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ON CONFORMAL TRANSFORMATIONS BETWEEN TWO ALMOST REGULAR (α, β)-METRICS

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Abstract. In this paper, we characterize the conformal transformations between two almost regular (α, β)-metrics. Suppose that F is a non-Riemannian (α, β)-metric and is conformally related to $\tilde{F}$, that is, $\tilde{F} = e^{\kappa(x)}F$, where $\kappa := \kappa(x)$ is a scalar function on the manifold. We obtain the necessary and sufficient conditions of the conformal transformation between F and $\tilde{F}$ preserving the mean Landsberg curvature. Further, when both F and $\tilde{F}$ are regular, the conformal transformation between F and $\tilde{F}$ preserving the mean Landsberg curvature must be a homothety.

1. Introduction

In Finsler geometry, the Weyl theorem states that the projective and conformal properties of a Finsler space determine the metric properties uniquely ([14, 16]). Therefore the conformal properties of a Finsler metric deserve extra attention. The study of conformal geometry is a recent popular trend in Finsler geometry. Let F and $\tilde{F}$ be two Finsler metrics on a manifold M. The *conformal transformation* between F and $\tilde{F}$ is defined by $L: \ F \to \tilde{F}, \ \tilde{F} = e^{\kappa(x)}F$, where $\kappa := \kappa(x)$ is a scalar function on M. We call such two metrics F and $\tilde{F}$ are *conformally related*. A Finsler metric which is conformally related to a Minkowski metric is called *conformally flat* Finsler metric.

In conformal geometry, it is one important problem how to characterise conformally flat Finsler metrics. M. Hashuiguchi and Y. Ichijyō defined a conformally invariant linear connection in a Finsler space with an (α, β)-metric and gave a condition that a Randers metric is conformally flat based on their connection ([11]). Later, S. Kikuchi found a conformally invariant Finsler connection and gave a necessary and sufficient condition for a Finsler metric to be conformally flat by a system of partial differential equations under an extra condition ([13]). But people are unable to know the local structure of conformal flat Finsler metrics by those results. In [12], L. Kang has proved that

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any conformally flat Randers metric of scalar flag curvature is projectively flat and classified completely such metrics. The first author and X. Y. Cheng have proved that, if a conformally flat (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ on a manifold M with the dimension $n \geq 3$ is a weak Einstein metric, then it is either a locally Minkowski metric or a Riemannian metric, where $\phi(s)$ is a polynomial in s ([3]). They have also characterized conformally flat (α, β)-metrics with isotropic S-curvature (See [3]). Further, the first author, Q. He and Z. Shen prove that conformally flat (α, β)-metrics with constant flag curvature must be either a locally Minkowski metric or a Riemannian metric ([5]). However, it is unfortunate that the local structure of conformal flat Finsler metrics is still unknown, even if conformal flat (α, β)-metrics.

There is the other important problem that, given a Finsler metric on a manifold M, we would like to determine all Finsler metrics which are conformally related to the given one. In [1], X. Y. Cheng and S. Bácsó characterized the conformal transformations which preserve Riemann curvature, Ricci curvature, (mean) Landsberg curvature and S-curvature respectively. In particular, they proved that, if the conformal transformation $\tilde{F}(x, y) = e^{\kappa(x)}F(x, y)$ preserves the geodesics, then it must be a homothety, that is, κ =constant. Recently, the first author, X. Cheng and Y. Zou characterize the conformal transformations between two regular (α, β)-metrics. They prove that if both conformally related (α, β)-metrics F and $\tilde{F}$ are Douglas metrics, then the conformal transformation between them is a homothety ([4]).

The (α, β)-*metrics* are those Finsler metrics which are defined by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a 1-form $\beta = b_i(x)y^i$ on an n-dimensional manifold M. They are expressed in the form

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)$ is a C^∞ positive function on $(-b_0, b_0)$. It is known that $F = \alpha\phi(\beta/\alpha)$ is a positive definite Finsler metric for any α and β with $\|\beta\|_\alpha < b_0$ if and only if ϕ satisfies the following ([2])

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0.$$

For the above function $\phi = \phi(s)$, if we consider a 1-form β with $\|\beta\|_\alpha \leq b_0$, then $F = \alpha\phi(\beta/\alpha)$ might be singular at a point x with $b(x) = b_0$. Such metrics are called almost regular (α, β)-metrics.

In this paper, we are mainly concerned with the conformal transformations between two almost regular (α, β)-metrics. In the following, we assume that both F and $\tilde{F}$ are non-Riemannian. We can prove:

Theorem 1.1. *Let F and $\tilde{F}$ be two conformally related almost regular non-Riemannian (α, β)-metrics on a manifold M of dimension $n \geq 3$. Let $\Delta := 1 + sQ + (b^2 - s^2)Q'$ and $Q := \phi'/(\phi - s\phi')$. Then both F and $\tilde{F}$ have the same weak Landsberg curvature if and only if one of the following holds*:

(1) *The conformal transformation between F and $\tilde{F}$ is a homothety, regardless of the choice of a particular ϕ.*
(2) *The conformal factor $\kappa(x)$ satisfies that $\kappa_i(x)$ is proportional to $b_i(x)$, where $\kappa_i(x) := \frac{\partial \kappa}{\partial x_i}(x)$ and ϕ satisfies*

$$(Q - sQ')\{n\Delta + 1 + sQ\} + (b^2 - s^2)(1 + sQ)Q'' = \frac{\lambda}{\sqrt{b^2 - s^2}}\Delta^{\frac{3}{2}}, \tag{1.1}$$

where λ is a constant.

By Theorem 1.1, it is easy to obtain the following corollary.

Corollary 1.2. *Let F and $\tilde{F}$ be two conformally related almost regular non-Riemannian (α, β)-metrics on a manifold M of dimension $n \geq 3$. Assume that F is weak Landsberg metric. Then $\tilde{F}$ is also weak Landsberg metric if and only if the following holds*:

(1) *The conformal transformation between F and $\tilde{F}$ is a homothety, regardless of the choice of a particular ϕ.*
(2) *The conformal factor $\kappa(x)$ satisfies that $\kappa_i(x)$ is proportional to $b_i(x)$ and ϕ satisfies* (1.1).

Further, we also get:

Corollary 1.3. *Let F be a conformally flat almost regular non-Riemannian (α, β)-metrics on a manifold M of dimension $n \geq 3$. Then F is weak Landsberg metric if and only if the following holds*:

(1) *F is a locally Minkowski metric, regardless of the choice of a particular ϕ.*
(2) *The conformal factor $\kappa(x)$ satisfies that $\kappa_i(x)$ is proportional to $b_i(x)$ and ϕ satisfies* (1.1).

2. Preliminaries

For a given Finsler $F = F(x, y)$, the geodesics of F are characterized locally by a system of 2nd ODEs as follows ([10]):

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big\},$$

and $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$. G^i are called the *geodesic coefficients* of F.

There are many interesting quantities in Finsler geometry which vanish in Riemann geometry. We call them non-Riemannian quantities. For a non-zero vector $y \in T_pM$, the Cartan torsion $\mathbf{C_y} = C_{ijk}dx^i \otimes dx^j \otimes dx^k : T_pM \otimes TpM \otimes TpM \longrightarrow \mathbb{R}$ is defined by

$$C_{ijk} := \frac{1}{4}[F^2]_{y^i y^j y^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}(x, y).$$

The mean Cartan torsion $\mathbf{I_y} = I_i(x,y)dx^i : T_pM \longrightarrow \mathbb{R}$ is defined by

$$I_i := g^{jk}C_{ijk},$$

where $(g^{ij}) := (g_{ij})^{-1}$. It is obvious that $C_{ijk} = 0$ if and only if F is Riemannian. According to Deicke's theorem ([17]), a Finsler metric is Riemannian if and only if the mean Cartan torsion vanishes.

The Landsberg curvature $\mathbf{L} := L_{ijk}dx^i \otimes dx^j \otimes dx^k$ and the mean Landsberg curvature $\mathbf{J} := J_i dx^i$ are defined respectively by

$$L_{ijk} := -\frac{1}{2}FF_{y^m}\frac{\partial G^m}{\partial y^i y^j y^k}, \quad J_i := g^{jk}L_{ijk}.$$

Finsler metrics with $(\mathbf{J} = 0)\mathbf{L} = 0$ are called (weak)Landsberg metrics.

Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$b^i := a^{ij}b_j, \ s_i := b^j s_{ji}, \ s^i{}_j := a^{il}s_{lj}, \ r_i := b^l r_{li},$$
$$s_0 := s_i y^i, \ s^i{}_0 := s^i{}_j y^j, \ r_{00} := r_{ij}y^i y^j,$$

where "$|$" denotes the horizontal covariant derivative with respect to α. Consider (α,β)-metrics $F = \alpha\phi(s), \ s = \beta/\alpha$ on a manifold. Let G^i and G^i_α denote the spray coefficients of F and α, respectively, then we have [10]

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \{-2Q\alpha s_0 + r_{00}\}\{\Psi b^i + \Theta\alpha^{-1}y^i\}, \tag{2.1}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \ \Theta := \frac{Q - sQ'}{2\Delta}, \ \Psi := \frac{Q'}{2\Delta}. \tag{2.2}$$

Put

$$y_i := a_{ij}y^j, \ h_i := \alpha b_i - s y_i,$$
$$\Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'',$$
$$\Psi_1 := \sqrt{b^2 - s^2}\,\Delta^{\frac{1}{2}}\Big[\frac{\sqrt{b^2 - s^2}\,\Phi}{\Delta^{\frac{3}{2}}}\Big]', \ \Psi_2 := 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta}.$$

By a direct computation, we obtain the following formula about the mean Cartan torsion of (α,β)-metrics [9]

$$I_i := -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}h_i. \tag{2.3}$$

By Deicke's theorem, an (α,β)-metric is Riemannian if and only if $\Phi \equiv 0$.

Further, the mean Landsberg curvature of an (α,β)-metric is given by [15]

$$\begin{aligned} J_i = \ & -\frac{1}{2\Delta\alpha^4}\Big\{\frac{2\alpha^2}{b^2 - s^2}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](r_0 + s_0)h_i \\ & + \frac{\alpha}{b^2 - s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_i + \alpha\big[-\alpha Q' s_0 h_i + \alpha Q(\alpha^2 s_i - y_i s_0) \\ & + \alpha^2\Delta s_{i0} + \alpha^2(r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0)y_i\big]\frac{\Phi}{\Delta}\Big\}. \end{aligned} \tag{2.4}$$

Contracting J_i with b^i, we obtain

$$\bar{J} := J_i b^i = -\frac{1}{2\Delta\alpha^2}\{\Psi_1(r_{00} - 2\alpha Q s_0) + \alpha\Psi_2(r_0 + s_0)\}. \tag{2.5}$$

3. The proof of Theorem 1.1

Let $F = \alpha\phi(s), s = \beta/\alpha$ be an (α, β)-metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Assume that $\tilde{F}$ is conformally related to F on M, $\tilde{F} = e^{\kappa(x)}F$. It is easy to see that $\tilde{F} = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$ is also an (α, β)-metric, where $\tilde{\alpha} = e^{\kappa(x)}\alpha$, $\tilde{\beta} = e^{\kappa(x)}\beta$. Write $\tilde{\alpha} = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$, $\tilde{\beta} = \tilde{b}_i(x)y^i$. Then $\tilde{a}_{ij} = e^{2\kappa(x)}a_{ij}$, $\tilde{b}_i = e^{\kappa(x)}b_i$. Further, we have ([8])

$$\tilde{b}_{j||k} = e^{\kappa(x)}\Big(b_{j|k} - b_k\kappa_j + f a_{jk}\Big). \tag{3.1}$$

Here $\tilde{b}_{j||k}$ denote the covariant derivative of $\tilde{b}_j$ with respect to $\tilde{\alpha}$ and $f := \kappa_m b^m$.

In the following, we always use symbols with tilde and corresponding indices to denote the corresponding quantities of the metric $\tilde{F}$. First, we have:

Proposition 3.1. *Let F and $\tilde{F}$ be two conformally related almost regular (α, β)-metrics on a manifold M of dimension $n \geq 3$. If F and $\tilde{F}$ have the same weak Landsberg curvature, then the conformal transformation between F and $\tilde{F}$ is a homothety or ϕ satisfies* (1.1).

Proof. By (2.5), we have

$$\bar{\tilde{J}} := \tilde{J}_i \tilde{b}^i = -\frac{1}{2\Delta\tilde{\alpha}^2}\{\Psi_1(\tilde{r}_{00} - 2\tilde{\alpha}Q\tilde{s}_0) + \tilde{\alpha}\Psi_2(\tilde{r}_0 + \tilde{s}_0)\}. \tag{3.2}$$

It follows from (3.1) that

$$\begin{aligned}
\tilde{r}_{ij} &= e^{\kappa(x)}[r_{ij} - \frac{1}{2}(\kappa_i b_j + \kappa_j b_i) + f a_{ij}] \\
\tilde{s}_{ij} &= e^{\kappa(x)}[s_{ij} - \frac{1}{2}(\kappa_i b_j - \kappa_j b_i)] \\
\tilde{r}_i &= r_i - \frac{1}{2}(b^2\kappa_i - f b_i) \\
\tilde{s}_i &= s_i - \frac{1}{2}(f b_i - b^2\kappa_i).
\end{aligned} \tag{3.3}$$

Using (3.3), by a direction computation, it is easy to obtain that

$$e^{\kappa(x)}\bar{\tilde{J}} - \bar{J} = -\frac{\Psi_1}{2\Delta\alpha^2}[\alpha f(1 + sQ) - (s + b^2 Q)\kappa_0],$$

where $\kappa_0 := \kappa_i y^i$. By assumption, we have $\bar{\tilde{J}} = \tilde{J}_i\tilde{b}^i = J_i\tilde{b}^i = e^{-\kappa(x)}J_i b^i = e^{-\kappa(x)}\bar{J}$. Thus one has

$$\Psi_1[\alpha f(1 + sQ) - (s + b^2 Q)\kappa_0] = 0. \tag{3.4}$$

It is obvious that $\Psi_1 = 0$ from (3.4), i.e., ϕ satisfies (1.1), or

$$\alpha f(1+sQ) - (s+b^2Q)\kappa_0 = 0. \tag{3.5}$$

In the next step, we prove that the conformal transformation between F and $\tilde{F}$ is a homothety by (3.5). To simplify the computations, take an orthonormal basis at x with respect to α such that

$$\alpha = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \qquad \beta = by^1.$$

Further, we take the following coordinate transformation ([7]) in T_xM, $\psi : (s, u^A) \to (y^i)$:

$$y^1 = \frac{s}{\sqrt{b^2-s^2}}\bar{\alpha}, \qquad y^A = u^A,$$

where $\bar{\alpha} = \sqrt{\sum_{i=2}^{n}(u^A)^2}$. Here, our index conventions are

$$1 \leq i, j, k, \ldots \leq n, \quad 2 \leq A, B, C, \ldots \leq n.$$

We have

$$\alpha = \frac{b}{\sqrt{b^2-s^2}}\bar{\alpha}, \qquad \beta = \frac{bs}{\sqrt{b^2-s^2}}\bar{\alpha}.$$

Then, from (3.5), we have

$$\frac{b^2\bar{\alpha}}{\sqrt{b^2-s^2}}(1+sQ)\kappa_1 - (s+b^2Q)(\frac{s\bar{\alpha}}{\sqrt{b^2-s^2}}\kappa_1 + \bar{\kappa}_0) = 0. \tag{3.6}$$

The above equation is equivalent to the following two equations.

$$(b^2-s^2)\kappa_1 = 0, \tag{3.7}$$

$$(s+b^2Q)\kappa_A = 0. \tag{3.8}$$

If $s + b^2Q = 0$, we can easily get $\phi(s) = a_0\sqrt{b^2-s^2}$, where a_0 is a constant. In this case, it is clear that $\phi - s\phi' + (b^2-s^2)\phi'' = 0$. Then (α,β)-metric F is not a positive definite Finsler metric. Thus (3.8) implies $\kappa_A = 0$. By (3.7), one has $\kappa_1 = 0$. Hence $\kappa_i = 0$, i.e., the conformal transformation between F and $\tilde{F}$ is a homothety. □

Further, we have:

Proposition 3.2. *Let F and $\tilde{F}$ be two conformally related almost regular (α,β)-metrics on a manifold M of dimension $n \geq 3$. If F and $\tilde{F}$ have the same weak Landsberg curvature and ϕ satisfies* (1.1)*, then the conformal factor $\kappa(x)$ satisfies that $\kappa_i(x)$ is proportional to $b_i(x)$.*

Proof. From (2.4) and (3.3), by a series of direct computations, we have

$$\begin{aligned} J_i - \tilde{J}_i = & -\frac{1}{2\Delta\alpha^4}\{\frac{\alpha}{b^2-s^2}[\Psi_1 + s\frac{\Phi}{\Delta}][f\alpha^2 - \beta\kappa_0 + \alpha Q(f\beta - b^2\kappa_0)]h_i \\ & + \alpha[\frac{1}{2}\alpha Q'(f\beta - b^2\kappa_0)h_i - \frac{1}{2}\alpha^3 Q(fb_i - b^2\kappa_i) + \frac{1}{2}\alpha Q(f\beta \end{aligned}$$

$$- b^2\kappa_0)y_i - \frac{1}{2}\alpha^2\Delta(\kappa_i\beta - b_i\kappa_0) - \frac{1}{2}\alpha^2(\kappa_i\beta + b_i\kappa_0) + \alpha^3 Q(fb_i$$

$$- b^2\kappa_i) + \kappa_0\beta y_i - \alpha Q(f\beta - b^2\kappa_0)y_i]\frac{\Phi}{\Delta}\}. \tag{3.9}$$

By assumption and (3.9), one can obtain

$$0 = \frac{s}{b^2 - s^2}[f\alpha^2 - \beta\kappa_0 + \alpha Q(f\beta - b^2\kappa_0)]h_i + \frac{1}{2}\alpha Q'(f\beta - b^2\kappa_0)h_i$$

$$- \frac{1}{2}\alpha^3 Q(fb_i - b^2\kappa_i) - \frac{1}{2}\alpha^2\Delta(\kappa_i\beta - b_i\kappa_0) - \frac{1}{2}\alpha^2(\kappa_i\beta + b_i\kappa_0)$$

$$+ \alpha^3 Q(fb_i - b^2\kappa_i) + \kappa_0\beta y_i - \frac{1}{2}\alpha Q(f\beta - b^2\kappa_0)y_i. \tag{3.10}$$

(3.10)$\times(b^2 - s^2)$ yields

$$0 = -\frac{1}{2}\alpha^2(s + s\Delta + b^2Q)(b^2 - s^2)\kappa_i + \alpha b_i\{s[\alpha f(1 + sQ) - (s + b^2Q)\kappa_0]$$

$$+ (b^2 - s^2)\frac{Q'}{2}(\alpha fs - b^2\kappa_0) + \frac{\Delta}{2}\kappa_0(b^2 - s^2) + \frac{1}{2}(\alpha fQ - \kappa_0)(b^2 - s^2)\}$$

$$+ y_i\{(b^2 - s^2)[\kappa_0 s - \frac{Q}{2}(\alpha fs - b^2\kappa_0)] - (b^2 - s^2)\frac{Q'}{2}s(\alpha fs - b^2\kappa_0)$$

$$- s^2[\alpha f(1 + sQ) - (s + b^2Q)\kappa_0]\}. \tag{3.11}$$

Simplifying further, we get

$$0 = -\frac{1}{2}\alpha^2(s + s\Delta + b^2Q)(b^2 - s^2)\kappa_i + \alpha b_i\{\alpha f[s(1 + sQ)$$

$$+ \frac{1}{2}sQ'(b^2 - s^2) + \frac{1}{2}Q(b^2 - s^2)] + \kappa_0[\frac{\Delta}{2}(b^2 - s^2) - \frac{1}{2}Q'b^2(b^2 - s^2)$$

$$- s(s + b^2Q) - \frac{1}{2}(b^2 - s^2)]\} + y_i\{\kappa_0[(b^2 - s^2)(s + \frac{b^2}{2}Q)$$

$$+ (b^2 - s^2)\frac{Q'}{2}sb^2 + s^2(s + b^2Q)] - \alpha f[(b^2 - s^2)\frac{1}{2}sQ$$

$$+ (b^2 - s^2)\frac{Q'}{2}s^2 + s^2(1 + sQ)]\}. \tag{3.12}$$

Let

$$M_1 := -\frac{1}{2}(s + s\Delta + b^2Q)(b^2 - s^2),$$

$$M_2 := s(1 + sQ) + \frac{1}{2}sQ'(b^2 - s^2) + \frac{1}{2}Q(b^2 - s^2),$$

$$M_3 := \frac{\Delta}{2}(b^2 - s^2) - \frac{1}{2}Q'b^2(b^2 - s^2) - s(s + b^2Q) - \frac{1}{2}(b^2 - s^2),$$

$$M_4 := (b^2 - s^2)(s + \frac{b^2}{2}Q) + (b^2 - s^2)\frac{Q'}{2}sb^2 + s^2(s + b^2Q),$$

$$M_5 := (b^2 - s^2)\frac{1}{2}sQ + (b^2 - s^2)\frac{Q'}{2}s^2 + s^2(1 + sQ). \tag{3.13}$$

Noting that the expression of Δ, by a direction computation, it is surprising that $M_1 = -M(b^2 - s^2)$, $M_2 = M$, $M_3 = -sM$, $M_4 = b^2 M$, $M_5 = sM$, where

$$M := s + \frac{1}{2} s Q'(b^2 - s^2) + \frac{1}{2} Q(b^2 + s^2).$$

Then (3.12) can be reduced to

$$M\{(b^2 - s^2)\alpha^2 \kappa_i - \alpha b_i(\alpha f - s\kappa_0) + y_i(\alpha f s - \kappa_0 b^2)\} = 0. \tag{3.14}$$

We claim $M \neq 0$. Suppose that $M = 0$, we get the solution of ordinary differential equation $M = 0$,

$$Q = \frac{k(b^2 - s^2) - 1}{s},$$

where k is a number independent of s. Then we have $1 + sQ = k(b^2 - s^2)$, i.e.,

$$\frac{\phi}{\phi - s\phi'} = k(b^2 - s^2).$$

Taking $s = 0$ in above equation, we get $k = 1/b^2$. Then the above equation becomes

$$\frac{\phi}{\phi - s\phi'} = \frac{1}{b^2}(b^2 - s^2) \tag{3.15}$$

which implies $\phi = a_1\sqrt{b^2 - s^2}$, where a_1 is a number independent of s. It contradicts that F is a positive definite Finsler metric. Thus $M \neq 0$.

By (3.14), we have

$$(b^2 - s^2)\alpha^2 \kappa_i - \alpha b_i(\alpha f - s\kappa_0) + y_i(\alpha f s - \kappa_0 b^2) = 0. \tag{3.16}$$

Then we can obtain

$$\alpha^2(f b_i - b^2 \kappa_i) + \beta^2 \kappa_i - \beta\kappa_0 b_i + b^2 \kappa_0 y_i - \beta f y_i = 0. \tag{3.17}$$

Differentiating (3.17) with respect to y^j yields

$$\begin{aligned} &2y_j(f b_i - b^2\kappa_i) + 2\beta\kappa_i b_j - \kappa_0 b_i b_j - \beta b_i \kappa_j + a_{ij} b^2 \kappa_0 \\ &+ y_i b^2 \kappa_j - \beta f a_{ij} - f y_i b_j = 0. \end{aligned} \tag{3.18}$$

Contracting (3.18) with a^{ij} yields

$$(n-2)(b^2\kappa_0 - \beta f) = 0. \tag{3.19}$$

Thus the conformal factor $\kappa(x)$ satisfies that $\kappa_i(x)$ is proportional to $b_i(x)$, i.e., $\kappa_i(x) = l b_i(x)$, where $l := l(x)$ is a scalar function on manifold M. □

4. The regular case

In this section, we consider the regular (α, β)-metrics. Firstly, we have:

Lemma 4.1. *Let $F = \alpha\phi(s)$ be a regular (α, β)-metric on a manifold M. If $\phi(s)$ satisfies* (1.1)*, then F is Riemannian.*

Proof. Because F is a positive definite Finsler metric, noting that the expression of Δ, we have

$$\Delta = \frac{\phi[\phi - s\phi' + (b^2 - s^2)\phi'']}{(\phi - s\phi')^2} > 0. \tag{4.1}$$

By (1.1), one obtains

$$\Phi\sqrt{b^2 - s^2} = -\lambda\Delta^{\frac{3}{2}}.$$

Let $s = b$ in above equation and by (4.1), it is obvious that $\lambda = 0$. Thus we get $\Phi = 0$. Then (2.3) implies F is Riemannian. □

Further, by Theorem 1.1, we have:

Theorem 4.2. *Let F and $\tilde{F}$ be two conformally related regular (α, β)-metrics on a manifold M of dimension $n \geq 3$. Then both F and $\tilde{F}$ have the same weak Landsberg curvature if and only if the conformal transformation between F and $\tilde{F}$ is a homothety.*

It is easy to obtain the following corollary from Theorem 4.2.

Corollary 4.3. *Let F and $\tilde{F}$ be two conformally related regular (α, β)-metrics on a manifold M of dimension $n \geq 3$. Assume that F is weak Landsberg metric. Then $\tilde{F}$ is also weak Landsberg metric if and only if the conformal transformation between F and $\tilde{F}$ is a homothety.*

Finally, we also get:

Corollary 4.4. *Let F be a conformally flat regular (α, β)-metrics on a manifold M of dimension $n \geq 3$. If F is weak Landsberg metric, it is a locally Minkowski metric or Riemannian.*

Corollary 4.4 is just the main theorem in [6].

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