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NOTE ON THE DECOMPOSITION OF STATES

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Abstract. We derive a sharp decomposition formula for the state polytope of the Hilbert point and the Hilbert-Mumford index of reducible varieties by using the decomposition of characters and basic convex geometry. This proof captures the essence of the decomposition of the state polytopes in general, and considerably simplifies an earlier proof by the authors which uses a careful analysis of initial ideals of reducible varieties.

1. Introduction and preliminaries

In this article, we take a new look at the decomposition formula for state polytopes [2] from a more general point of view. We shall work over an algebraically closed field k of characteristic zero. Let G be a linear algebraic group and R be a maximal torus of it. Let W be a rational representation of G and $w \in W$ be a point. Recall that the *state* $\Xi_w(R)$ of w with respect to R is the set of the characters $\chi \in X(R)$ such that $w_\chi \neq 0$. Since $w = \sum_\chi w_\chi$ implies $cw = \sum_\chi cw_\chi$, we have $\Xi_w(R) = \Xi_{cw}(R)$ for any nonzero $c \in k$. Hence we may define the state $\Xi_p(R)$ of $p \in \mathbb{P}(W)$ to be $\Xi_w(R)$ for any affine point $w \in W$ over p. (We conflate a vector space W with the affine scheme Spec $\mathrm{Sym}(W^*)$.)

We shall be concerned with the states of Hilbert points of homogeneous ideals. Let V be an $(n+1)$ dimensional k-module and S be the symmetric algebra of the dual vector space V^*. Choose coordinates $x_0, \ldots, x_n$ and identify S with $k[x_0, \ldots, x_n]$. Let $P(u)$ be a rational polynomial in one variable u and $Q(u) = \binom{u+n}{n} - P(u)$. If m is at least the Gotzmann number of $P(u)$, then for any saturated homogeneous polynomial $I \subset k[x_0, \ldots, x_n]$ whose Hilbert

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polynomial is P, the *mth Hilbert point* $[I]_m$ of I is well defined as a $Q(m)$-dimensional subspace of the degree m piece S_m:

$$[I]_m : [I_m \hookrightarrow S_m] \in \mathrm{Gr}_{Q(m)} S_m \hookrightarrow \mathbb{P}\left(\bigwedge^{Q(m)} S_m\right).$$

The *dual mth Hilbert point* $[I]_m^*$ is defined as

$$[I]_m^* : [S_m \to S_m/I_m] \in \mathrm{Gr}^{P(m)} S_m \hookrightarrow \mathbb{P}\left(\bigwedge^{P(m)} S_m^*\right).$$

If $X \subset \mathbb{P}(V)$ is the projective scheme defined by a homogeneous ideal I_X, then we shall abuse notation and conflate $\mu([X]_m, \rho)$ with $\mu([I_X]_m, \rho)$. We shall also abuse the notations for the dual Hilbert points. The collection of mth Hilbert points form a closed subscheme $\mathrm{Hilb}_P\mathbb{P}(V)$, called the *Hilbert scheme*, of the Grassmannian. Working out the geometric invariant theory (GIT) of suitable Hilbert schemes give rise to various moduli spaces [4], and our main motivation for the study in this article is the construction of the moduli of curves. The link between GIT and the study of state polytopes is given by the following fundamental observation (numerical criterion): *If G is reductive and V is a rational representation, $v \in V$ is GIT unstable if and only if there is a torus R of G such that the convex hull of $\Xi_v(R)$ does not contain the trivial character.*

The *monomial basis* of $\bigwedge^{Q(m)} S_m$ consists of $x^\alpha := x^{\alpha(1)} \wedge \cdots \wedge x^{\alpha(Q(m))}$, the wedge product of $Q(m)$ degree m monomials $x^{\alpha(i)}$'s. The basis members are also the R-weight vectors of $\bigwedge^{Q(m)} S_m$, where R is the maximal torus of $G = GL(S_1)$ diagonalized by $x_0, \dots, x_n$: Indeed, let χ_i be the character of R determined by $t.x_i = \chi_i(t)x_i$. Then by letting t_i denote $\chi_i(t)$ and using the usual multivector notation $t^\gamma = \prod_{i=0}^n t_i^{\gamma_i}$, $t \in T$, $\gamma \in \mathbb{Z}^{n+1}$, we have $t.x^\alpha = t^{\sum_{i=1}^{Q(m)} \alpha(i)} x^\alpha$, which means precisely that $\left(\bigwedge^{Q(m)} S_m\right)_{\chi^\alpha} = k.x^\alpha$ where $\chi^\alpha = \prod_{i=1}^{Q(m)} \prod_{j=0}^n \chi_j^{\alpha(i)_j}$.

The Hilbert point $[I]_m$ has a nonzero x^α-coefficient if and only if the degree m monomials other than $x^{\alpha(1)}, \dots, x^{\alpha(Q(m))}$ form a k-basis of S_m/I_m. Following [3], we denote the set of states by $\Xi_{[I]_m}(R)$ and its convex hull by $\mathcal{P}_m(I)$. We call $\mathcal{P}_m(I)$ the *mth state polytope* of I, following [1].

For any fixed sufficiently large m, Bayer and Morrison proved that:

Theorem 1.1 ([1, Theorem 3.1]). *There is a canonical bijection between the vertices of $\mathcal{P}_m(I)$ and the initial ideals $\mathbf{in}_\prec(I)$ as $\prec$ runs through all term orders on $k[x_0, \dots, x_n]$.*

Using the Bayer-Morrison theorem and basic properties of monomial orders and initial ideals, decomposition formulae for initial ideals, state polytopes, and Hilbert-Mumford indices were obtained in [2]:

Theorem 1.2 ([2]). *Let X be a chain of projective varieties $X_1, \ldots, X_\ell$ defined by a saturated homogeneous ideal $I_X = \cap_i I_{X_i}$, i.e., $X = \cup_{i=1}^{\ell} X_i$ and X_i meets X_j when and only when $|i-j| = 1$. Suppose that there is a homogeneous coordinate system $x_0, \ldots, x_n$ and a sequence $n_0 = 0 < n_1 < \cdots < n_\ell = n$ such that*

$$X_i \subset \{x_0 = \cdots = x_{n_{i-1}-1} = 0, x_{n_i+1} = x_{n_i+2} = \cdots = x_n = 0\}.$$

Then the state polytope of X is given by the following decomposition formula

$$(1)\ \ \mathcal{P}_m(I_X) = \sum_{i=1}^{\ell} \mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}]) + \sum_{i=1}^{\ell-1} \mathcal{P}_m(T_i \cap k[x_{n_{i-1}}, \ldots, x_n])$$

where $T_i = \langle x_{n_{i-2}}, \ldots, x_{n_{i-1}-1}\rangle\langle x_{n_i+1}, \ldots, x_n\rangle$ for $2 \leq i \leq \ell - 1$, and $T_1 = \langle x_{n_1+1}, x_{n_1+2}, \ldots, x_n\rangle$ and $T_\ell = \langle x_{n_{\ell-2}}, x_{n_{\ell-2}+1}, \ldots, x_{n_{\ell-1}-1}\rangle$.

Here, $\mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}])$ is regarded as a convex polytope in the subspace $\{\mathbf{a} \in \mathbb{R}^{n+1} \mid a_j = 0, \forall j < n_{i-1}, \forall j > n_i\}$.

Similarly, $\mathcal{P}_m(T_i \cap k[x_{n_{i-1}}, \ldots, x_n])$ is also regarded as a convex polytope in the relevant vector subspace. Note that $\mathcal{P}_m(T_i \cap k[x_{n_{i-1}}, \ldots, x_n])$ is a point since T_i is a monomial ideal. We let τ denote the point $\sum_{i=1}^{\ell-1} \mathcal{P}_m(T_i \cap k[x_{n_{i-1}}, \ldots, x_n])$. We shall show in Section 2.1 that Theorem 1.2 is a consequence of the fact that the characters of a direct sum are the sums of the characters of the direct summands (Proposition 2.1).

It is also shown in [2] that this decomposition is sharp: the vertices of $\mathcal{P}_m(I_X)$ are precisely the sums of vertices of $\mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}])$ and τ (Corollary 2.4). The proof in [2] uses Theorem 1.1 and the initial ideal decomposition formula. We shall show in Section 2.2 that the sharpness of the decomposition is in fact a consequence of a general convex geometry phenomenon.

Finally, we also reprove in Section 3 the Hilbert-Mumford index decomposition formula below by using the decomposition of characters.

Proposition 1.3 ([2]). *Let X be as in Theorem 1.2 and $\rho : \mathbb{G}_m \to GL_{n+1}$ be a 1-parameter subgroup of GL_{n+1} diagonalized by $\{x_0, \ldots, x_n\}$ with weights $(r_0, \ldots, r_n)$ and ρ_i be the restriction of ρ to $GL(kx_{n_{i-1}} + \cdots + kx_{n_i})$. Then the Hilbert-Mumford index $\mu([X]^*_m, \rho)$ of the (dual) mth Hilbert point of X with respect to ρ is given by*

$$\begin{aligned} \mu([X]^*_m, \rho) = &\sum_{i=1}^{\ell} \mu([X_i]^*_m, \rho_i) - \sum_{i=1}^{\ell} \left(\frac{mP_i(m)}{n_i - n_{i-1} + 1} \sum_{k=n_{i-1}}^{n_i} r_k \right) \\ &+ \frac{mP(m)}{n+1} \sum_{i=0}^{n} r_i + m \sum_{i=1}^{\ell-1} r_{n_i}, \end{aligned}$$

where $P(m)$ is the Hilbert polynomial of $I_X \subset k[x_0, \ldots, x_n]$ and $P_i(m)$, the Hilbert polynomial of $I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}]$ regarded as an ideal in $k[x_{n_{i-1}}, \ldots, x_{n_i}]$.

We close this section with an observation that will be used in Section 3:

Lemma 1.4. *Retain the notations from above. Then* $\mu([X]^*_m, \rho) = \mu([X]_m, \rho)$. *That is, the Hilbert-Mumford index of* $[X]_m$ *and that of the dual Hilbert point* $[X]^*_m$ *are exactly the same.*

Proof. Let ρ be a one-parameter subgroup of $SL(S_1)$. We recall the fact that $\lim_{t\to 0}\rho(t).[I_X]_m = [\mathbf{in}_{\prec_\rho} I_X]_m$ where $\prec_\rho$ is the ρ-weight order with the reverse lexicographic tie-breaking [1]. Then the Hilbert-Mumford index is

$$\mu([I_X]_m, \rho) = \mu([\mathbf{in}_{\prec_\rho} I_X]_m, \rho) = -\sum_{x^\alpha \in (\mathbf{in}_{\prec_\rho} I_X)_m} \mathrm{wt}_\rho(x^\alpha).$$

Let $\{f_0, \dots, f_n\} \subset S_1^*$ be the dual basis of $\{x_0, \dots, x_n\} \subset S_1$. Use the multi-vector notation $f^\alpha = \prod_{i=0}^n f_i^{\alpha_i}$. Then $f^{\alpha(1)} \wedge \cdots \wedge f^{\alpha(P(m))}$ appears in $\bigwedge^{P(m)}(S/I_X)^*_m$ with a nonzero Plücker co-ordinate if and only if $x^{\alpha(1)}, \dots, x^{\alpha(P(m))}$ form a basis of $(S/I_X)_m$. Since ρ is a co-character of the special linear group, the weights of all monomials of S_m sum up to zero. Also, ρ acts on $\mathbb{P}(\bigwedge S_m)$ and $\mathbb{P}(\bigwedge S^*_m)$ with opposite weights. Hence we have

$$\begin{aligned}\mu([I_X]^*_m, \rho) &= \max\{-\sum_{x^\alpha\in\mathcal{B}} \mathrm{wt}_\rho(f^\alpha) \,|\, \mathcal{B} \text{ a monomial basis of } (S/I_X)_m\} \\ &= -\sum_{x^\alpha \in S_m \setminus \mathbf{in}_{\prec_\rho}(I_X)} (-\mathrm{wt}_\rho(x^\alpha)) = -\sum_{x^\alpha \in (\mathbf{in}_{\prec_\rho}(I_X))_m} \mathrm{wt}_\rho(x^\alpha).\end{aligned}$$

□

2. Decomposition of states

Let $G = GL(V)$ and let V_i, $i = 1, \dots, \nu$, be vector subspaces of V that span V. Note that $V = \sum_{i=1}^\nu V_i$ is *not necessarily a direct sum.* Then

$$S^m V = \sum_{i=1}^{\nu} S^m V_i + \sum_{\substack{\sum_j m_j = m \\ 0 < m_j < m}} \bigotimes_{j=1}^{\nu} S^{m_j} V_j.$$

For a notational convenience, we let W denote $S^m V$, $W_j = S^m V_j$ for $1 \le j \le \nu$ and

$$W_{\nu+1} = \sum_{\substack{\sum_j m_j = m \\ 0 < m_j < m}} \bigotimes_{j=1}^{\nu} S^{m_j} V_j.$$

Let R be a maximal torus of $GL(V)$ which preserves the subspaces V_i. Then one can choose a basis of $\mathcal{B} = \{v_1, \dots, v_M\}$ of V diagonalizing the R-action such that V_j is the linear subspace spanned by $\{v_{M_j}, v_{M_j+1}, \dots, v_{M'_j}\}$. We identify $GL(V_j)$ with the subgroup of $GL(V)$ which preserves V_j and acts trivially on $\mathrm{Span}\{v_i \,|\, i < M_j, i > M'_j\}$.

Let χ_i be the character of R determined by $t.v_i = \chi_i(t)v_i$. Set $R_j = R \cap GL(V_j)$. Then there is a natural projection $\pi_j : R \to R_j$ defined by

$$\chi_s(\pi_j(t)) = \begin{cases} \chi_s(t), & M_j \le s \le M'_j, \\ 1, & \text{else.} \end{cases}$$

Then π_j's induce injective group homomorphisms $\pi_j^* : X(R_j) \hookrightarrow X(R)$. We shall identify $X(R_j)$ with its image in $X(R)$ under π_j^*.

Proposition 2.1. *Let I be a subspace of W, $I_j = I \cap W_j$, and suppose that the sum $I = \sum_{j=1}^{\nu+1} I_j$ is direct. Let $\dim I = N$ and $\dim I_j = N_j$, $1 \le j \le \nu+1$. We have the decomposition of states*

$$\Xi_{[\bigwedge^N I]}(R) = \sum_{j=1}^{\nu} \Xi_{[\bigwedge^{N_j} I_j]}(R_j) + \Xi_{[\bigwedge^{N_{\nu+1}} I_{\nu+1}]}(R).$$

Proof. Let ξ be an affine point over $\left[\bigwedge^N I\right] \in \mathbb{P}(\bigwedge^N S^m V)$, and let ξ_j be an affine point over $\left[\bigwedge^{N_j} I_j\right]$ in $\bigwedge^{N_j} W_j$. The affine point ξ_j generates the one-dimensional subspace $\bigwedge^{N_j} I_j$. Let $j \le \nu$. Consider the R-weight decomposition of ξ_j. Since I_j is contained in the R-module $W_j = S^m V_j$, the R-weight decomposition of ξ_j is precisely the R_j-weight decomposition, i.e.,

$$\xi_j = \sum_{\chi \in X(R)} (\xi_j)_\chi = \sum_{\chi \in X(R_j)} (\xi_j)_\chi.$$

Since the sum $I = \sum_{j=1}^{\nu+1} I_j$ is direct, we have

$$\bigwedge^N I \simeq \bigotimes_{j=1}^{\nu+1} \left(\bigwedge^{N_j} I_j \right)$$

and hence the R-weight decomposition of ξ is given as

$$\xi = \sum_{\substack{\chi^{(j)} \in X(R_j) \\ \chi^{\nu+1} \in X(R)}} \bigotimes_{j=1}^{\nu+1} (\xi_j)_{\chi^{(j)}}. \tag{2}$$

The summand $\bigotimes_{j=1}^{\nu+1} (\xi_j)_{\chi^{(j)}}$ has a weight $\sum_{j=1}^{\nu+1} \chi^{(j)} \in X(R)$ and it is a state of ξ if and only if the weight vector $\sum_{j=1}^{\nu+1} (\xi_j)_{\chi^{(j)}}$ is nonzero if and only if $\chi^{(j)}$ is a state of ξ_j, $\forall j$. It follows that every state of ξ is a sum of states of ξ_j's and vice versa. □

2.1. Proof of Theorem 1.2

We shall now deduce Theorem 1.2 from Proposition 2.1. Let $X = X_1 \cup X_2 \subset \mathbb{P}(V^*)$ be a chain of subvarieties X_i and suppose that there exists a homogeneous coordinate system $x_0, \dots, x_{n_1}, \dots, x_n \in V$ such that

$$(\dagger) \qquad \begin{aligned} X_1 &\subset \{x_{n_1+1} = \cdots = x_n = 0\}, \\ X_2 &\subset \{x_0 = \cdots = x_{n_1-1} = 0\}. \end{aligned}$$

We also assume that $X_1 \cap X_2 \neq \emptyset$, and that X, X_1, X_2 are cut out by saturated homogeneous ideals I_{X_i} and $I_X = I_{X_1} \cap I_{X_2}$.

Let V_1 (resp. V_2) be the subspace of V spanned by $\{x_0, \dots, x_{n_1}\}$ (resp. $\{x_{n_1}, \dots, x_n\}$). Let $W = S^m V$, $W_i = S^m V_i$ for $i = 1, 2$, and

$$W_3 = \sum_{i+j=m, ij\neq 0} S^i V_1' \otimes S^j V_2',$$

where $V_1' = \sum_{i=0}^{n_1-1} kx_i$ and $V_2' = \sum_{i=n_1+1}^{n} kx_i$. Evidently we have $W = \sum_{i=1}^{3} W_i$. Let R be the maximal torus of $GL(V)$ diagonalized by $x_0, \dots, x_n$ and $R_i = R \cap GL(V_i)$ for each i, where $GL(V_i)$ is identified with a suitable subgroup of $GL(V)$ as in the discussion preceding Proposition 2.1. Of course, R_1 (resp. R_2) is identified with the maximal torus of $GL(V_1)$ (resp. $GL(V_2)$) diagonalized by $x_0, \dots, x_{n_1}$ (resp. $x_{n_1}, \dots, x_n$).

For each $m \geq 2$, we have

$$(3) \qquad (I_X)_m = (I_X \cap W_1) + (I_X \cap W_2) + (I_X \cap W_3).$$

We claim that the property of coordinates $(\dagger)$ implies that this is a direct sum decomposition. Indeed, since $W_1 \cap W_2$ is the 1-dimensional space $k.x_{n_1}^m$ and $x_{n_1}^m$ does not vanish at $\{p\} = X_1 \cap X_2$, $I_X \cap W_1 \cap W_2 = 0$. Since W_1 and W_2 meet W_3 trivially, the claim follows and we may apply Proposition 2.1.

Note that the three terms on the right hand side of (3) are

$$I_{X_1} \cap k[x_0, \dots, x_{n_1}]_m, \ I_{X_2} \cap k[x_{n_1}, \dots, x_n]_m \ \text{ and } T_m$$

respectively, where $T = \langle x_0, \dots, x_{n_1-1}\rangle\langle x_{n_1+1}, \dots, x_n\rangle$. Then by Proposition 2.1, we have

$$(\dagger\dagger) \ \Xi_{\bigwedge^N (I_X)_m}(R) = \Xi_{[\bigwedge^{N_1}(I_X \cap W_1)]}(R_1) + \Xi_{[\bigwedge^{N_2}(I_X \cap W_2)]}(R_2) + \Xi_{[\bigwedge^{N_3} T_m]}(R),$$

where N and N_i denote the appropriate dimensions. As observed in the introduction, we may naturally identify the characters $\prod_{i=0}^n \chi_i^{a_i}$ with the Laurent monomials $\prod_{i=0}^n x_i^{a_i} \in k(x_0, \dots, x_n)$, where χ_i is the projection $t = (t_0, \dots, t_N) \mapsto t_i$. Identifying the characters with monomials and taking the convex hull of both sides, we obtain Theorem 1.2 for the case $\ell = 2$ from which, as observed in [2], the general case follows by a simple induction.

Remark 2.2. Note that since T is a monomial ideal, $\Xi_{[\bigwedge^{N_3} T_m]}(R)$ consists of one point χ^τ where $\tau = \sum_{x^\alpha \in T_m} \alpha$.

2.2. Decomposition of vertices

Let $P_1, \ldots, P_r$ be polytopes in $\mathbb{R}^n$. In general, every *face* F of the Minkowski sum $\sum_{i=1}^r P_i$ has a unique decomposition $F = \sum_{i=1}^r F_i$ into a sum of faces F_i of P_i. The converse is easily seen to be false: If the origin $\mathbf{0}$ is a vertex of a polytope P, then $\mathbf{0} + v = v$ is not a vertex of $2P$ for any nonzero vertex v of P. The following lemma guarantees that vertices always sum up to be a vertex provided that the polytopes are positioned well enough.

Lemma 2.3. *Let P_1, P_2 be polytopes in $\mathbb{R}^n$. Suppose that P_1 and P_2 are contained in affine hyperplanes H_1 and H_2 respectively such that $H_1 \cap H_2$ is of dimension one. Then the vertices of the Minkowski sum $P_1 + P_2$ are precisely the sums of vertices of P_1 and P_2.*

Proof. It is evident that a vertex of $P_1 + P_2$ is a sum of vertices of P_1 and P_2 since for any subsets S_1 and S_2 of $\mathbb{R}^n$, the sum of their convex hulls is the convex hull of their sum $S_1 + S_2$. To prove the converse, we start by choosing affine coordinates $x_1, \ldots, x_n$ judiciously so that

$$H_1 = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n_1} x_i = N_1,\ x_i = 0,\ \forall i > n_1 \right\}$$

and

$$H_2 = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i=n_1}^{n} x_i = N_2,\ x_i = 0,\ \forall i < n_1 \right\}.$$

Let $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_s\}$ be the sets of vertices of P_1 and P_2, respectively. We aim to show that $v_i + w_j$ is a vertex of $P_1 + P_2$ for any i, j. Suppose it is not the case - suppose without losing generality $v_1 + w_1$ is not a vertex. Then there exist λ_{ij} for $i = 1, \ldots, r, j = 1, \ldots, s$ such that $\sum_{i,j} \lambda_{ij} = 1$, $0 \leq \lambda_{ij} \leq 1$, $\lambda_{11} = 0$ and

$$v_1 + w_1 = \sum_{i,j} \lambda_{ij}(v_i + w_j).$$

By rearranging the terms, we have

$$\left(1 - \sum_j \lambda_{1j}\right) v_1 - \sum_{i \neq 1} \lambda_{ij} v_i = \sum_{j \neq 1} \lambda_{ij} w_j + \left(\sum_i \lambda_{i1} - 1\right) w_1 \in H_1 \cap H_2$$

which implies that x_{n_1} is the only nonzero coordinate of each side. Moreover, $1 - \sum_j \lambda_{1j} \neq 0$ or $\sum_i \lambda_{i1} - 1 \neq 0$ since $\sum_{i,j} \lambda_{ij} = 1$. Suppose the $1 - \sum_j \lambda_{1j} \neq 0$ (the other case is proved similarly) and let $v_i = (v_{i1}, \ldots, v_{in_1})$. Then we have

$$v_{1k} = \sum_{\substack{i \neq 1 \\ 1 \leq j \leq s}} \mu_{ij} v_{ik}, \quad k \neq n_1,$$

where $\mu_{ij} = \lambda_{ij}/(1 - \sum_{j'=1}^s \lambda_{1j'})$ and $\sum_{\substack{i \neq 1 \\ 1 \leq j \leq s}} \mu_{ij} = 1$.

The n_1th coordinate also satisfies the above condition because

$$\begin{aligned}
v_{1n_1} &= N_1 - \sum_{k=1}^{n_1-1} v_{1k} \\
&= N_1 - \sum_{k=1}^{n_1-1} \sum_{\substack{i\neq 1\\ 1\leq j\leq s}} \mu_{ij} v_{ik} \\
&= \sum_{\substack{i\neq 1\\ 1\leq j\leq s}} \mu_{ij} N_1 - \sum_{\substack{i\neq 1\\ 1\leq j\leq s}} \sum_{k=1}^{n_1-1} \mu_{ij} v_{ik} \\
&= \sum_{\substack{i\neq 1\\ 1\leq j\leq s}} \mu_{ij} \left(N_1 - \sum_{k=1}^{n_1-1} v_{ik} \right) \\
&= \sum_{\substack{i\neq 1\\ 1\leq j\leq s}} \mu_{ij} v_{in_1}
\end{aligned}$$

which means that $v_1 = \sum_{i\neq 1} \left(\sum_{j=1}^{s} \mu_{ij} \right) v_i$. But this is a contradiction since v_1 is a vertex of P_1. □

As an immediate corollary, we obtain:

Corollary 2.4. *Retain notations from Theorem 1.2. Let $\mathcal{V}_i$ denote the set of vertices of $\mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}])$, $i = 1, \ldots, \ell$. Then the vertices of $\mathcal{P}_m(I_X)$ are precisely*

$$\left\{ \tau + \sum_{i=1}^{\ell} v_i \;\middle|\; v_i \in \mathcal{V}_i \right\}.$$

Proof. The $\ell = 2$ case follows from Lemma 2.3: The state polytopes $\mathcal{P}_m(I_{X_1} \cap k[x_0, x_1, \ldots, x_{n_1}])$ and $\mathcal{P}_m(I_{X_2} \cap k[x_{n_1}, x_{n_1+1} \ldots, x_n])$ are in the affine hyperplanes

$$H_1 = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=0}^{n_1} x_i = Q_1(m),\, x_i = 0,\, \forall i > n_1 \right\}$$

and

$$H_2 = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=n_1}^{n} x_i = Q_2(m),\, x_i = 0,\, \forall i < n_1 \right\}$$

respectively, where $Q_1(m) = \dim I_{X_1} \cap k[x_0, x_1, \ldots, x_{n_1}]_m$ and $Q_2(m) = \dim I_{X_2} \cap k[x_{n_1}, x_{n_1+1} \ldots, x_n]_m$. Since $H_1 \cap H_2$ is one dimensional, Lemma 2.3 applies. The general, $\ell \geq 2$ case follows by induction. □

3. Decomposition of Hilbert-Mumford index

Retain the notations from Section 2.1. To prove Proposition 1.3, as in [2] we shall assume that $\ell = 2$ as the general case follows by a simple induction. We let $Q(m) = \dim(I_X)_m$ and $P(m) = \dim(S/I_X)_m = \dim S_m - Q(m)$. Likewise, $Q_i(m) = \dim(I_{X_i})_m$ and $P_i(m) = \dim S^m V_i - Q_i(m)$. Let ρ be a 1-PS of R and let ρ_i be the induced 1-PS of R_i, $i = 1, 2$. These are obtained by composing with the projections $R \to R_i$. Recall that, if the sum of the ρ-weights is zero, the Hilbert-Mumford index is given by

$$\mu([I_X]_m, \rho) = \max\{-\langle \chi, \rho\rangle \mid \chi \in \Xi_{[I_X]_m}(R)\},$$

where $\langle\ ,\ \rangle$ denotes the natural pairing of the character group and the 1-PS group, i.e., $\chi \circ \rho(t) = t^{\langle \chi, \rho\rangle}$ for any $t \in \mathbb{G}_m(k)$.

For any $\chi \in \Xi_{[I_X]_m}(R)$, due to (††), we have $\chi = \chi_1 + \chi_2 + \tau$, $\chi_i = \iota_i \circ \chi$ where $\iota_i : R_i \to R$ is the inclusion. And τ is the character with which ρ acts on

$$\bigwedge^{\max}\left(I \cap \sum_{i+j=m, ij\neq 0} S^i V_1 \otimes S^j V_2\right) = T_m,$$

where $T = \langle x_0, \ldots, x_{n_1-1}\rangle\langle x_{n_1+1}, \ldots, x_n\rangle$.

Hence we have

$$\langle \chi, \rho\rangle = \langle \chi_1, \rho_1\rangle + \langle \chi_2, \rho_2\rangle + \langle \tau, \rho\rangle.$$

Clearly, the minimum of $\langle \chi, \rho\rangle$ is achieved precisely when each χ_i pairs minimally with ρ. Let ρ' be the 1-ps of $SL(V)$ associated to ρ, i.e., if r_i are the weights of ρ, then ρ' is the 1-ps with weights $r_i - w$ where w is the average of the weight $\frac{1}{\dim V}\sum_{i=1}^n r_i$. Conflating a 1-ps with its weight vector, we may write $\rho = \rho' + (w, w, \ldots, w)$.

The minimum of $\langle \chi, \rho\rangle$ is achieved by

$$\sum_{x^\alpha \in \left(\mathbf{in}_{\prec_\rho} I_X\right)_m} \chi^\alpha,$$

where we used the multiplicative multi-vector notation $\chi^\alpha = \prod \chi_i^{\alpha_i}$ as in the discussion preceding Theorem 1.1. Note that

$$\begin{aligned}\langle \chi, \rho\rangle &= \langle \chi, \rho'\rangle + \langle \chi, (w, w, \ldots, w)\rangle \\ &= -\mu([I_X]_m, \rho) + w \sum_{x^\alpha \in \left(\mathbf{in}_{\prec_\rho} I_X\right)_m} (\sum_{i=0}^n \alpha_i) \\ &= -\mu([I_X]_m, \rho) + wmQ(m).\end{aligned}$$

Similarly, let ρ_i' denote the 1-ps of $SL(V_i)$ associated to ρ_i, $i = 1, 2$, whose weights are shifted by the average weight $w_i = \frac{\sum_{x_j \in V_i} \mathrm{wt}_\rho(x_j)}{\dim V_i}$. Clearly, χ_i pairs with ρ_i minimally if and only if it pairs with ρ_i' minimally, and

$$\min\langle \chi_i, \rho_i\rangle = -\mu([I_{X_i} \cap S^m V_i]_m, \rho_i) + w_i m Q_i(m).$$

Hence we have

$$\begin{aligned}\mu([I_X]_m, \rho) = &-\min\langle \chi_1, \rho_1\rangle - \min\langle \chi_2, \rho_2\rangle - \langle \tau, \rho\rangle + mwQ(m)\\ =& \ \mu([I_{X_1} \cap k[x_0, \dots, x_{n_1}]]_m, \rho_1) + \mu([I_{X_2} \cap k[x_{n_1}, \dots, x_n]]_m, \rho_2)\\ &- w_1 m Q_1(m) - w_2 m Q_2(m) - \sum_{x^\alpha \in T_m} \mathrm{wt}_\rho(x^\alpha) + wmQ(m). \qquad (4)\end{aligned}$$

Substitute $\dim S^m V - Q(m) = P(m)$ and $\dim S^m V_i - P_i(m) = Q_i(m)$. Subsequently, substitute $\sum_{x^\alpha \in S^m V} \mathrm{wt}_\rho(x^\alpha)$ for $mw \dim S^m V$ and $\sum_{x^\alpha \in S^m V_i} \mathrm{wt}_\rho(x^\alpha)$ for $mw_i \dim S^m V_i$. Then we get

$$\begin{aligned}\mu([I_X]_m, \rho) = &\ \mu([I_{X_1} \cap k[x_0, \dots, x_{n_1}]]_m, \rho_1) + \mu([I_{X_2} \cap k[x_{n_1}, \dots, x_n]]_m, \rho_2)\\ &+ w_1 m P_1(m) + w_2 m P_2(m) + \mathrm{wt}_\rho(x_{n_1}^m) - wmP(m), \qquad (\ddagger)\end{aligned}$$

since $S^m V_1 \coprod (S^m V_2 \setminus \{x_{n_1}^m\}) \coprod T_m = S^m V$. Recall from Lemma 1.4 and its proof that $[I_X]_m$ and $[I_X]_m^*$ have the same Hilbert-Mumford indices, and that ρ acts with opposite weights on $\mathbb{P}(\bigwedge S_m^*)$ in which the dual Hilbert points live. That is, if w^* is the average of the weights for the ρ action on $\mathbb{P}(\bigwedge S_m^*)$, then $w* = -w$ so that the signs of the terms $wmP(m)$, $w_i m P_i(m)$ are reversed. Hence (‡) is precisely the assertion of Proposition 1.3 for the case $\ell = 2$.

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