat{S}_T = \prod_{i=1}^n (1 + \Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H}))$$

and

$$E[|S_T - \widehat{S}_T|] = E[I_A|S_T - \widehat{S}_T|] + E[I_{A^c}|S_T - \widehat{S}_T|].$$

By Lemma 2.6, $E[I_A|S_T - \widehat{S}_T|]$ goes to zero faster than polynomials of any degree for small Δt. In case A^c, $1 + \Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H})$ is nonnegative and thus taking log gives

$$I_{A^c}\log(\widehat{S}_T) = \sum_{i=1}^n \log(1 + \Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H})).$$

Using the expansion $\log(1+\epsilon) = \epsilon - \epsilon^2/2 + \cdots$,

$$\begin{aligned} I_{A^c}\log(\widehat{S}_T) = &\sum_{i=1}^n (\Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H}) \\ &- \frac{1}{2}((\Delta B_i^H)^2 - \Delta B_i^H(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H}) \\ &+ \frac{1}{4}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H})^2) + \cdots). \end{aligned}$$

Note that

$$\sum_{i=1}^n \Delta B_i^H - \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H}) = B_T^H - \frac{1}{2}T^{2H} = \log S_T.$$

Hence it follows that

$$\begin{aligned} I_{A^c}|\log(\widehat{S}_T) - \log S_T| \le & \left|\frac{1}{2}\sum_{i=1}^{n}(\Delta B_i^H)^2 - (t_i - t_{i-1})^{2H}\right| \\ & + \sum_{i=1}^{n}|\Delta B_i^H(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H})| \\ & + \sum_{i=1}^{n}\frac{1}{4}(t_i^{2H} - t_{i-1}^{2H} - (t_i - t_{i-1})^{2H})^2) + C\Delta t \\ =& I + II + III + C(T+1)\Delta t \end{aligned}$$

and

$$E[I_{A^c}|\log(\widehat{S}_T) - \log S_T|^2] \le 2E[I^2] + 2E[II^2] + 2E[III^2] + C(T+1)^2\Delta t^2$$

for a C. By Lemma 2.5 and $4H - 1 \ge 2H$, we have

$$E[I^2] \le C(T+1)^2\Delta t^{2H}$$

and

$$E[III^2] \le C(T+1)^2\Delta t^2.$$

Finally by Cauchy-Schwarz inequality, we have

$$E[II^2] \le E[\sum_{i=1}^{n}(\Delta B_i^H)^2]\sum_{i=1}^{n}\Delta t^2 \le C(T+1)^2\Delta t^{2H}.$$

Therefore

$$E[I_{A^c}|\log S_T - \log \widehat{S}_T|^2] \le C(T+1)^2\Delta t^{2H}.$$

If we let $a = \log S_T$ and $b = \log \widehat{S}_T$, $S_T = e^a$ and $\widehat{S}_T = e^b$, and

$$\begin{aligned} I_{A^c}|\widehat{S}_T - S_T| &= I_{A^c}|\int_0^1 e^{ra+(1-r)b}dr(a-b)| \le I_{A^c}\int_0^1 re^a + (1-r)e^b dr|a-b| \\ &= \int_0^1 rS_T + (1-r)\widehat{S}_T dr(I_{A^c}|\log S_T - \log \widehat{S}_T|). \end{aligned}$$

Thus, it follows that from Cauchy-Schwarz inequality

$$E[I_{A^c}|\widehat{S}_T - S_T|] \le \left(\frac{1}{2}(E[S_T^2] + E[\widehat{S}_T^2]\right)^{\frac{1}{2}} E[I_{A^c}|\log S_T - \log \widehat{S}_T|^2]^{\frac{1}{2}}.$$

Since B_T^H has normal distribution $N(0, T^{2H})$, we have

$$E[S_T^2] = \int_R e^{2x - T^{2H}}\frac{1}{\sqrt{\pi}T^H}e^{-\frac{x^2}{2T^{2H}}} = Ce^{T^{2H}}.$$

We have

$$\widehat{S}_T^2 \le \left(\Pi_{i=1}^n(1 + \Delta B_i^H))\right)^2 \le e^{2\sum_{i=1}^n \log(1+\Delta B_i^H)} \le e^{2B_T^H}$$

and

$$E[\widehat{S}_T^2] \le Ce^{2T^{2H}}.$$

Therefore we conclude that there is a constant C depending only on H such that

$$E[I_{A^c}|\widehat{S}_T - S_T|] \leq C(T+1)e^{T^{2H}}\Delta t^H.$$

Since $E[I_{A^c}|\widehat{S}_T - S_T|]$ dominate $E[I_A|S_T - \widehat{S}_T|]$ for small Δt, we obtain

$$E[|\widehat{S}_T - S_T|] \leq C(T+1)e^{T^{2H}}\Delta t^H. \qquad \square$$

If $H = \frac{1}{2}$, then our recursive Equation (17) becomes Euler-Maruyama method and it's known that Euler-Maruyama method has order $\frac{1}{2}$ strong convergence. See Kloeden and Platen [17].

We can extend Equation (17) to general case (10), i.e, $\mu \neq 0$, $\sigma \neq 1$ case. Then we can obtain discrete asset model of the fractional Black-Scholes market (10) as

$$\widehat{S}_n = \widehat{S}_{n-1}(1+\mu(t_n - t_{n-1}) + \sigma\Delta B_n^H - \sigma^2\frac{1}{2}(t_n^{2H} - t_{n-1}^{2H} - (t_n - t_{n-1})^{2H})). \tag{25}$$

3. Numerical result

As in the well-known case of classical geometric Brownian motion, we generate a recursive multiplicative path starting with a value S_0 using recursion equation (25) in the fractional Black-Scholes market.

First we have to generate fractional Brownian motion B_n^H. Sottinen[26] constructed approximation procedure of fractional Brownian motion as binary random walk. In this paper we generate fractional Brownian motion through the FFT method (see [16]). Using FFT method we can generate fractional Brownian motion even faster.

We first simulate a path of stock price (S_t^H) with $\mu = 0.02$, $\sigma = 0.3$ $H = 0.7$ and initial price $S_0 = 100$ in the fractional Brownian market over time interval [0 T] by (25)(step size $t_k - t_{k-1} = \frac{T}{n}$). We then plot exact solution (13) in Figure 1.

If Wick product is replaced with ordinary product in (14) the recursion law is just

$$S_n = S_{n-1}(1 + \mu\Delta t + \sigma\Delta B^H) \tag{26}$$

and it converges to (12) for small Δt. Figure 2 shows the stock paths based on Wick product and on the ordinary product. We can observe the Wick product effectuates a correction of the values generated by pathwise multiplication.

The fractional Brownian motion has memory effect for $H \neq \frac{1}{2}$. In Figure 3 the early increase effect of B^H persists in the corresponding path B^H and S^H. For the smaller Hurst parameter, early increase of B^H effects more on the after-path behavior of B^H and S^H at the neighbor time steps. For the larger Hurst parameter the early increase is less influential at the beginning. However the tendency to increase persists up to the later time steps. This comes from the memory effect of non-Markovian process and the Hurst parameter is connected to how the past path affects future path.

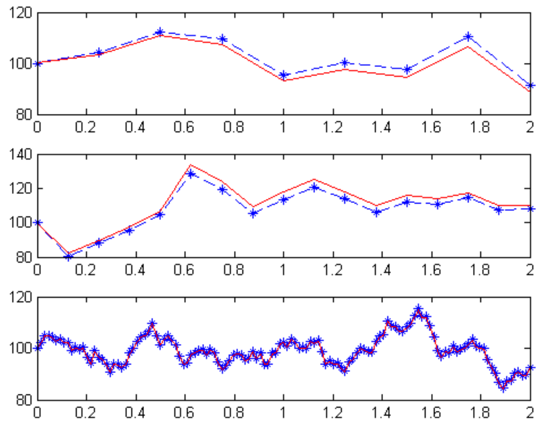

FIGURE 1. Sample path of equation (25) (dashed asterisk) and exact solution (13) (solid line); from the top to the bottom for $n = 8$, $n = 16$, $n = 128$.

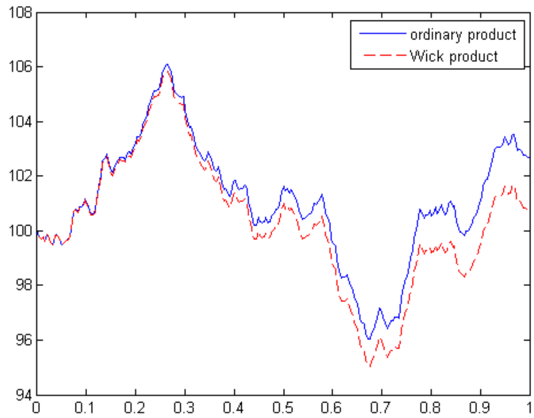

FIGURE 2. Sample paths of the geometric fractional Brownian motion for $H = 0.8$, $n = 256$, $T = 1$, $\mu = 0.02$, $\sigma = 0.2$ based on the Wick product (dashed line) and on the ordinary product (solid line).

We now examine, how well the distribution of S_n^H in (25) approximates the distribution of the $S_0 \exp(\mu T - \frac{1}{2}\sigma^2 T^{2H} + \sigma B_T^H)$, which has log-normal distribution. We generate sample paths through algorithm (25) $M = 100, 1000, 10000$

times with $n = 128$. Then we draw histogram of S_n. Figure 4 shows the histogram of S_n with $H = 0.7$.

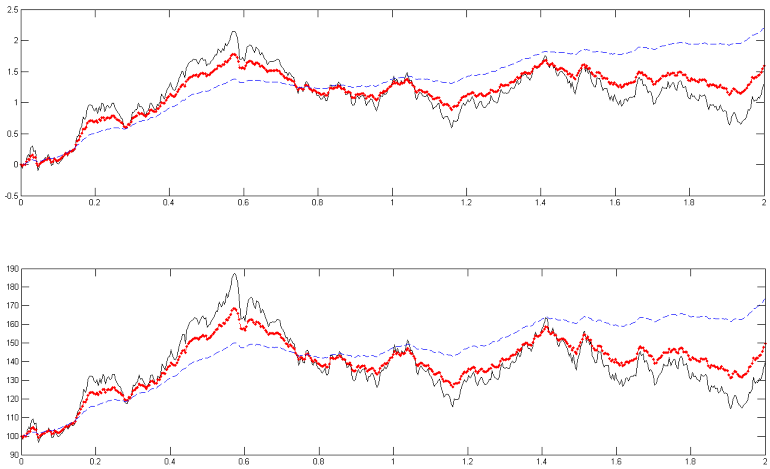

FIGURE 3. Top graph is fractional Brownian path through FFT method with Hurst parameters $H = 0.5$ (solid line), 0.65 (dot line), and 0.85 (dashed line). Bottom graph is corresponding geometric fractional Brownian path through Equation (25).

We finally evaluate European option by Monte Carlo method through path algorithm (25). Suppose first that stock price is governed by (10), i.e., fractional Brownian market. Then the value of the European call option at time 0 is

$$C(0, S_0) = S_0\Phi(y_+) - Ke^{-rT}\Phi(y_-), \tag{27}$$

where r is the interest rate, Φ the $N(0,1)$ cumulative distribution function and K is the exercise price and

$$y_\pm = \frac{\log\frac{S_0}{K} + rT \pm \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H}. \tag{28}$$

See Hu and Øksendal [15].

In Figure 5, option value$(E[e^{-rT}\max(S(T) - K, 0)])$ is obtained by Monte Carlo method using (25) in $n = 128, T = 0.75, S0 = 100, K = 100, r = 0.02, \sigma = 0.2, H = 0.8$, sampling number 10^2 to 10^5. The exact value is 7.0571.

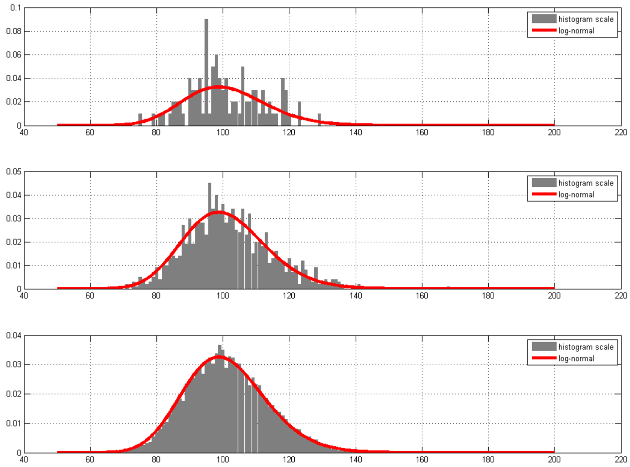

FIGURE 4. Histogram of $\widehat{S}_n$ through (25) for $n = 128$, $T = 0.5$, $S0 = 100$, $\mu = 0.02$, $\sigma = 0.2$, $H = 0.7$ of 100, 1000, 10000 sampling from the top to the bottom and the exact log-normal density.

4. Conclusion

Our approximation of the geometric fractional Brownian motion makes use of directly generated fractional Brownian motion through FFT, not using construction of binary random walk. Then Malliavin derivative term can be changed to explicit expectation of past fractional Brownian motion. The discrete Wick product of binary random walk cannot be calculated recursively as we note in Bender [3] and it needs to keep a large amount of memories of past paths to proceed one more step. In this paper we introduced an algorithm using explicit expectation of past fractional Brownian motion recursively. It is Malliavin type correction of pathwise multiplication.

Through the path generation of the geometric factional Brownian motion, Markovian market will be extended to the non-Markovian market and more delicate hedge will be possible in the fractional Brownian market.

5. Further work

In this paper, the generating method is applicable to the fractional Doléans-Dade SDE. with the geometric fractional Brownian motion. But one can think about more general Wick fractional SDEs. Similar general Malliavin concept

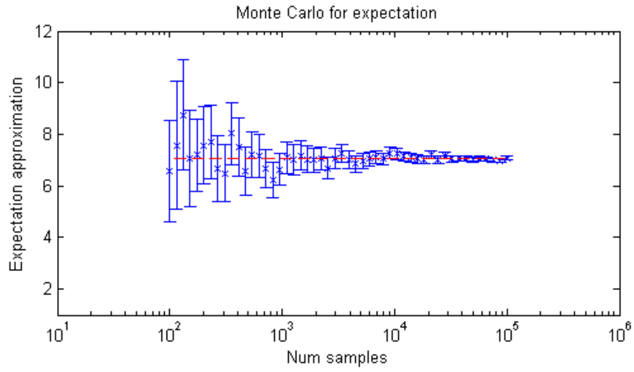

FIGURE 5. Monte Carlo approximations to a European call option value. Crosses are the approximation, vertical bars give computed 95% confidence intervals. Horizontal dashed line is at height give by (27), (28) in $n = 128$, $T = 0.75$, $S0 = 100$, $K = 100$, $r = 0.02$, $\sigma = 0.2$, $H = 0.8$, sampling number 10^2 to 10^5. The exact value is 7.0571 (for $H = 0.5$ the value is 7.6199).

help to derive the conjecture of derivation of approximation with Wick-free version. The convergence of the approximation in general Wick fractional SDEs will be remarkable issue at the numerical SDE theory. It will need more complex non-semimartingale stochastic calculus and general Malliavin calculus and white noise theory.

There is a rich limit theory like the *Malliavin-Stein* approach [19] recently. In the Main Theorem 2.7, it will be possible to find the right sequence $a_n \to \infty$ as $n \to \infty$ in such way

$$a_n(S_T - \hat{S}_{T,n}) \xrightarrow{law} S_\infty$$

where S_∞ is a non-degenerate random variable. If we find the $a_n \to \infty$, we could quantify the error in the approximation.

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