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ON A CLASS OF CONSTACYCLIC CODES OF LENGTH $2p^s$ OVER $\frac{\mathbb{F}_{p^m}[u]}{\langle u^a \rangle}$

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Abstract. The aim of this paper is to study the class of Λ-constacyclic codes of length $2p^s$ over the finite commutative chain ring $\mathcal{R}_a = \frac{\mathbb{F}_{p^m}[u]}{\langle u^a \rangle} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{a-1}\mathbb{F}_{p^m}$, for all units Λ of $\mathcal{R}_a$ that have the form $\Lambda = \Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1}$, where $\Lambda_0, \Lambda_1, \ldots, \Lambda_{a-1} \in \mathbb{F}_{p^m}$, $\Lambda_0 \neq 0$, $\Lambda_1 \neq 0$. The algebraic structure of all Λ-constacyclic codes of length $2p^s$ over $\mathcal{R}_a$ and their duals are established. As an application, this structure is used to determine the Rosenbloom-Tsfasman (RT) distance and weight distributions of all such codes. Among such constacyclic codes, the unique MDS code with respect to the RT distance is obtained.

1. Introduction

The classes of cyclic and negacyclic codes in particular, and constacyclic codes in general, play a very significant role in the theory of error-correcting codes. Let $\mathbb{F}$ be a finite field of characteristic p and λ be a nonzero element of $\mathbb{F}$. λ-constacyclic codes of length n over $\mathbb{F}$ are classified as the ideals $\langle g(x) \rangle$ of the quotient ring $\mathbb{F}[x]/\langle x^n - \lambda \rangle$, where the generator polynomial $g(x)$ is the unique monic polynomial of minimum degree in the code, which is a divisor of $x^n - \lambda$.

In fact, cyclic codes are the most studied of all codes. Many well known codes, such as BCH, Kerdock, Golay, Reed-Muller, Preparata, Justesen, and binary Hamming codes, are either cyclic codes or constructed from cyclic codes. Cyclic codes over finite fields were first studied in the late 1950s by Prange [29], [30], [31], [32], while negacyclic codes over finite fields were initiated by Berlekamp in the late 1960s [2], [3]. The case when the code length n is divisible by the characteristic p of the field yields the so-called repeated-root codes, which were first studied since 1967 by Berman [4], and then in the 1970s and 1980s by several authors such as Massey et al. [25], Falkner et al. [20], Roth and Seroussi [34]. However, repeated-root codes were investigated in the most generality in the 1990's by Castagnoli et al. [7], and van Lint [38], where they

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showed that repeated-root cyclic codes have a concatenated construction, and are asymptotically bad. Nevertheless, such codes are optimal in a few cases, that motivates researchers to further study this class of codes.

After the realization in the 1990's [6,21,26] by Nechaev and Hammons et al., codes over $\mathbb{Z}_4$ in particular, and codes over finite rings in general, has developed rapidly in recent decade years. Constacyclic codes over a finite commutative chain ring have been studied by many authors (see, for example, [1], [5], [27], and [36]). The structure of constacyclic codes is also investigated over a special family of finite chain rings of the form $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$. For example, the structure of $\frac{\mathbb{F}_2[u]}{\langle u^2\rangle}$ is interesting, because this ring lies between $\mathbb{F}_4$ and $\mathbb{Z}_4$ in the sense that it is additively analogous to $\mathbb{F}_4$, and multiplicatively analogous to $\mathbb{Z}_4$. Codes over $\frac{\mathbb{F}_2[u]}{\langle u^2\rangle}$ have been extensively studied by many researchers, whose work includes cyclic and self-dual codes [5], decoding of cyclic codes [37], Type II codes [18], duadic codes [24], repeated-root constacyclic codes [10].

Recently, Dinh, in a series of papers ([12], [13], [14]), determined the generator polynomials of all constacyclic codes of lengths $2p^s$, $3p^s$ and $6p^s$ over finite fields $\mathbb{F}_{p^m}$. We also have been studying certain classes of repeated-root constacyclic codes over finite chain rings. For example, Dinh [11] classified all constacyclic codes of length p^s over $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$. Moreover, in 2015, Dinh et al. [17] studied negacyclic codes of length $2p^s$ over the ring $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$. Recently, Chen et al. [8] determined the algebraic structures of all λ-constacyclic codes of length $2p^s$ over the finite commutative chain ring $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$ and provided the number of codewords and the dual of every λ-constacyclic code. As a generalization of finite chain rings $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$ ($u^2=0$), finite chain rings of the form $\frac{\mathbb{F}_{p^m}[u]}{\langle u^a\rangle}=\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+\cdots+u^{a-1}\mathbb{F}_{p^m}$ ($u^a=0$) have been developed as code alphabet as well. In a recent paper [15], we partitioned the units of the chain ring $\mathcal{R}_a=\frac{\mathbb{F}_{p^m}[u]}{\langle u^a\rangle}=\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}+\cdots+u^{a-1}\mathbb{F}_{p^m}$ into a distinct types, and studied Type 1 constacyclic codes of length p^s over $\mathcal{R}_a$ in details. From this, we showed that self-dual Λ-constacyclic codes of length p^s over $\mathcal{R}_a$ exist if and only if a is even, and in such case, it is unique.

Motivated by these, in this paper, we consider all Type 1 Λ-constacyclic codes of length $2p^s$ over $\mathcal{R}_a$. The case that Λ is a square, say $\Lambda=\alpha^2$, is easy, as the ambient ring $\frac{\mathcal{R}_a[x]}{\langle x^{2p^s}-\Lambda\rangle}$ can be decomposed as $\frac{\mathcal{R}_a[x]}{\langle x^{p^s}+\alpha\rangle}\oplus\frac{\mathcal{R}_a[x]}{\langle x^{p^s}-\alpha\rangle}$. This follows that each Type 1 Λ-constacyclic code of length $2p^s$ over $\mathcal{R}_a$ is expressed as a direct sum of C_+ and C_-, where C_+ and C_- are α and $-\alpha$ constacyclic codes length p^s over $\mathcal{R}_a$, respectively. The classification, detailed structure, and number of codewords of such α and $-\alpha$ constacyclic codes length p^s were investigated and provided in [15]. Thus, our main study concentrates in the situation where the unit Λ is not a square in $\mathcal{R}_a$. With the observation that this condition is equivalent to the condition that Λ_0 is not a square in $\mathbb{F}_{p^m}$, we can show that the ambient ring $\frac{\mathcal{R}_a[x]}{\langle x^{2p^s}-\Lambda\rangle}$, in this case, is a chain ring. From that, we get the algebraic structure of all Type 1 Λ-constacyclic codes of length $2p^s$ and their duals. This structure is applied to establish the Rosenbloom-Tsfasman distance and weight distributions of all such codes.

The rest of this paper is organized as follows. Preliminary concepts and some properties of constacyclic codes over finite commutative rings are shown in Section 2. We also give some results about the rings $\mathcal{R}_a = \frac{\mathbb{F}_{p^m}[u]}{\langle u^a \rangle}$ and their units in this section. The algebraic structures of Type 1 Λ-constacyclic codes of length $2p^s$ over $\mathcal{R}_a$ and their duals are presented in Section 3. In Section 4, these structures are used to obtain the Rosenbloom-Tsfasman distance and weight distributions of all such codes. The only MDS code, with respect to the RT distance, among these constacyclic codes, is also identified.

2. Constacyclic codes over finite commutative rings

Let R be a finite commutative ring. An ideal I of R is called *principal* if it is generated by one element. A ring R is a *principal ideal ring* if its ideals are principal. A ring R is called a *local ring* if it has a unique maximal ideal. A local ring is a *chain ring* if its lattice of ideals is a chain. A ring R is a finite commutative chain ring if and only if R is a local ring and its maximal ideal is principal.

The following equivalent conditions are well-known for the class of finite commutative chain rings.

Proposition 2.1 (cf. [16, Proposition 2.1])**.** *For a finite commutative ring R the following conditions are equivalent*:

(i) *R is a local ring and the maximal ideal M of R is principal*;

(ii) *R is a local principal ideal ring*;

(iii) *R is a chain ring.*

Let a be a fixed generator of the maximal ideal M of a finite commutative chain ring R. Then a is nilpotent and we denote its nilpotency index by t. The ideals of R form a chain:

$$R = \langle a^0 \rangle \supsetneq \langle a^1 \rangle \supsetneq \cdots \supsetneq \langle a^{t-1} \rangle \supsetneq \langle a^t \rangle = \langle 0 \rangle.$$

Let $\bar{R} = \frac{R}{M}$. We consider the natural ring homomorphism $\nu : R[x] \longrightarrow \bar{R}[x]$ that maps $r \to r + M$ and the variable x to x. The following result is a well-known fact about finite commutative chain rings.

Proposition 2.2. *Let R be a finite commutative chain ring, with maximal ideal $M = \langle a \rangle$, and let t be the nilpotency of a. Then*

(i) *For some prime p and positive integers k, l $(k \geq l), |R| = p^k, |\bar{R}| = p^l$, and the characteristic of R and $\bar{R}$ are powers of p*;

(ii) *For $i = 0, 1, \ldots, t, |\langle a^i \rangle| = |\bar{R}|^{t-i}$. In particular, $|R| = |\bar{R}|^t$, i.e., $k = lt$.*

Given n-tuples $x = (x_0, x_1, \ldots, x_{n-1})$, $y = (y_0, y_1, \ldots, y_{n-1}) \in R^n$, their inner product or dot product is defined in the usual way:

$$x \cdot y = x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1},$$

evaluated in R. Two words x, y are called *orthogonal* if $x \cdot y = 0$. For a linear code C over R, its *dual code* $C^\perp$ is the set of n-tuples over R that are orthogonal to all codewords of C, i.e.,

$$C^\perp = \{x \mid x \cdot y = 0, \forall y \in C\}.$$

A code C is said to be *self-orthogonal* if $C \subseteq C^\perp$, and it is said to be *self-dual* if $C = C^\perp$. The following result is appeared in [16, 22, 28].

Proposition 2.3. *Let R be a finite chain ring of size p^α. The number of codewords in any linear code C of length n over R is p^k, for some integer k, $0 \leq k \leq \alpha n$. Moreover, the dual code $C^\perp$ has $p^{\alpha n - k}$ codewords, so that $|C| \cdot |C^\perp| = |R|^n$.*

The *Hamming weight* of x is the number of nonzero components of x, denoted by $\mathrm{wt}(x)$ for every word $x = (x_0, x_1, \ldots, x_{n-1}) \in R^n$. The Hamming distance $d(x, y)$ of two words x, y is the number of components in which they differ, which is the Hamming weight $\mathrm{wt}(x - y)$ of $x - y$. For a nonzero linear code C, the Hamming weight and the Hamming distance $d(C)$ are the same. They are defined as the smallest Hamming weight of nonzero codewords of C:

$$d(C) = \min\{\mathrm{wt}(x) \mid x \neq \mathbf{0},\ x \in C\}.$$

The zero code is conventionally said to have Hamming distance 0.

Given an n-tuple $(x_0, x_1, \ldots, x_{n-1}) \in R^n$, the *cyclic shift* τ and *negacyclic shift* ν on R^n are defined as usual, i.e.,

$$\tau(x_0, x_1, \ldots, x_{n-1}) = (x_{n-1}, x_0, x_1, \ldots, x_{n-2}),$$

and

$$\nu(x_0, x_1, \ldots, x_{n-1}) = (-x_{n-1}, x_0, x_1, \ldots, x_{n-2}).$$

A code C is called *cyclic* if $\tau(C) = C$, and C is called *negacyclic* if $\nu(C) = C$. More generally, if λ is a unit of the ring R, then the λ-*constacyclic* (λ-twisted) *shift* τ_λ on R^n is the shift

$$\tau_\lambda(x_0, x_1, \ldots, x_{n-1}) = (\lambda x_{n-1}, x_0, x_1, \ldots, x_{n-2}),$$

and a code C is said to be λ-*constacyclic* if $\tau_\lambda(C) = C$, i.e., if C is closed under the λ-constacyclic shift τ_λ. From this definition, when $\lambda = 1$, λ-constacyclic codes are cyclic codes, and when $\lambda = -1$, λ-constacyclic codes are just negacyclic codes.

Each codeword $c = (c_0, c_1, \ldots, c_{n-1})$ is customarily identified with its polynomial representation $c(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$, and the code C is in turn identified with the set of all polynomial representations of its codewords. Then in the ring $\frac{R[x]}{\langle x^n - \lambda \rangle}$, $xc(x)$ corresponds to a λ-constacyclic shift of $c(x)$. From this, the following fact is straightforward:

Proposition 2.4. *A linear code C of length n is λ-constacyclic over R if and only if C is an ideal of $\frac{R[x]}{\langle x^n - \lambda \rangle}$.*

We knew that the dual of a cyclic code is a cyclic code, and the dual of a negacyclic code is a negacyclic code. In general, the dual of a λ-constacyclic code is a λ^{-1}-constacyclic code (see, for example, [11], [13]).

The following result is also a fact appeared in [11], [13].

Proposition 2.5. *Let R be a finite commutative ring, λ be a unit of R and*

$$a(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1},\ b(x) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1} \in R[x].$$

Then $a(x)b(x) = 0$ in $\frac{R[x]}{\langle x^n-\lambda\rangle}$ if and only if $(a_0, a_1, \ldots, a_{n-1})$ is orthogonal to $(b_{n-1}, b_{n-2}, \ldots, b_0)$ and all its λ^{-1}-constacyclic shifts.

For a nonempty subset S of the ring R, the *annihilator* of S, denoted by $\mathrm{ann}(S)$, is the set

$$\mathrm{ann}(S) = \{f \in R \mid fg = 0 \text{ for all } g \in S\}.$$

Then $\mathrm{ann}(S)$ is an ideal of R.

For a polynomial f of degree k, the polynomial $x^kf(x^{-1})$ is called *reciprocal polynomial* of polynomial f. The reciprocal polynomial of f will be denoted by f^*. Suppose that $f(x) = a_0 + a_1x + \cdots + a_{k-1}x^{k-1} + a_kx^k$. Then $f^*(x) = x^k(a_0+a_1x^{-1}+\cdots+a_{k-1}x^{-(k-1)}+a_kx^{-k}) = a_k+a_{k-1}x+\cdots+a_1x^{k-1}+a_0x^k$. Note that $(f^*)^* = f$ if and only if the constant term of f is nonzero, if and only if $\deg(f) = \deg(f^*)$. We denote $A^* = \{f^*(x) \mid f(x) \in A\}$. It is easy to see that if A is an ideal, then A^* is also an ideal. Since the dual of a λ-constacyclic code is a λ^{-1}-constacyclic code, $C^\perp$ is a λ^{-1}-constacyclic codes of length n over R, and hence, $C^\perp$ is an ideal of the ring $\frac{R[x]}{\langle x^n-\lambda^{-1}\rangle}$, by Proposition 2.4. It is clear that $\mathrm{ann}^*(C)$ is also an ideal of $\frac{R[x]}{\langle x^n-\lambda^{-1}\rangle}$. Therefore, applying Proposition 2.5, we can conclude that $g(x) \in \mathrm{ann}^*(C)$ if and only if $g(x) = f^*(x)$ for some $f(x) \in \mathrm{ann}(C)$, if and only if $g(x) \in C^\perp$. Then, we have a following result.

Proposition 2.6. *Let R be a finite commutative ring, and λ be a unit of R. Assume that C is a λ-constacyclic code of length n over R. Then the dual $C^\perp$ of C is $\mathrm{ann}^*(C)$.*

We provide some results about the rings $\mathcal{R}_a = \frac{\mathbb{F}_{p^m}[u]}{\langle u^a\rangle} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{a-1}\mathbb{F}_{p^m}$ and its unit. The following result is introduced in [15].

Proposition 2.7 ([15, Proposition 3.1]). *Let $\mathcal{R}_a = \frac{\mathbb{F}_{p^m}[u]}{\langle u^a\rangle} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{a-1}\mathbb{F}_{p^m}$. Then*

(i) *$\mathcal{R}_a$ is a chain ring with maximal ideal $\langle u\rangle_{\mathcal{R}_a}$, and residue field $\mathbb{F}_{p^m}$.*

(ii) *The ideals of $\mathcal{R}_a$ are $\langle u^i\rangle_{\mathcal{R}_a} = u^i\mathcal{R}_a$, $0 \le i \le a$, each ideal $\langle u^i\rangle_{\mathcal{R}_a}$ contains $p^{m(a-i)}$ elements.*

(iii) *$\mathcal{R}_a$ has $(p^m-1)p^{m(a-1)}$ units, they are of the form*

$$\alpha_0 + u\alpha_1 + \cdots + u^{a-1}\alpha_{a-1},$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{a-1} \in \mathbb{F}_{p^m}$, $\alpha_0 \neq 0$.

Suppose that C is a code of length n over $\mathcal{R}_a$. We denote i_C, the smallest integer such that there is a nonzero component of a codeword of C belonging to $\langle u^{i_C}\rangle_{\mathcal{R}_a} \setminus \langle u^{i_C+1}\rangle_{\mathcal{R}_a}$. Clearly, $0 \le i_C \le a-1$, and $C \subseteq \langle u^{i_C}\rangle_{\mathcal{R}_a}^n \subseteq \mathcal{R}_a^n$.

In the following results, constacyclic codes over the chain ring $\mathcal{R}_a$ are investigated.

Proposition 2.8 ([15, Proposition 3.2])**.** *Let Λ be a unit of $\mathcal{R}_a$. If a code C of length n is Λ-constacyclic over $\mathcal{R}_a$, then C is also Γ-constacyclic for any unit Γ such that $\Gamma - \Lambda \in \langle u^j\rangle_{\mathcal{R}_a}$ for every $j \ge a - i_C$.*

Proposition 2.9 ([15, Proposition 3.3])**.** *Let C be a code of length n over $\mathcal{R}_a$, and Λ, Λ' be units of $\mathcal{R}_a$ such that $\Lambda - \Lambda' \in \langle u^j\rangle_{\mathcal{R}_a} \setminus \langle u^{j+1}\rangle_{\mathcal{R}_a}$, $0 \le j \le a - i_C$. If C is both Λ- and Λ'-constacyclic over $\mathcal{R}_a$, then $\langle u^{j+i_C}\rangle_{\mathcal{R}_a}^n \subseteq C$. In particular, if $\Lambda - \Lambda'$ is a unit, then $C = \langle u^{i_C}\rangle_{\mathcal{R}_a}^n$.*

In [15], the units of $\mathcal{R}_a$ are separated into a distinct types. A unit $\alpha = \alpha_0 + u\alpha_1 + \cdots + u^{a-1}\alpha_{a-1}$ of $\mathcal{R}_a$ is said to be of Type k, if k is the smallest index such that $\alpha_k \ne 0$ for an integer $k \in \{1, \ldots, a-1\}$. Moreover, if $\alpha_0 = 1$, then $1 + u\alpha_1 + \cdots + u^{a-1}\alpha_{a-1}$ is said to be of Type k^*. If $\alpha_i = 0$ for all $1 \le i \le a-1$, i.e., the unit is of the form $\alpha = \alpha_0 \in \mathbb{F}_{p^m}$, we say that α is of Type 0 (or Type 0^* if $\alpha_0 = 1$). $\mathcal{R}_a$ has $p^m - 1$ units of Type 0, and $(p^m-1)^2 p^{m(a-k-1)}$ units of Type k, showing that $\mathcal{R}_a$ has $p^m - 1$ Type 0 constacyclic codes and $(p^m-1)^2 p^{m(a-k-1)}$ Type k constacyclic codes.

We now suppose that Λ is a unit of Type k of $\mathcal{R}_a$. Then Λ can be expressed as following:

$$\Lambda = \Lambda_0 + u^k\Lambda_k + \cdots + u^{a-1}\Lambda_{a-1},$$

where $\Lambda_0, \Lambda_k, \ldots, \Lambda_{a-1} \in \mathbb{F}_{p^m}$, $\Lambda_0 \ne 0$, $\Lambda_k \ne 0$, and $1 \le k \le a-1$. Let $\lambda = 1 + u^k\lambda_k + \cdots + u^{a-1}\lambda_{a-1}$, for $k \le i \le a-1$, $\lambda_i = \Lambda_i\Lambda_0^{-1} \in \mathbb{F}_{p^m}$. Then we can see that λ is a unit of Type k^* such that $\Lambda = \Lambda_0\lambda$. It is easy to verify that in the case of Λ is a unit of Type 0 and λ is of Type 0^*, we also have $\Lambda = \Lambda_0\lambda$. The unit of Λ is determined in the following proposition.

Proposition 2.10 ([15])**.** *Let $\Lambda = \Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1}$ be a unit of $\mathcal{R}_a$, and t be the smallest positive integer such that $p^{tm} \ge a$. Then*

(a) $\Lambda^{-1} = \Lambda^{p^{tm}-1}\Lambda_0^{-1}$.

(b) *If Λ is of Type k, for $1 \le k \le a-1$, i.e., $\Lambda = \Lambda_0 + u^k\Lambda_k + \cdots + u^{a-1}\Lambda_{a-1}$, where $\Lambda_0 \ne 0$, $\Lambda_k \ne 0$, then Λ^{-1} is also of Type k. More precisely,*

$$\Lambda^{-1} = \Lambda_0^{-1} + u^k\Lambda'_k + \cdots + u^{a-1}\Lambda'_{a-1},$$

where $\Lambda'_k \ne 0$. If Λ is of Type 0, i.e., $\Lambda = \Lambda_0$, then $\Lambda^{-1} = \Lambda_0^{-1}$, which is of Type 0. In particular, for $0 \le \ell \le a-1$, Λ is of Type ℓ (resp. Type ℓ^*) if and only if Λ^{-1} is of Type ℓ (resp. Type ℓ^*).

(c) *Let Λ be of Type k, for $1 \le k \le a-1$. If $\Lambda = \Lambda^{-1}$, then $p = 2$, and $k \ge a/2$ (if a is even) or $k \ge \lfloor a/2 \rfloor + 1$ (if a is odd). More precisely,* *in such case, the units Λ such that $\Lambda = \Lambda^{-1}$ are precisely units of the form*

$$\Lambda = 1 + \sum_{i=a/2}^{a-1} u^i \Lambda_i^i \qquad \textit{if } a \textit{ is even, or}$$

$$\Lambda = 1 + \sum_{i=\lfloor a/2 \rfloor + 1}^{a-1} u^i \Lambda_i^i \qquad \textit{if } a \textit{ is odd,}$$

where $\Lambda_i \in \mathbb{F}_{2^m}$.

Let Λ-constacyclic code of length n over $\mathcal{R}_a$. We give the necessary and sufficient condition to have a self-dual of Λ-constacyclic code introduced in [15].

Proposition 2.11 ([15])**.** *Let $\Lambda = \Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1}$ be a unit of $\mathcal{R}_a$ such that $\Lambda_0^2 \neq 1$. Then there is a self-dual Λ-constacyclic code of length n over $\mathcal{R}_a$ if and only if a is even. In such case, $\langle u^{a/2} \rangle_{\mathcal{R}_a}^n$ is the unique self-dual Λ-constacyclic code of length n over $\mathcal{R}_a$.*

Remark 2.12 ([15]). When a is even, $\langle u^{a/2} \rangle_{\mathcal{R}_a}^n$ is always a self-dual Λ-constacyclic codes of length n over $\mathcal{R}_a$ for any unit Λ, without the condition $\Lambda_0^2 \neq 1$. However, when $\Lambda_0^2 = 1$, $\langle u^{a/2} \rangle_{\mathcal{R}_a}^n$ may not be the only self-dual Λ-constacyclic codes.

3. Type 1 Λ-constacyclic codes of length $2p^s$ over $\mathcal{R}_a$

The algebraic structure of all Type 1 Λ-constacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ is recently obtained in [8]. Motivated by this, we can study the Λ-constacyclic codes of length $2p^s$ over $\mathcal{R}_a$.

In this paper, we study Λ-constacyclic codes of length $2p^s$ over $\mathcal{R}_a$ and its dual, where Λ is a unit of Type 1 of $\mathcal{R}_a$. This means that Λ has the following form:

$$\Lambda = \Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1},$$

where $\Lambda_0, \Lambda_1, \ldots, \Lambda_{a-1} \in \mathbb{F}_{p^m}$, $\Lambda_0 \neq 0$, $\Lambda_1 \neq 0$. By Proposition 2.4, we know that these codes are ideals of the ring

$$\mathcal{S}_a(s, \Lambda) = \frac{\mathcal{R}_a[x]}{\langle x^{2p^s} - \Lambda \rangle}.$$

From Proposition 2.10, Λ^{-1} is also a unit of Type 1, which can be written as follows.

$$\Lambda^{-1} = \Lambda_0^{-1} + u\Lambda_1' + \cdots + u^{a-1}\Lambda_{a-1}',$$

where $\Lambda_1' \neq 0$.

If the unit Λ is a square in $\mathcal{R}_a$, i.e., there exists a unit $\alpha \in \mathcal{R}_a$ such that $\Lambda = \alpha^2$. Then we have

$$x^{2p^s} - \Lambda = x^{2p^s} - \alpha^2 = (x^{p^s} + \alpha)(x^{p^s} - \alpha).$$

Applying the Chinese remainder theorem, we can see that

$$\mathcal{S}_a(s,\Lambda) = \frac{\mathcal{R}_a[x]}{\langle x^{p^s}+\alpha\rangle} \oplus \frac{\mathcal{R}_a[x]}{\langle x^{p^s}-\alpha\rangle}.$$

This follows that all ideals of $\mathcal{S}_a(s,\Lambda)$ are of the form $A \oplus B$, where A and B are ideals of $\frac{\mathcal{R}_a[x]}{\langle x^{p^s}+\alpha\rangle}$ and $\frac{\mathcal{R}_a[x]}{\langle x^{p^s}-\alpha\rangle}$, respectively, i.e., they are $-\alpha$- and α-constacyclic codes of length p^s over $\mathcal{R}_a$. Hence, if Λ is a square in $\mathcal{R}_a$, a Type 1 Λ-constacyclic code of length $2p^s$ over $\mathcal{R}_a$ is expressed as a direct sum of C_+ and C_-:

$$C = C_+ \oplus C_-,$$

where C_+ and C_- are ideals of $\frac{\mathcal{R}_a[x]}{\langle x^{p^s}+\alpha\rangle}$ and $\frac{\mathcal{R}_a[x]}{\langle x^{p^s}-\alpha\rangle}$, respectively. The classification, detailed structure, and number of codewords of α and $-\alpha$ constacyclic codes length p^s were investigated in [15]. Thus, when Λ is a square in $\mathcal{R}_a$, we can obtain Λ-constacyclic codes C of length $2p^s$ over $\mathcal{R}_a$ from that of the direct summands C_+ and C_- (cf. [15]). Hence, we can prove that the dual code $C^\perp$ of C is also a direct sum of the dual codes of the direct summand $C_+^\perp$ and $C_-^\perp$.

Theorem 3.1. *Let the unit $\Lambda = \alpha^2 \in \mathcal{R}_a$, and $C = C_+ \oplus C_-$ be a constacyclic code of length $2p^s$ over $\mathcal{R}_a$, where C_+, C_- are ideals of $\frac{\mathcal{R}_a[x]}{\langle x^{p^s}+\alpha\rangle}$, $\frac{\mathcal{R}_a[x]}{\langle x^{p^s}-\alpha\rangle}$, respectively. Then*

$$C^\perp = C_+^\perp \oplus C_-^\perp.$$

In particular, C is a self-dual Λ-constacyclic code of length $2p^s$ over $\mathcal{R}_a$ if and only if C_+, C_- are self-dual $-\alpha$-constacyclic code and self-dual α-constacyclic code of length p^s over $\mathcal{R}_a$, respectively.

Proof. It is easy to verify that $C_+^\perp \oplus C_-^\perp \subseteq C^\perp$. On the other hand,

$$|C_+^\perp \oplus C_-^\perp| = |C_+^\perp| \cdot |C_-^\perp| = \frac{|\mathcal{R}_a|^{p^s}}{|C_+|} \cdot \frac{|\mathcal{R}_a|^{p^s}}{|C_-|} = \frac{|\mathcal{R}_a|^{2p^s}}{|C_+| \cdot |C_-|} = \frac{|\mathcal{R}_a|^{2p^s}}{|C|} = |C^\perp|.$$

This implies that $C^\perp = C_+^\perp \oplus C_-^\perp$. □

Therefore, we only need to concentrate on the main case where Λ is not a square in $\mathcal{R}_a$. We first start by characterizing this condition.

Proposition 3.2. *Let $\Lambda = \Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1}$, $\Lambda_0, \Lambda_1, \ldots, \Lambda_{a-1} \in \mathbb{F}_{p^m}$, $\Lambda_0 \neq 0$, $\Lambda_1 \neq 0$, be a unit of Type 1 of $\mathcal{R}_a$. Then Λ is not a square if and only if Λ_0 is not a square.*

Proof. Suppose that $\Lambda_0'^2 = \Lambda_0$, we consider $(\Lambda_0' + u\Lambda_1' + u^2\Lambda_2' + u^3\Lambda_3' + \cdots + u^{a-1}\Lambda_{a-1}')^2$, where $\Lambda_i' \in \mathbb{F}_{p^m}$. Assume that $(\Lambda_0' + u\Lambda_1' + u^2\Lambda_2' + u^3\Lambda_3' + \cdots + u^{a-1}\Lambda_{a-1}')^2 = \Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1}$.

Case 1: a is even. We have

$$\begin{aligned}
\Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1} &= (\Lambda'_0 + u\Lambda'_1 + u^2\Lambda'_2 + u^3\Lambda'_3 \cdots + u^{a-1}\Lambda'_{a-1})^2 \\
&= \Lambda'^2_0 + u(2\Lambda'_0\Lambda'_1) + u^2(\Lambda'^2_1 + 2\Lambda'_0\Lambda'_2) + \cdots \\
&\quad + u^{2t}(2\Lambda'_{2t}\Lambda'_0 + \Lambda'^2_t + 2\sum_{j+k=2t}\Lambda'_j\Lambda'_k) + \\
&\quad + u^{2t+1}(2\Lambda'_{2t+1}\Lambda'_0 + 2\sum_{c+d=2t+1}\Lambda'_c\Lambda'_d) + \cdots \\
&\quad + u^{a-1}(2\Lambda'_{a-1}\Lambda'_0 + 2\sum_{l+h=a-1}\Lambda'_l\Lambda'_h),
\end{aligned}$$

where $0 < j,k < 2t < a-1$, $0 < c,d < 2t+1 < a-1$, and $0 < l,h < a-1$. Comparing coefficients, we have

$$\begin{aligned}
\Lambda_0 &= \Lambda'^2_0, \\
\Lambda_1 &= 2\Lambda'_0\Lambda'_1, \\
\Lambda_2 &= \Lambda'^2_1 + 2\Lambda'_0\Lambda'_2, \\
&\vdots \\
\Lambda_{2t} &= 2\Lambda'_{2t}\Lambda'_0 + \Lambda'^2_t + 2\sum_{j+k=2t}\Lambda'_j\Lambda'_k, \\
\Lambda_{2t+1} &= 2\Lambda'_{2t+1}\Lambda'_0 + 2\sum_{c+d=2t+1}\Lambda'_c\Lambda'_d, \\
&\vdots \\
\Lambda_{a-1} &= 2\Lambda'_{a-1}\Lambda'_0 + 2\sum_{l+h=a-1}\Lambda'_l\Lambda'_h.
\end{aligned}$$

Since Λ'^{-1}_0 exists, we can compute

$$\begin{aligned}
\Lambda'_1 &= 2^{-1}\Lambda'^{-1}_0\Lambda_1, \\
\Lambda'_2 &= 2^{-1}\Lambda'^{-1}_0(\Lambda_2 - \Lambda'^2_1), \\
\Lambda'_3 &= 2^{-1}\Lambda'^{-1}_0(\Lambda_3 - 2\Lambda'_1\Lambda'_2), \\
&\vdots \\
\Lambda'_{2t} &= 2^{-1}\Lambda'^{-1}_0(\Lambda_{2t} - \Lambda'^2_t - 2\sum_{j+k=2t}\Lambda'_j\Lambda'_k), \\
\Lambda'_{2t+1} &= 2^{-1}\Lambda'^{-1}_0(\Lambda_{2t+1} - 2\sum_{c+d=2t+1}\Lambda'_c\Lambda'_d), \\
&\vdots
\end{aligned}$$

$$\Lambda'_{a-1} = 2^{-1}\Lambda_0'^{-1}(\Lambda_{a-1} - 2\sum_{l+h=a-1}\Lambda'_l\Lambda'_h),$$

where $0 < j,k < 2t < a-1$, $0 < c,d < 2t+1 < a-1$, and $0 < l,h < a-1$. Therefore, for each Λ_i, we can determine Λ'_i such that $\Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1} = (\Lambda'_0 + u\Lambda'_1 + \cdots + u^{a-1}\Lambda'_{a-1})^2$, proving that Λ is a square.

Case 2: a is odd. Suppose that

$$\begin{aligned}\Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1} &= (\Lambda'_0 + u\Lambda'_1 + u^2\Lambda'_2 + u^3\Lambda'_3 \cdots + u^{a-1}\Lambda'_{a-1})^2\\ &= \Lambda_0'^2 + u(2\Lambda'_0\Lambda'_1) + u^2(\Lambda_1'^2 + 2\Lambda'_0\Lambda'_2) + \cdots\\ &\quad + u^{2t}(2\Lambda'_{2t}\Lambda'_0 + \Lambda_t'^2 + 2\sum_{j+k=2t}\Lambda'_j\Lambda'_k) +\\ &\quad + u^{2t+1}(2\Lambda'_{2t+1}\Lambda'_0 + 2\sum_{c+d=2t+1}\Lambda'_c\Lambda'_d) + \cdots\\ &\quad + u^{a-1}(2\Lambda'_{a-1}\Lambda'_0 + \Lambda'^2_{\frac{a-1}{2}} + 2\sum_{j+k=a-1}\Lambda'_j\Lambda'_k).\end{aligned}$$

Similar to the case 1, we compare coefficients. Then we have

$$\begin{aligned}\Lambda'_1 &= 2^{-1}\Lambda_0'^{-1}\Lambda_1,\\ \Lambda'_2 &= 2^{-1}\Lambda_0'^{-1}(\Lambda_2 - \Lambda_1'^2),\\ \Lambda'_3 &= 2^{-1}\Lambda_0'^{-1}(\Lambda_3 - 2\Lambda'_1\Lambda'_2),\\ &\vdots\\ \Lambda'_{2t} &= 2^{-1}\Lambda_0'^{-1}(\Lambda_{2t} - \Lambda_t'^2 - 2\sum_{j+k=2t}\Lambda'_j\Lambda'_k),\\ \Lambda'_{2t+1} &= 2^{-1}\Lambda_0'^{-1}(\Lambda_{2t+1} - 2\sum_{c+d=2t+1}\Lambda'_c\Lambda'_d),\\ &\vdots\\ \Lambda'_{a-1} &= 2^{-1}\Lambda_0'^{-1}(\Lambda_{a-1} - \Lambda'^2_{\frac{a-1}{2}} - 2\sum_{l+h=a-1}\Lambda'_l\Lambda'_h),\end{aligned}$$

where $0 < j,k < 2t < a-1$, $0 < c,d < 2t+1 < a-1$, and $0 < l,h < a-1$. Hence, we can express $\Lambda_0 + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1} = (\Lambda'_0 + u\Lambda'_1 + \cdots + u^{a-1}\Lambda'_{a-1})^2$.

Combining Cases 1 and 2, Λ is not a square if and only if Λ_0 is not a square. □

From this, we can prove the following result.

Proposition 3.3. *Any nonzero linear polynomial $cx + d \in \mathbb{F}_{p^m}[x]$ is invertible in $\mathcal{S}_a(s,\Lambda)$.*

Proof. In $\mathcal{S}_a(s,\Lambda)$, we have

$$(x+d)^{p^s}(x-d)^{p^s} = (x^2-d^2)^{p^s} = x^{2p^s}-d^{2p^s} = (\Lambda_0-d^{2p^s})+u\Lambda_1+\cdots+u^{a-1}\Lambda_{a-1}.$$

Since Λ_0 is not a square in $\mathbb{F}_{p^m}$, $\Lambda_0 - d^{2p^s}$ is invertible in $\mathbb{F}_{p^m}$. This follows that $(\Lambda_0 - d^{2p^s}) + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1}$ is invertible in $\mathcal{S}_a(s,\Lambda)$. Thus,

$$(x+d)^{-1} = (x+d)^{p^s-1}(x-d)^{p^s}(\Lambda_0 - d^{2p^s} + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1})^{-1}.$$

Therefore, for any $c \neq 0$ in $\mathbb{F}_{p^m}$,

$$\begin{aligned}(cx+d)^{-1} &= c^{-1}(x+c^{-1}d)^{-1} \\ &= c^{-1}(x+c^{-1}d)^{p^s-1}(x-c^{-1}d)^{p^s} \\ &\quad (\Lambda_0 - c^{-2p^s}d^{2p^s} + u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1})^{-1}. \qquad \square\end{aligned}$$

Since $\Lambda_0 \in \mathbb{F}_{p^m}$, we have $\Lambda_0^{p^{tm}} = \Lambda_0$ for any positive integer t. By the Division Algorithm, there exist nonnegative integers α_q, α_r such that $s = \alpha_q m + \alpha_r$, and $0 \leq \alpha_r \leq m-1$. Let $\alpha_0 = \Lambda_0^{p^{(\alpha_q+1)m-s}} = \Lambda_0^{p^{m-\alpha_r}}$. Then $\alpha_0^{p^s} = \Lambda_0^{p^{(\alpha_q+1)m}} = \Lambda_0$. The following provides the key to prove that the ring $\mathcal{S}_a(s,\Lambda)$ is a chain ring.

Lemma 3.4. *In $\mathcal{S}_a(s,\Lambda)$, we have $\langle (x^2-\alpha_0)^{p^s} \rangle = \langle u \rangle$. In particular, $x^2-\alpha_0$ is nilpotent with nipotency index ap^s.*

Proof. The results follow from the fact that in $\mathcal{S}_a(s,\Lambda)$, $(x^2-\alpha_0)^{p^s} = x^{2p^s} - \Lambda_0 = u\Lambda_1 + \cdots + u^{a-1}\Lambda_{a-1}$. $\square$

Any element $f(x)$ of $\mathcal{S}_a(s,\Lambda)$ can be expressed as a polynomial of degree up to $2p^s-1$ of $\mathcal{R}_a[x]$, and so $f(x) = f_1(x) + uf_2(x) + \cdots + u^{a-1}f_a(x)$, where $f_1(x), f_2(x), \ldots, f_a(x)$ are polynomials of degrees up to $2p^s-1$ of $\mathbb{F}_{p^m}[x]$. Thus, $f(x)$ can be uniquely represented as

$$\begin{aligned}f(x) = &\sum_{i=0}^{p^s-1}(c_{0i}x+d_{0i})(x^2-\alpha_0)^i + u\sum_{i=0}^{p^s-1}(c_{1i}x+d_{1i})(x^2-\alpha_0)^i + \cdots \\ &+ u^{a-1}\sum_{i=0}^{p^s-1}(c_{(a-1)i}x+d_{(a-1)i})(x^2-\alpha_0)^i \\ = &(c_{00}x+d_{00}) + (x^2-\alpha_0)\sum_{i=1}^{p^s-1}(c_{0i}x+d_{0i})(x^2-\alpha_0)^{i-1} \\ &+ u\sum_{i=0}^{p^s-1}(c_{1i}x+d_{1i})(x^2-\alpha_0)^i + \cdots \\ &+ u^{a-1}\sum_{i=0}^{p^s-1}(c_{(a-1)i}x+d_{(a-1)i})(x^2-\alpha_0)^i,\end{aligned}$$

where $c_{0i}, d_{0i}, \ldots, c_{(a-1)i}, d_{(a-1)i} \in \mathbb{F}_{p^m}$. By Lemma 3.4, $u \in \langle x^2-\alpha_0 \rangle$, and so $f(x)$ can be written as

$$f(x) = (c_{00}x+d_{00}) + (x^2-\alpha_0)g(x).$$

Thus, $f(x)$ is non-invertible if and only if $c_{00} = d_{00} = 0$, i.e., $f(x) \in \langle x^2 - \alpha_0 \rangle$. It means that $\langle x^2 - \alpha_0 \rangle$ forms the set of all non-invertible elements of $\mathcal{R}_a$. Thus, $\mathcal{S}_a(s, \Lambda)$ is a local ring with maximal ideal $\langle x^2 - \alpha_0 \rangle$, hence, by Proposition 2.1, $\mathcal{S}_a(s, \Lambda)$ is a chain ring. We summarize the discussion above in the following theorem.

Theorem 3.5. *The ring $\mathcal{S}_a(s, \Lambda)$ is a chain ring with maximal ideal $\langle x^2 - \alpha_0 \rangle$, whose ideals are*

$$\mathcal{S}_a(s, \Lambda) = \langle 1 \rangle \supsetneq \langle x^2 - \alpha_0 \rangle \supsetneq \cdots \supsetneq \langle (x^2 - \alpha_0)^{ap^s - 1} \rangle \supsetneq \langle (x^2 - \alpha_0)^{ap^s} \rangle = \langle 0 \rangle.$$

From Theorem 3.5, we now can give the structure of Type 1 Λ-constacyclic codes of length $2p^s$ over $\mathcal{R}_a$, and their sizes as follows.

Theorem 3.6. *Type 1 Λ-constacyclic codes of length $2p^s$ over $\mathcal{R}_a$ are precisely the ideals $\langle (x^2 - \alpha_0)^i \rangle \subseteq \mathcal{R}_a$, where $0 \le i \le ap^s$. Each Type 1 Λ-constacyclic code $\langle (x^2 - \alpha_0)^i \rangle$ has $p^{2m(ap^s - i)}$ codewords.*

For a Type 1 Λ-constacyclic code $C = \langle (x^2 - \alpha_0)^i \rangle \subseteq \mathcal{R}_a$ of length $2p^s$ over $\mathcal{R}_a$, by Proposition 2.5 and Proposition 2.10, its dual $C^\perp$ is a Type 1 Λ^{-1}-constacyclic code of length $2p^s$ over $\mathcal{R}_a$. This means

$$C^\perp \subseteq \mathcal{S}_a(s, \Lambda^{-1}) = \frac{\mathcal{R}_a[x]}{\langle x^{2p^s} - \Lambda^{-1} \rangle}.$$

Hence, Lemma 3.4 and Theorem 3.5 are applicable for $C^\perp$ and $\mathcal{S}_a(s, \Lambda^{-1})$. Therefore, similar to the case of $\mathcal{S}_a(s, \Lambda)$, we can prove that $\mathcal{S}_a(s, \Lambda^{-1})$ is a chain ring.

Theorem 3.7. *The ring $\mathcal{S}_a(s, \Lambda^{-1})$ is a chain ring with maximal ideal $\langle x^2 - \alpha_0^{-1} \rangle$, whose ideals are*

$$\mathcal{S}_a(s, \Lambda^{-1}) = \langle 1 \rangle \supsetneq \langle x^2 - \alpha_0^{-1} \rangle \supsetneq \cdots \supsetneq \langle (x^2 - \alpha_0^{-1})^{ap^s - 1} \rangle \supsetneq \langle (x^2 - \alpha_0^{-1})^{ap^s} \rangle = \langle 0 \rangle.$$

In other words, Type 1 Λ^{-1}-constacyclic codes of length $2p^s$ over $\mathcal{R}_a$ are precisely the ideals $\langle (x^2 - \alpha_0^{-1})^i \rangle \subseteq \mathcal{S}_a(s, \Lambda^{-1})$, where $0 \le i \le ap^s$. Each Type 1 Λ^{-1}-constacyclic code $\langle (x^2 - \alpha_0^{-1})^i \rangle \subseteq \mathcal{S}_a(s, \Lambda^{-1})$ has p^{2mi} codewords.

Applying Theorem 3.7, we now can describe the duals of Type 1 Λ-constacyclic codes in the following corollary.

Corollary 3.8. *Let C be a Type 1 Λ-constacyclic code of length $2p^s$ over $\mathcal{R}_a$. Then $C = \langle (x^2 - \alpha_0)^i \rangle \subseteq \mathcal{R}_a$, for some $i \in \{0, 1, \ldots, ap^s\}$, and its dual $C^\perp$ is the Type 1 Λ^{-1}-constacyclic code*

$$C^\perp = \left\langle (x^2 - \alpha_0^{-1})^{ap^s - i} \right\rangle \subseteq \mathcal{R}_a.$$

Proof. Let $C = \langle (x^2 - \alpha_0)^i \rangle \subseteq \mathcal{S}_a(s, \Lambda)$ be a Type 1 Λ-constacyclic code of length $2p^s$ over $\mathcal{R}_a$. Then, $C^\perp$ is an ideal of $\mathcal{S}_a(s, \Lambda^{-1})$. By Theorem 3.7, $|C| = p^{2m(ap^s - i)}$, and hence, by Proposition 2.3,

$$|C^\perp| = \frac{|\mathcal{R}_a|^{2p^s}}{|C|} = \frac{p^{2map^s}}{p^{2m(ap^s - i)}} = p^{2mi}.$$

From Theorem 3.7, we have $C^{\perp} = \left\langle (x^2 - \alpha_0^{-1})^{ap^s - i} \right\rangle \subseteq \mathcal{S}_a(s, \Lambda^{-1})$. □

4. Rosenbloom-Tsfasman distance

In 1997, Rosenbloom and Tsfasman [33] introduced a new distance in coding theory, which was later named after them as the Rosenbloom-Tsfasman (RT) distance. Well-known bounds for distances such as the Singleton bound, the Plotkin bound, the Hamming bound, and the Gilbert bound were derived for the RT distance. Since then, there are many other studies focusing on codes with respect to this RT metric (see, for example, [9, 19, 23, 35]).

For any finite commutative ring R, the *Rosenbloom-Tsfasman weight* (RT weight) (see [33]) of an n-tuple $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in R^n$ is defined as follows:

$$\mathrm{wt}_{\mathrm{RT}}(\mathbf{x}) = \begin{cases} 1 + \max\{j \mid x_j \neq 0\}, & \text{if } \mathbf{x} \neq \mathbf{0}; \\ 0, & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

The *Rosenbloom-Tsfasman distance* (RT distance) of any two n-tuples $\mathbf{x}, \mathbf{y}$ of R^n is defined as:

$$d_{\mathrm{RT}}(\mathbf{x}, \mathbf{y}) = \mathrm{wt}_{\mathrm{RT}}(\mathbf{x} - \mathbf{y}).$$

Let C be a code of length n over R. Then

$$d_{\mathrm{RT}}(C) = \min\{d_{\mathrm{RT}}(\mathbf{c}, \mathbf{c}') \mid \mathbf{c} \neq \mathbf{c}' \in C\}$$

is called the *RT distance* of C.

In this section we proceed to compute the Rosenbloom-Tsfasman distances of all Λ-constacyclic codes of length $2p^s$ over the ring $\mathcal{R}_a$ for any unit Λ of Type 1 of $\mathcal{R}_a$ such that Λ is not a square. We start with an observation, that is followed readily from the definition of the RT weight.

Proposition 4.1. *Let $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in R^n$ be a word of length n over R, and $c(x)$ be its polynomial presentation. Then*

$$\mathrm{wt}_{\mathrm{RT}}(\mathbf{c}) = \begin{cases} 1 + \deg(c(x)), & \text{if } \mathbf{c} \neq \mathbf{0}; \\ 0, & \text{if } \mathbf{c} = \mathbf{0}. \end{cases}$$

Theorem 4.2. *Let Λ be a unit of Type 1 of $\mathcal{R}_a$ such that Λ is not a square. Assume that C is a Λ-constacyclic code of length $2p^s$ over $\mathcal{R}_a$, i.e., $C = \langle (x^2 - \alpha_0)^i \rangle \subseteq \mathcal{S}_a(s, \Lambda)$ for some $i \in \{0, 1, \ldots, ap^s\}$. Then the Rosenbloom-Tsfasman distance $d_{\mathrm{RT}}(C)$ of C is completely determined as follows.*

$$d_{\mathrm{RT}}(C) = \begin{cases} 0 & \text{if } \; i = ap^s \;, \\ 1 & \text{if } \; 0 \leq i \leq (a-1)p^s \;, \\ 2i - 2(a-1)p^s + 1 & \text{if } \; (a-1)p^s + 1 \leq i \leq ap^s - 1. \end{cases}$$

Proof. If $i = ap^s$, the code C is just the zero code, and the result follows trivially. By Lemma 3.4 and Theorem 3.5, when $0 \leq i \leq (a-1)p^s$,

$$\left\langle (x^2 - \alpha_0)^i \right\rangle \supseteq \left\langle (x^2 - \alpha_0)^{(a-1)p^s} \right\rangle = \left\langle u^{a-1} \right\rangle,$$

which implies that the RT distance of the code $\left\langle (x^2-\alpha_0)^i \right\rangle$ is 1. Consider the case $(a-1)p^s+1 \le i \le ap^s-1$, we have

$$\begin{aligned}\left\langle (x^2-\alpha_0)^i \right\rangle &= \left\langle (x^2-\alpha_0)^{(a-1)p^s}(x^2-\alpha_0)^{i-(a-1)p^s} \right\rangle \\ &= \left\langle u^{a-1}(x^2-\alpha_0)^{i-(a-1)p^s} \right\rangle .\end{aligned}$$

It suffices to show that, in each ideal $\left\langle u^{a-1}(x^2-\alpha_0)^{i-(a-1)p^s} \right\rangle$, the generator polynomial $u^{a-1}(x^2-\alpha_0)^{i-(a-1)p^s}$ is of smallest degree, which is $2i-2(a-1)p^s$. Hence, in light of Proposition 4.1, its RT distance is $2i-2(a-1)p^s+1$. Suppose that $f(x)$ is a nonzero polynomial in $\left\langle u^{a-1}(x^2-\alpha_0)^{i-(a-1)p^s} \right\rangle$ of degree $0 \le k < 2i-2(a-1)p^s$, then $f(x)$ can be expressed as

$$f(x) = \sum_{j=0}^{k} (c_j x + d_j)(x^2-\alpha_0)^j,$$

where $c_j, d_j \in \mathcal{R}_a$. Let ℓ $(0 \le \ell \le k)$ be the smallest index such that $c_j x + d_j \neq 0$, then

$$\begin{aligned}f(x) &= (x^2-\alpha_0)^\ell \sum_{j=\ell}^{k} (c_j x + d_j)(x^2-\alpha_0)^{j-\ell} \\ &= (x^2-\alpha_0)^\ell (c_\ell x + d_\ell)\left[1 + (x^2-\alpha_0)g(x)\right],\end{aligned}$$

where

$$g(x) = \begin{cases} 0, & \text{if } \ell = k, \\ (c_\ell x + d_\ell)^{-1} \sum\limits_{j=\ell+1}^{k} (c_j x + d_j)(x^2-\alpha_0)^{j-\ell-1}, & \text{if } 0 \le \ell < k, \end{cases}$$
$$\in \mathcal{S}_a(s,\Lambda).$$

Since, in $\mathcal{S}_a(s,\Lambda)$, $x^2-\alpha_0$ is nilpotent, there is an odd integer t such that $(x^2-\alpha_0)^t = 0$, we get

$$\begin{aligned}1 &= 1 + \left[(x^2-\alpha_0)g(x)\right]^t \\ &= \left[1 + (x^2-\alpha_0)g(x)\right] \\ &\quad \left[1 - (x^2-\alpha_0)g(x) + (x^2-\alpha_0)^2 g(x)^2 - \cdots + (x^2-\alpha_0)^{t-1} g(x)^{t-1}\right],\end{aligned}$$

implying that $1 + (x^2-\alpha_0)g(x)$ is invertible in $\mathcal{S}_a(s,\Lambda)$. Therefore, $f(x) = (x^2-\alpha_0)^\ell h(x)$ for some unit $h(x)$ of $\mathcal{S}_a(s,\Lambda)$. That means $f(x) \in \langle (x^2-\alpha_0)^\ell \rangle$, but $f(x) \notin \langle (x^2-\alpha_0)^{\ell+1} \rangle$, and in particular, $f(x) \notin C$. Thus, we have shown that any nonzero polynomial of degree less than $2i-2(a-1)p^s$ is not in C, i.e., the smallest degree of nonzero polynomials in C is $2i-2(a-1)p^s$, as desired. □

Proposition 4.3. *For $(a-1)p^s+1 \le i \le ap^s-1$, the RT weight distribution of the Type 1 Λ-constacyclic code $\langle (x^2-\alpha_0)^i \rangle \subseteq \mathcal{S}_a(s,\Lambda)$ is as follows.*

$$A_j = \begin{cases} 1 & \text{if } j=0, \\ 0 & \text{if } 1 \le j \le 2i-2(a-1)p^s, \\ (p^m-1)p^{mk} & \text{if } j = 2i-2(a-1)p^s+1+k \\ & \quad \text{for } 0 \le k \le 2ap^s-2i-1, \end{cases}$$

where A_j is the number of codewords of RT weight j of $\langle (x^2-\alpha_0)^i \rangle$.

Proof. As in the proof of Theorem 4.2, when $(a-1)p^s+1 \le i \le ap^s-1$, $\left\langle (x^2-\alpha_0)^i \right\rangle = \left\langle u^{a-1}(x^2-\alpha_0)^{i-(a-1)p^s} \right\rangle$, and so $A_j = 0$ for $1 \le j \le 2i-2(a-1)p^s$. When $2i-2(a-1)p^s+1 \le j \le 2p^s$, say, $j = 2i-2(a-1)p^s+1+k$, for $0 \le k \le 2ap^s-2i-1$, then A_j is the number of distinct polynomials of degree k in $\mathbb{F}_{p^m}[x]$. Thus, $A_j = (p^m-1)p^{mk}$. ☐

When $i = p^s t$, $0 \le t \le a-1$, by Lemma 3.4, the ideals $\langle (x^2-\alpha_0)^i \rangle = \langle u^t \rangle \subseteq \mathcal{S}_a(s,\Lambda)$. Thus, we get their weight distributions as follows.

Proposition 4.4. *For $i = p^s t$, $0 \le t \le a-1$, the RT weight distribution of the Λ-constacyclic code $\langle (x^2-\alpha_0)^i \rangle \subseteq \mathcal{S}_a(s,\Lambda)$ is as follows.*

$$A_j = \begin{cases} 1 & \text{if } j=0, \\ \left(p^{m(a-t)}-1\right) p^{m(a-t)(j-1)} & \text{if } 1 \le j \le 2p^s, \end{cases}$$

where A_j is the number of codewords of RT weight j of $\langle (x^2-\alpha_0)^i \rangle$.

Proposition 4.5. *Let $1 \le b \le a-1$. For $(b-1)p^s+1 \le i \le bp^s-1$, the RT weight distribution of the Λ-constacyclic code $\langle (x^2-\alpha_0)^i \rangle \subseteq \mathcal{S}_a(s,\Lambda)$ is as follows.*

$$A_j = \begin{cases} 1 \text{ if } j=0, \\ \left(p^{m(a-b)}-1\right) p^{m(a-b)(j-1)} \text{ if } 1 \le j \le 2i-2(b-1)p^s, \\ p^{2m(a-b)p^s}(p^m-1)p^{mk} + \left(p^{m(a-b)}-1\right) p^{m(a-b)(j-1)} \\ \text{if } j = 2i-2(b-1)p^s+1+k \text{ for } 0 \le k \le 2bp^s-2i-1, \end{cases}$$

where A_j is the number of codewords of RT weight j of $\langle (x^2-\alpha_0)^i \rangle$.

Proof. We have $(b-1)p^s+1 \le i \le (b-1)p^s+p^s-1$, i.e., $1 \le i-(b-1)p^s \le p^s-1$, so by Lemma 3.4,

$$\begin{aligned} \left\langle u^{b-1}(x^2-\alpha_0) \right\rangle \supseteq \left\langle (x^2-\alpha_0)^i \right\rangle &= \left\langle u^{b-1}(x^2-\alpha_0)^{i-p^s(b-1)} \right\rangle \\ &\supseteq \left\langle u^{b-1}(x^2-\alpha_0)^{p^s-1} \right\rangle \supsetneq \left\langle u^b \right\rangle. \end{aligned}$$

Let B_j be the number of codewords of RT weight j of $\langle (x^2-\alpha_0)^i \rangle$, which are not in $\langle u^b \rangle$; and B'_j be the number of codewords of RT weight j of $\langle u^b \rangle$. Then, for all j, $A_j = B_j + B'_j$. Similar to Proposition 4.3, we get

$$B_j = \begin{cases} 0 & \text{if } j = 0, \\ 0 & \text{if } 1 \le j \le 2i - 2(b-1)p^s, \\ p^{2m(a-b)p^s}(p^m - 1)p^{mk} & \text{if } j = 2i - 2(b-1)p^s + 1 + k \\ & \quad \text{for } 0 \le k \le 2bp^s - 2i - 1. \end{cases}$$

Clearly, by Proposition 4.4,

$$B'_j = \begin{cases} 1 & \text{if } j = 0, \\ \left(p^{m(a-b)} - 1\right) p^{m(a-b)(j-1)} & \text{if } 1 \le j \le 2p^s. \end{cases}$$

Therefore,

$$A_j = \begin{cases} 1 \text{ if } j = 0, \\ \left(p^{m(a-b)} - 1\right) p^{m(a-b)(j-1)} \text{ if } 1 \le j \le 2i - 2(b-1)p^s, \\ p^{2m(a-b)p^s}(p^m - 1)p^{mk} + \left(p^{m(a-b)} - 1\right) p^{m(a-b)(j-1)} \\ \text{if } j = 2i - 2(b-1)p^s + 1 + k \text{ for } 0 \le k \le 2bp^s - 2i - 1. \end{cases}$$

□

Remark 4.6. Propositions 4.3, 4.4, and 4.5 give us the RT weight distributions for all λ-constacyclic codes $C_i = \langle (x^2 - \alpha_0)^i \rangle \subseteq \mathcal{S}_a(s, \Lambda)$ of length $2p^s$ over $\mathcal{R}_a$. By Theorem 3.5, $|C_i| = p^{2m(ap^s - i)}$. As $|C_i| = \sum_{j=0}^{2p^s} A_j$, these RT weight distributions can be used to verify the size $|C_i|$ of such codes.

- If $(a-1)p^s + 1 \le i \le ap^s - 1$, then

$$\begin{aligned} |C_i| &= \sum_{j=0}^{2p^s} A_j \\ &= 1 + \sum_{k=0}^{2ap^s - 2i - 1} (p^m - 1)p^{mk} \\ &= 1 + (p^m - 1) \sum_{k=0}^{2ap^s - 2i - 1} (p^m)^k \\ &= 1 + (p^m - 1) \frac{p^{m(2ap^s - 2i)} - 1}{p^m - 1} \\ &= p^{2m(ap^s - i)}. \end{aligned}$$

- If $i = p^s t$, $0 \le t \le a - 1$, then

$$|C_i| = \sum_{j=0}^{2p^s} A_j$$

$$= 1 + \sum_{j=1}^{2p^s} \left(p^{m(a-t)} - 1\right) p^{m(a-t)(j-1)}$$
$$= 1 + \left(p^{m(a-t)} - 1\right) \sum_{j=0}^{2p^s-1} p^{m(a-t)j}$$
$$= 1 + \left(p^{m(a-t)} - 1\right) \frac{p^{m(a-t)2p^s} - 1}{p^{m(a-t)} - 1}$$
$$= p^{2m(a-t)p^s}$$
$$= p^{2m(ap^s-i)}.$$

- If $(b-1)p^s + 1 \leq i \leq bp^s - 1$, where $1 \leq b \leq a-1$, then

$$|C_i| = \sum_{j=0}^{2p^s} A_j$$
$$= 1 + \sum_{j=1}^{2i-2(b-1)p^s} \left(p^{m(a-b)} - 1\right) p^{m(a-b)(j-1)} +$$
$$+ \sum_{k=0}^{2bp^s-2i-1} p^{2m(a-b)p^s}(p^m - 1)p^{mk}$$
$$+ \sum_{j=2i-2(b-1)p^s+1}^{2p^s} \left(p^{m(a-b)} - 1\right) p^{m(a-b)(j-1)}$$
$$= 1 + \left(p^{m(a-b)} - 1\right) \sum_{j=0}^{2p^s-1} p^{m(a-b)j} + p^{2m(a-b)p^s}(p^m - 1) \sum_{k=0}^{2bp^s-2i-1} p^{mk}$$
$$= 1 + \left(p^{m(a-b)} - 1\right) \frac{p^{m(a-b)2p^s} - 1}{p^{m(a-b)} - 1} + p^{2m(a-b)p^s}(p^m - 1)\frac{p^{m(2bp^s-2i)} - 1}{p^m - 1}$$
$$= 1 + \left(p^{m(a-b)2p^s} - 1\right) + p^{2m(a-b)p^s}\left(p^{m(2bp^s-2i)} - 1\right)$$
$$= p^{2m(ap^s-i)}.$$

In fact, as discussed in [33], the Singleton Bound of RT distance is quite straightforward from the definition. Let C be a linear code of length n over $\mathcal{R}_a$ with Rosenbloom-Tsfasman distance $d_{\mathrm{RT}}(C)$. Mark the first $d_{\mathrm{RT}}(C)-1$ entries of each codeword of C, then two different codewords of C can not coincide in all other $n - d_{\mathrm{RT}}(C) + 1$ entries, otherwise C would have had a nonzero codeword of RT weight less than or equal to $d_{\mathrm{RT}}(C) - 1$. Thus, C can contain at most $p^{am(n-d_{\mathrm{RT}}(C)+1)}$ codewords.

Theorem 4.7 (Singleton Bound for RT distance)**.** *Let C be a linear code of length n over $\mathcal{R}_a$ with Rosenbloom-Tsfasman distance $d_{\mathrm{RT}}(C)$. Then $|C| \leq p^{am(n-d_{\mathrm{RT}}(C)+1)}$.*

When a code C attains this Singleton Bound, i.e., $|C| = p^{am(n-d_{\mathrm{RT}}+1)}$, it is said to be a *Maximum Distance Separable* (MDS) code (with respect to the RT distance). We now point out the unique MDS Type 1 constacyclic codes of length $2p^s$ over $\mathcal{R}_a$ with respect to the RT distance.

Theorem 4.8. *The only maximum distance separable Type 1 Λ-constacyclic code of length $2p^s$ over $\mathcal{R}_a$, with respect to the RT distance, is the whole ambient ring $\mathcal{S}_a(s,\Lambda)$.*

Proof. Let C be a nonzero Type 1 Λ-constacyclic code of length $2p^s$ over $\mathcal{R}_a$. By Theorem 3.5, $C = \langle (x^2-\alpha_0)^i \rangle \subseteq \mathcal{S}_a(s,\Lambda)$ for some integer $i \in \{0, 1, \dots, ap^s-1\}$, and $|C| = p^{2m(ap^s-i)}$.

If $0 \leq i \leq (a-1)p^s$, then, by Theorem 4.2, $d_{\mathrm{RT}}(C) = 1$, and $2p^s - d_{\mathrm{RT}}(C) + 1 = 2p^s$. Thus,

$$\begin{aligned} C \text{ is MDS} &\leftrightarrow |C| = p^{am(2p^s - d_{\mathrm{RT}}(C)+1)} \\ &\leftrightarrow p^{2m(ap^s-i)} = p^{2amp^s} \\ &\leftrightarrow ap^s - i = ap^s \\ &\leftrightarrow i = 0. \end{aligned}$$

If $(a-1)p^s + 1 \leq i \leq ap^s - 1$, then, using Theorem 4.2 again, we get $d_{\mathrm{RT}}(C) = 2i - 2(a-1)p^s + 1$, and $2p^s - d_{\mathrm{RT}}(C) + 1 = 2ap^s - 2i$. Hence,

$$\begin{aligned} C \text{ is MDS} &\leftrightarrow |C| = p^{am(2p^s - d_{\mathrm{RT}}(C)+1)} \\ &\leftrightarrow p^{2m(ap^s-i)} = p^{am(2ap^s-2i)} \\ &\leftrightarrow ap^s - i = a^2p^s - ai \\ &\leftrightarrow (a-1)i = (a^2-a)p^s \\ &\leftrightarrow i = ap^s, \end{aligned}$$

which is impossible since $(a-1)p^s + 1 \leq i \leq ap^s - 1$.

Therefore, the code $C = \langle (x^2-\alpha_0)^i \rangle \subseteq \mathcal{S}_a(s,\Lambda)$ is MDS if and only if $i = 0$, i.e., $C = \mathcal{S}_a(s,\Lambda)$. □

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