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GLOBAL SOLUTIONS FOR A CLASS OF NONLINEAR SIXTH-ORDER WAVE EQUATION

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Abstract. In this paper, we consider the Cauchy problem for a class of nonlinear sixth-order wave equation. The global existence and the finite time blow-up for the problem are proved by the potential well method at both low and critical initial energy levels. Furthermore, we present some sufficient conditions on initial data such that the weak solution exists globally at supercritical initial energy level by introducing a new stable set.

1. Introduction

This paper is concerned with the Cauchy problem for the following nonlinear sixth-order wave equation

$$u_{tt} - u_{xxtt} - u_{xx} + u_{xxxx} + u_{xxxxtt} = f(u_x)_x, \tag{1.1}$$

$$u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), \tag{1.2}$$

where $x \in \mathbb{R}$, $u(x, t)$ is the unknown function, $f(s)$ is a given nonlinear function.

In order to investigate the water wave problem with surface tension, Schneider and Wayne [12] considered a class of Boussinesq equation which models the water wave problem with surface tension as follows

$$u_{tt} = u_{xx} + u_{xxtt} + \mu u_{xxxx} - u_{xxxxtt} + (u^2)_{xx}, \tag{1.3}$$

where $x, t, \mu \in \mathbb{R}$ and $u(t, x) \in \mathbb{R}$. The model can also be formally derived from the 2D water wave problem. Eq. (1.3) with $\mu > 0$ is known as the "good" Boussinesq equation because of its linear instability. For a degenerate case, they have proved that the long wave limit can be described approximately by two decoupled Kawahara-equations. The authors included the term with the sixth-order derivatives since they were interested precisely in the case when the coefficient in front of the term with the fourth-order derivative is small, i.e., $1 + \mu = \varepsilon^2\nu$, with $\nu \in \mathbb{R}$ fixed [12]. The lowest order nonlinear terms in the water wave problem remain unchanged from the classical equation because they

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are independent of surface tension. Wang [20,21] studied the well-posedness of the local and global solution, the blow-up of solutions and nonlinear scattering for small amplitude solutions to Eq. (1.3) in $\mathbb{R}$ and $\mathbb{R}^n$. The global existence and finite time blow-up of the solutions for Eq. (1.3) are established by the potential well method [17]. Eq. (1.1) is similar to the generalized Boussinesq equation and various generalized of the Boussinesq equations have been studied from many aspects [5,6,8–10,13–15].

Wang and Xu [16] considered the Cauchy problem for the following Rosenau equation

$$u_{tt} + u_{xxxx} + u_{xxxxtt} - ru_{xx} = f(u)_{xx}. \tag{1.4}$$

Under some conditions, the well-posedness of the solution and the nonexistence of global solution to the problem are proved with the aid of the potential well method. The authors [23] studied the following nonlinear wave equation

$$u_{tt} + u_{xxxx} + u_{xxxxtt} - ru_{xx} = \phi(u_x)_x. \tag{1.5}$$

The existence and nonexistence of global solutions to Eq. (1.5) are obtained by the potential well method. Most of the authors proved the global existence and finite time blow-up of the solution at the sub-critical initial energy level ($E(0) < d$) and critical initial energy level ($E(0) = d$) by the potential well-method (the definitions of $E(t)$ and d will be given later). In particular, to our knowledge, there have only been a few results up to now on the global existence of a solution to the Cauchy problem (1.1) and (1.2) at the high initial energy level ($E(0) > 0$).

Hatice et al. [18,19] considered the following Rosenau equation

$$u_{tt} - u_{xx} + u_{xxxx} + u_{xxxxtt} = (f(u))_{xx},\ x \in \mathbb{R},\ t > 0, \tag{1.6}$$

where $f(u) = \gamma|u|^p$, $\gamma > 0$. By defining new functionals and using potential well method, they established the existence of global weak solutions for Eq. (1.6) with supercritical initial energy ($E(0) > 0$). Furthermore, the authors generalized the results from one dimension to n-dimensional spaces. Then, Kutev et al. [4] considered the Cauchy problem for the Boussinesq paradigm equation

$$u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \beta_2 \Delta^2 u = \Delta f(u),\ (x,t) \in \mathbb{R}^n \times (0,+\infty), \tag{1.7}$$

where $f(u) = \alpha|u|^p$, $\alpha > 0$, $\beta_1 \geq 0$, $\beta_2 > 0$, and Δ denotes the Laplace operator in $\mathbb{R}^n$. They introduced some new functionals and gave the existence of global weak solution with supercritical initial energy $E(0) > 0$. But the method in [4,18,19] is not valid for the nonlinear term $f(u) = \beta|u|^p u$, $\beta < 0$ at arbitrarily positive initial energy level $E(0) > 0$.

Wang and Mu [21] considered the following Boussinesq equation

$$u_{tt} - \Delta u_{tt} + \Delta^2 u_{tt} - \Delta u + \Delta^2 u = \Delta f(u),\ (x,t) \in \mathbb{R}^n \times (0,+\infty). \tag{1.8}$$

They obtained the existence and the uniqueness of the global solution and blow-up of the solution under some restrictions on the nonlinear source term $f(u)$. When $f(u) = \pm\beta|u|^p$ or $-\beta|u|^{p-1}u$, $\beta > 0$, Xu et al. [24] considered the Cauchy problem of Eq. (1.8) at three different initial energy levels. They established the global existence and blow-up solutions at low and critical initial energy levels, and also proved the global existence of the weak solution at supercritical initial energy level. In this paper, we apply the potential well method ([7, 22, 24]) to study the global existence and nonexistence of the solution to the problem (1.1) and (1.2) at the sub-critical initial energy level, the critical initial energy level and the high initial energy level.

In the present paper, using the contraction mapping principle, the well-posedness for problem (1.1) and (1.2) was established in Section 2. Under some conditions of $f(u)$, we also prove the existence and uniqueness of the global solution for the problem (1.1) and (1.2) in Section 2. In sections 3 and 4, we fix the nonlinear terms to be $f(u) = r|u|^p$, $r \neq 0$ to classify the initial data for the global existence and non-global existence. Following the main idea of potential well method introduced in [7, 22, 24], we will construct the stable and unstable sets and study the global existence and non-existence of the weak solutions for problem (1.1) and (1.2) at three different initial energy levels: $E(0) < d$, $E(0) = d$ and $E(0) > 0$, where d is the potential well depth, and the three cases will be respectively dealt with different methods.

Throughout this paper, L^p denotes the usual space of all $L^p(\mathbb{R})$-functions with norm $\|\cdot\|_p$ and $\|u\| = \|u\|_2$, H^s denotes the Sobolev space $H^s(\mathbb{R})$ with norm $\|\cdot\|_{H^s}$ and $H_0^1(\mathbb{R})$ denotes the closure of $C_c^\infty(\mathbb{R})$ in $H^1(\mathbb{R})$. If not specified throughout this paper, we denote C as a generic constant varying line by line and depending only on the norms of initial data and absolute constant.

At first, by using the contraction mapping theorem, we obtain the following existence and uniqueness of the local solution to problem (1.1) and (1.2).

Theorem 1.1. *Suppose that $s > \frac{1}{2}, \phi, \psi \in H^s$ and $f \in C^N(\mathbb{R})$, then there exists a maximal time T_0 which depends on ϕ and ψ such that for each $T < T_0$, the Cauchy problem* (1.1) *and* (1.2) *has a unique solution $u \in C^1([0,T], H^s)$. Moreover, if*

$$\sup_{t\in[0,T_0)} [\|u(\cdot,t)\|_{H^s} + \|u_t(\cdot,t)\|_{H^s}] < \infty,$$

then $T_0 = \infty$.

Theorem 1.2. *Suppose that $s \geq 1, \phi, \psi \in H^s, F(u) = \int_0^u f(z)dz, F(\phi_x) \in L^1$ and $T_0 > 0$ is the maximal existence time of corresponding solution $u(t) \in C^1([0,T_0), H^s)$ to the Cauchy problem* (1.1) *and* (1.2)*. Then the equality*

$$\begin{aligned} E(t) &= \frac{1}{2}[\|u_t\|^2 + \|u_{xt}\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxt}\|^2] + \int_{\mathbb{R}} F(u_x)dx \\ &= E(0), \forall t \in (0,T_0), \end{aligned} \tag{1.9}$$

holds.

Under some assumptions, the well-posedness of the global solution for the Cauchy problem (1.1) and (1.2) is established.

Theorem 1.3. *Suppose that the assumptions of Theorem 1.1 hold, and $T_0 > 0$ is the maximal existence time of the corresponding solution $u(t) \in C^2([0,T_0], H^s)$ to the problem* (1.1) *and* (1.2). *Then $T_0 < \infty$ if and only if*

$$\lim_{t\to T_0} \sup \|u_x(t)\|_{L^\infty} = \infty.$$

For the case $f(s) = r|u|^p$, $r \neq 0$, we firstly introduce the potential energy functional

$$J(u) = \frac{1}{2}(\|u_x\|^2 + \|u_{xx}\|^2) + \frac{r}{p+1}\int_{\mathbb{R}} |u_x|^p u_x dx \tag{1.10}$$

and the Nehari functional

$$I(u) = (\|u_x\|^2 + \|u_{xx}\|^2) + r\int_{\mathbb{R}} |u_x|^p u_x dx. \tag{1.11}$$

We define the stable set

$$K_1 = \{u \in H^2 \,|\, I(u) > 0\} \cup \{0\}, \tag{1.12}$$

the unstable set

$$K_2 = \{u \in H^2 \,|\, I(u) < 0\} \tag{1.13}$$

and the depth of potential well as

$$d = \inf_{u\in NE} J(u),$$

where the Nehari manifold $NE = \{u \in H^2 \setminus \{0\} \,|\, I(u) = 0\}$. And for the function $u(x,t)$ satisfying $u \in C^1((0,T), H^2)$, $u_t \in C((0,T), H^2)$, we define a new functional space

$$Y_T := \{u | I(u(t)) > \|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2\} \cup \{0\},$$

which will be used in Section 4.

For the low initial energy case and the critical energy case, we prove the global existence and finite-time blow-up for the problem (1.1) and (1.2) by the potential well method.

Theorem 1.4. *Suppose that $2 \leq s < p+1$ and $\phi, \psi \in H^s$. If $E(0) \leq d$ and $\phi \in K_1 \cup \partial K_1$, then problem* (1.1) *and* (1.2) *has a unique solution $u \in C^1([0,\infty); H^s)$ and $u(t) \in K_1 \cup \partial K_1$ for $t \in [0,\infty)$.*

Theorem 1.5. *Suppose that $2 \leq s < p+1$ and $\phi, \psi \in H^s$. If $E(0) < d$, $\phi \in K_2$ and $(\phi,\psi) + (\phi_x,\psi_x) + (\phi_{xx},\psi_{xx}) \geq 0$ when $E(0) = d$, then the solution $u(x,t)$ of problem* (1.1) *and* (1.2) *ceases to exist in finite time.*

For the supercritical initial energy case, by utilizing the method of [4,24], we obtain the global existence of weak solution for problem (1.1) and (1.2) under some sufficient conditions on the initial data.

Theorem 1.6. *Suppose that $2 \le s < p+1$ and $\phi \in H^2 \cap H_0^1$, $\psi \in H^2 \cap H_0^1$. If*

$$E(0) > 0, \tag{1.14}$$

$$I(\phi) > \|\psi\|^2 + \|\psi_x\|^2 + \|\psi_{xx}\|^2, \tag{1.15}$$

$$\begin{aligned} &2(\phi,\psi) + 2(\phi_x,\psi_x) + 2(\phi_{xx},\psi_{xx}) \\ &\quad + \|\phi\|^2 + \|\phi_x\|^2 + \|\phi_{xx}\|^2 + \frac{2(p+1)}{p+3}E(0) < 0, \end{aligned} \tag{1.16}$$

then the solution $u(x,t)$ of problem (1.1) *and* (1.2) *exists globally.*

2. Well-posedness of solutions

In this section, we study the local and global well-posedness of solutions to the problem (1.1) and (1.2). We first consider the following linear wave equation

$$u_{tt} - u_{xxtt} - u_{xx} + u_{xxxx} + u_{xxxxtt} = q(x,t), \tag{2.1}$$

with initial value (1.2). By the Fourier transform and Duhamel's principle, the solution $u(x,t)$ of problem (2.1) and (1.2) can be written as

$$u(x,t) = (\partial_t S(t)\phi)(x) + (S(t)\psi)(x) + \int_0^t \Gamma(t-\tau)q(u(\tau))d\tau. \tag{2.2}$$

Here $\Gamma(t) = S(t)(1-\partial_x^2+\partial_x^4)^{-1}$ and

$$(\partial_t S(t)\phi)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\cos\Big(\frac{|\xi|\sqrt{1+\xi^2}t}{\sqrt{1+\xi^2+\xi^4}}\Big)\hat{\phi(\xi)})d\xi,$$

$$(S(t)\psi)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\sin\Big(\frac{|\xi|\sqrt{1+\xi^2}t}{\sqrt{1+\xi^2+\xi^4}}\Big)\frac{\sqrt{1+\xi^2+\xi^4}}{|\xi|\sqrt{1+\xi^2}}\hat{\psi(\xi)})d\xi,$$

where $\hat{\phi(\xi)} = F(\phi)(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Lemma 2.1. *For the operators $\partial_t S(t), S(t)$ and $\Gamma(t)$, we have the following estimates*

$$\|\partial_t S(t)\phi\|_{H^s} \le \|\phi\|_{H^s}, \forall \phi \in H^s, \tag{2.3}$$

$$\|S(t)\psi\|_{H^s} \le 2(1+t)\|\psi\|_{H^s}, \forall \psi \in H^s, \tag{2.4}$$

$$\|\partial_{tt} S(t)\phi\|_{H^{s-2}} \le \sqrt{2}\|\phi\|_{H^s}, \forall \phi \in H^s, \tag{2.5}$$

$$\|\Gamma(t)q\|_{H^s} \le \sqrt{2}\|q\|_{H^{s-4}}, \forall q \in H^{s-4}, \tag{2.6}$$

$$\|\partial_t \Gamma(t)q\|_{H^{s-2}} \le \|q\|_{H^{s-4}}, \forall q \in H^{s-4}. \tag{2.7}$$

Proof. We only prove (2.4) and (2.6), the proof of other inequalities are similar. By Plancherel's theorem, we obtain

$$\|S(t)\psi\|_{H^s}^2 = \int_{\mathbb{R}} (1+\xi^2)^s \frac{1+\xi^2+\xi^4}{\xi^2(1+\xi^2)}\sin^2\Big(\frac{t|\xi|\sqrt{1+\xi^2}}{\sqrt{1+\xi^2+\xi^4}}\Big)|\hat{\psi}(\xi)|^2 d\xi$$

$$
\begin{aligned}
&\le \int_{|\xi|\le 1}(1+\xi^2)^s t^2|\hat{\psi}|^2 d\xi + \int_{|\xi|>1}(1+\xi^2)^s \frac{(1+\xi^2+\xi^4)}{\xi^2(1+\xi^2)}|\hat{\psi}(\xi)|^2 d\xi \\
&\le t^2\int_{|\xi|\le 1}(1+\xi^2)^s|\hat{\psi}(\xi)|^2 d\xi + 4\int_{|\xi|>1}(1+\xi^2)^s|\hat{\psi}(\xi)|^2 d\xi \\
&\le 4(1+t)^2\|\psi\|_{H^s}^2,
\end{aligned}
$$

$$
\begin{aligned}
\|\Gamma(t)q\|_{H^s}^2 &= \int_{\mathbb{R}}(1+\xi^2)^s \sin^2\!\Big(\tfrac{t|\xi|\sqrt{1+\xi^2}}{\sqrt{1+\xi^2+\xi^4}}\Big)\frac{1+\xi^2+\xi^4}{\xi^2(1+\xi^2)}\frac{1}{(1+\xi^2+\xi^4)^2}|\hat{q}(\xi)|^2 d\xi \\
&\le 2\int_{\mathbb{R}}(1+\xi^2)^{s-4}|\hat{q}(\xi)|^2 d\xi = 2\|q\|_{H^{s-4}}^2.
\end{aligned}
$$

Therefore (2.4) and (2.6) hold. This completes the proof of the lemma. □

Lemma 2.2 ([1])**.** *Suppose that $g(u)\in C^N(\mathbb{R})$ is a function vanishing at zero, where $N\ge 0$ is an integer. Then for any s with $0\le s\le N$ and any $u,v\in H^s\cap L^\infty$, we have*

$$
\begin{aligned}
\|g(u)\|_{H^s} &\le G(\|u\|_{L^\infty})\|u\|_{H^s}, \\
\|g(u)-g(v)\|_{H^s} &\le \bar{G}(\|u\|_{L^\infty},\|v\|_{L^\infty})\|u-v\|_{H^s},
\end{aligned}
$$

where $G:[0,\infty)\to\mathbb{R}$ and $\bar{G}:[0,\infty)\times[0,\infty)\to\mathbb{R}$ are continuous functions.

Lemma 2.3 (Sobolev Lemma [2])**.** *If $s>\frac{n}{2}+k$, where k is a nonnegative integer, then*

$$H^s(\mathbb{R}^n)\subset C^k(\mathbb{R}^n)\cap L^\infty,$$

where the inclusion is continuous. In fact,

$$\Sigma_{|\beta|\le k}\|\partial_x^\beta u\|_{L^\infty}\le C\|u\|_{H^s},$$

where C is dependent on s, independent of u.

Lemma 2.4 ([11])**.** *Assume $u\in H^s\cap L^\infty$, $0<s<p$. Then there exists a constant C such that*

$$\||u|^p\|_{H^s}\le C\|u\|_{L^\infty}^{p-1}\|u\|_{H^s}.$$

Lemma 2.5 ([3])**.** *If $s>0$, then $H^s\cap L^\infty$ is an algebra. Moreover,*

$$\|uv\|_{H^s}\le C(\|u\|_{L^\infty}\|v\|_{H^s}+\|v\|_{L^\infty}\|u\|_{H^s}).$$

Lemma 2.6. *The operator $P=(1-\partial_x^2+\partial_x^4)^{-1}$ is bounded from H^{s-4} to H^s, i.e.,*

$$\|Pu\|_{H^s}\le C\|u\|_{H^{s-4}}.$$

Proof.

$$
\begin{aligned}
\|Pu\|_{H^s} &= \|(1+\xi^2)^{\frac{s}{2}}\frac{1}{1+\xi^2+\xi^4}\hat{u}\| \\
&\le C\|(1+\xi^2)^{\frac{s-4}{2}}\hat{u}\| = C\|u\|_{H^{s-4}}.
\end{aligned}
$$

□

Proof of Theorem 1.1. Now we are going to prove the existence and uniqueness of local solutions for problem (1.1) and (1.2) by contraction mapping argumentation. For this purpose, we define the function space $X(T) = C^1([0,T]; H^s)$ with $s > \frac{1}{2}$, equipped with the norm defined by

$$\|u\|_{X(T)} = \max_{t\in[0,T]}[\|u(\cdot,t)\|_{H^s} + \|u_t(\cdot,t)\|_{H^s}].$$

Since $H^s \hookrightarrow L^\infty$ for $s > \frac{1}{2}$, we have $u \in L^\infty$ if $u \in X(T)$. Let $B_R(T)$ be the ball of radius R centered at the origin in $X(T)$, i.e.,

$$B_R(T) = \{u \in X(T) : \|u\|_{X(T)} \le R\}.$$

For $\phi \in H^s, \psi \in H^s, u \in X(T)$, we define the map

$$\Theta(u(t)) = \partial_t S(t)\phi + S(t)\psi + \int_0^t S(t-\tau)(I - \partial_x^2 + \partial_x^4)^{-1} f(u_x)_x(\tau)d\tau. \tag{2.8}$$

It will be shown that $\Theta : B_R(T) \to B_R(T)$ is contractive if R and T are well chosen.

Let $\|\phi\|_{H^s} + \|\psi\|_{H^s} \le \rho,\ u, v \in B_R(T)$. Using the estimates in Lemmas 2.1-2.5, we have

$$\begin{aligned}\|\Theta(u)\|_{H^s} &\le \|\phi\|_{H^s} + 2(1+T)\|\psi\|_{H^{s-2}} + \sqrt{2}\int_0^T \|f(u_x))\|_{H^{s-3}}d\tau \\ &\le 2\rho + (2\rho + \sqrt{2}G(R))RT \end{aligned}\tag{2.9}$$

and

$$\|\Theta(u) - \Theta(v)\|_{H^s} \le \sqrt{2}\int_0^T \|f(u_x)_x - f(v_x)_x\|_{H^s}d\tau \le \sqrt{2}\tilde{G}T\|u - v\|_{X(T)}, \tag{2.10}$$

where $\tilde{G}(R) = \bar{G}(R,R)$. Differentiating with respect to t, we see that

$$u_t(x,t) = (\partial_{tt}S(t)\phi)(x) + (\partial_t S(t)\psi)(x) + \int_0^T \partial_t\Theta(t-\tau)f(u_x)_x d\tau. \tag{2.11}$$

Using the estimates in Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\|\Theta(u)_t\|_{H^s} &\le 2\|\phi\|_{H^s} + \|\psi\|_{H^{s-2}} + \int_0^T \|f(u_x)_x\|_{H^{s-4}}d\tau \\ &\le 2(\rho + G(R)RT)\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}\|\Theta(u)_t - \Theta(v)_t\|_{H^s} &\le \int_0^T \|f(u_x)_x - f(v_x)_x\|_{H^{s-4}}d\tau \\ &\le G(R)T\|u - v\|_{X(T)}.\end{aligned}\tag{2.13}$$

From (2.9)-(2.13), we have

$$\|\Theta(u)\|_{X(T)} \le 4\rho + (2\rho + 2\sqrt{3}G(R)R)T,$$
$$\|\Theta(u) - \Theta(v)\|_{X(T)} \le 3\bar{G}(R)T\|u - v\|_{X(T)}.$$

Choosing $R = 8\rho$ and fixing T so small enough such that

$$T < \min\left(\frac{2\rho}{\rho+\sqrt{3}G(R)R}, \frac{1}{4G(R)}\right), \tag{2.14}$$

therefore, Θ is a contraction map on $B_R(T)$. It follows from the contraction mapping theorem that problem (2.1) and (1.2) has a unique solution $u \in B_R(T)$. Similar to that of [20], we can prove uniqueness and local Lipschitz dependence with respect to the initial data in the space $B_R(T)$. Using uniqueness we can extend the result in the space $C([0,T_0]; H^s(\mathbb{R}))$ by a standard technique. □

Proof of Theorem 1.2. It follows from (1.1) that

$$\begin{aligned}\frac{d}{dt}E(t) &= (u_{tt}, u_t) + (u_x, u_{xt}) \\ &\quad + (u_{xx}, u_{xxt}) + (u_{xtt}, u_{xt}) + (u_{xxtt}, u_{xxt}) + (f(u_x), u_{xt}) \\ &= \langle (u_{tt} - u_{xxtt} + u_{xxxxtt} - u_{xx} + u_{xxxx} - f(u_x)_x, u_t \rangle_{X*X} = 0,\end{aligned}$$

where $(\cdot,\cdot)$ denotes the inner product of L^2-space and $\langle\cdot,\cdot\rangle_{X*X}$ means the usual duality of X and X with $X = H^2$. Integrating the above equality with respect to t, we have (1.9). Theorem 1.2 is proved. □

Next we study the existence of global solutions to the problem (1.1) and (1.2).

Proof of Theorem 1.3. One implication is obvious in view of Theorem 1.1. Let us prove that if

$$\lim_{t\in[0,T_0)} \sup \|u_x(\cdot,t)\|_{L^\infty} = M < \infty, \tag{2.15}$$

then $T_0 = \infty$. (1.1) can be written as follows:

$$u_{tt} + u = Pu + Pf(u_x)_x,$$

where $P = (1 - \partial_x^2 + \partial_x^4)^{-1}$.

So, it follows from Hölder inequality that for $t \in (0,T)$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|_{H^s}^2 + \|u_t\|_{H^s}^2) \\ &= ((I-\partial_x^2)^{\frac{s}{2}}u_{tt}, (I-\partial_x^2)^{\frac{s}{2}}u_t) + ((I-\partial_x^2)^{\frac{s}{2}}u, (I-\partial_x^2)^{\frac{s}{2}}u_t) \\ &= ((I-\partial_x^2)^s u_{tt} + (I-\partial_x^2)^s u, u_t) \\ &= ((I-\partial_x^2)^s (u_{tt}+u), u_t) \\ &= ((I-\partial_x^2)^{\frac{s}{2}}(Pu + P(\partial_x f(u_x))), (I-\partial_x^2)^{\frac{s}{2}}u_t) \\ &\le \|(I-\partial_x^2)^{\frac{s}{2}}(Pu + P(\partial_x f(u_x)))\|_{H^s}\|u_t\|_{H^s} \\ &\le (\|(I-\partial_x^2)^{\frac{s}{2}}Pu\|_{H^s} + \|(I-\partial_x^2)^{\frac{s}{2}}P(\partial_x f(u_x))\|_{H^s})\|u_t\|_{H^s}.\end{aligned}$$

From Lemma 2.4, Lemma 2.6 and (3.1), we obtain

$$(\|(I-\partial_x^2)^{\frac{s}{2}}Pu\|_{H^s} + \|(I-\partial_x^2)^{\frac{s}{2}}P(\partial_x f(u_x))\|_{H^s}) \le (G(R)+C)\|u\|_{H^s}.$$

It follows from the Cauchy inequality that

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{H^s}^2 + \|u_t\|_{H^s}^2) \le (G(R)^2 + C)(\|u\|_{H^s}^2 + \|u_t\|_{H^s}^2), t \in (0,T).$$

Then previous relation shows by Gronwall' inequality that $\|u(t)\|_{H^s}^2 + \|u_t(t)\|_{H^s}^2$ do not blow-up in finite time. Then $T = \infty$ by Theorem 1.1. The theorem is proved. □

3. Existence and nonexistence of global solutions for $f(u) = r|u|^p$

In this section, we discuss the existence and nonexistence of global solution for the low initial energy case and the critical energy case when $f(s) = r|u|^p, r \neq 0$. We start with the following elementary statement.

Lemma 3.1. *The depth of potential well $d = \frac{p-1}{2(p+1)}(|r|S_*^{p+1})^{-\frac{2}{p-1}}$, where S_* is the optimal Sobolev constant, i.e.,*

$$S_* = \sup_{u \in H^2 \setminus \{0\}} \frac{\|u_x\|_{p+1}}{(\|u_x\|^2 + \|u_{xx}\|^2)^{\frac{1}{2}}}. \tag{3.1}$$

Proof. From the definition of d, we obtain $u \in NE$, i.e., $I(u) = 0$, which yields

$$\|u_x\|^2 + \|u_{xx}\|^2 = -r\|u_x\|_{p+1}^{p+1} \le |r|S_*^{p+1}(\|u_x\|^2 + \|u_{xx}\|^2)^{\frac{p-1}{2}}(\|u_x\|^2 + \|u_{xx}\|^2).$$

From (3.1), we get

$$\|u_x\|^2 + \|u_{xx}\|^2 \ge (|r|S_*^{p+1})^{-\frac{2}{p-1}}. \tag{3.2}$$

On the other hand, from (1.11), (1.12), (3.2) and $I(u) = 0$, we obtain

$$\begin{aligned} J(u) &= (\frac{1}{2} - \frac{1}{p+1})(\|u_x\|^2 + \|u_{xx}\|^2) \\ &\ge \frac{p-1}{2(p+1)}(|r|S_*^{p+1})^{-\frac{2}{p-1}}, \end{aligned} \tag{3.3}$$

which completes the proof. □

Lemma 3.2. *Let $\phi, \psi \in H^2$, and $u \in C^1([0,T_0); H^2)$ is the unique solution of the Cauchy problem* (1.1) *and* (1.2)*, where T_0 is the maximal existence time of $u(t)$. Assume that $E(0) < d$, then for all $t \in [0,T_0)$,*

(i) *$u(t) \in K_1$ and $\|u_x(t)\|^2 + \|u_{xx}(t)\|^2 < \frac{2(p+1)}{p-1}d$ if $\phi \in K_1$;*

(ii) *$u(t) \in K_2$ and $\|u_x(t)\|^2 + \|u_{xx}(t)\|^2 > \frac{2(p+1)}{p-1}d$ if $\phi \in K_2$.*

Proof. Since the proof of (i) and (ii) are similar, we only prove (ii). Let $u(t)$ be any local weak solution of problem (1.1) and (1.2) with $E(0) < d, \phi \in K_2$ and T_0 be the maximum existence time of $u(t)$. Then, it follows from Theorem 1.2 that $E(u(t)) = E(0) < d$. Thus, it suffices to show that $I(u(t)) < 0$ for $0 < t < T_0$. Arguing by contradiction, we suppose that there exists a $t_1 \in (0,T_0)$ such that

$I(u(t_1)) \geq 0$. From the continuity of $I(u(t))$ in time, there exists a $t_* \in (0, T_0)$ such that $I(u(t_*)) = 0$. Then, from the definition of d, we get

$$d \leq J(u(t_*)) \leq E(u(t_*)) = E(0) < d,$$

which is a contradiction, that is $\|u_x\|^2 + \|u_{xx}\|^2 < -r\|u_x\|_{p+1}^{p+1}$ for any $t \in [0, T_0)$. From (3.1), we arrive at

$$\frac{1}{S_*^2} \leq \frac{\|u_x\|^2 + \|u_{xx}\|^2}{\|u_x\|_{p+1}^2} < \frac{-r\|u_x\|_{p+1}^{p+1}}{\|u_x\|_{p+1}^2} \leq |r|\|u_x\|_{p+1}^{p-1}. \tag{3.4}$$

Then, from Lemma 3.1 and (3.4), yields

$$\begin{aligned} d &= \frac{p-1}{2(p+1)}|r|^{-\frac{2}{p-1}} S_*^{-2} S_*^{-\frac{4}{p-1}} \\ &< \frac{p-1}{2(p+1)} \frac{\|u_x\|^2 + \|u_{xx}\|^2}{\|u_x\|_{p+1}^2} (\|u_x\|_{p+1}^{p-1})^{\frac{2}{p-1}} \\ &= \frac{p-1}{2(p+1)}(\|u_x\|^2 + \|u_{xx}\|^2), \end{aligned}$$

which means

$$\|u_x\|^2 + \|u_{xx}\|^2 > \frac{2(p+1)}{p-1} d.$$

In fact, for (i), from the definition of $I(t)$ and $E(t)$, we get

$$\begin{aligned} E(t) &= \frac{1}{2}[\|u_t\|^2 + \|u_{xt}\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxt}\|^2] + \frac{r}{p+1}\int_{\mathbb{R}} |u_x|^p u_x dx \\ &= \frac{1}{2}(\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2) + \frac{p-1}{2(p+1)}(\|u_x\|^2 + \|u_{xx}\|^2) + \frac{1}{p+1} I(t). \end{aligned}$$

If $I(u(t)) > 0$, we obtain

$$\frac{p-1}{2(p+1)}(\|u_x\|^2 + \|u_{xx}\|^2) < E(t) = E(0) < d.$$

So,

$$\|u_x\|^2 + \|u_{xx}\|^2 < \frac{2(p+1)}{p-1} d.$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Let $\phi, \psi \in H^2$, and $u \in C^1([0, T_0); H^2)$ is the unique solution of the Cauchy problem* (1.1) *and* (1.2)*, where T_0 is the maximal existence time of $u(t)$. Assume that $E(0) = d$ and $(\phi_{xx}, \psi_{xx}) + (\phi_x, \psi_x) + (\phi, \psi) \geq 0$, then for all $t \in [0, T_0), u(t) \in K_2$ if $u(0) \in K_2$.*

Proof. If the result is false, there would exist a $t_0 \in (0, T_0)$ such that $I(t_0) = 0$ from the continuity of $I(t)$. Hence we have $J(u(t_0)) \geq d$, which together with $E(t_0) = E(0) = d$ gives $J(u(t_0)) = d$ and

$$\|u_t(t_0)\| + \|u_{xt}(t_0)\| + \|u_{xxt}(t_0)\| = 0. \tag{3.5}$$

On the other hand, let

$$L(t) = \|u(t)\|^2 + \|u_x(t)\|^2 + \|u_{xx}(t)\|^2. \tag{3.6}$$

Then

$$L'(t) = 2(u, u_t) + 2(u_x, u_{xt}) + 2(u_{xx}, u_{xxt}), \tag{3.7}$$

with $L'(0) \geq 0$. From (1.1) and (1.8), we get

$$\begin{aligned} L''(t) &= 2\langle u_{tt} - u_{xxtt} + u_{xxxxtt}, u\rangle_{X*X} + 2\|u_t\|^2 + 2\|u_{xt}\|^2 + 2\|u_{xxt}\|^2 \\ &= -2\|u_x\|^2 - 2\|u_{xx}\|^2 - 2r\int_R |u_x|^p u_x dx + 2\|u_t\|^2 + 2\|u_{xt}\|^2 + 2\|u_{xxt}\|^2 \\ &= 2\|u_t\|^2 + 2\|u_{xt}\|^2 + 2\|u_{xxt}\|^2 - 2I(u) \\ &> 0, \forall t \in (0, t_0). \end{aligned} \tag{3.8}$$

Hence $L'(t)$ is strictly increasing on $[0, t_0]$, and $L'(t_0) > 0$ which contradicts (3.5). The Lemma is proved. □

Lemma 3.4. *Let $\phi \in H^s, \psi \in H^s, 2 \leq s \leq p+1$. Assume that $E(0) < d$, then when $\phi \in K_1$, problem* (1.1) *and* (1.2) *has a unique solution $u \in C^1([0,\infty); H^s)$ and $u \in K_1$ for $0 \leq t < \infty$.*

Proof. From Lemma 3.2, we can obtain

$$\|u_x(t)\|^2 + \|u_{xx}(t)\|^2 < \frac{2(p+1)d}{p-1}.$$

It follows from above inequality and the Sobolev imbedding theorem that

$$\sup_{t\in[0,T_0)} \|u_x(\cdot,t)\|^2_{L^\infty} < C\frac{2(p+1)d}{p-1}.$$

Therefore, problem (1.1) and (1.2) has a unique global solution $u \in C^1([0,\infty); H^s)$ by Theorem 1.3. □

Proof of Theorem 1.4. First, it follows from Theorem 1.3 that problem (1.1) and (1.2) has a unique local solution $u \in C^1([0,T_0), H^s)$, where T_0 is the maximal existence time of $u(t)$. Next we prove $T_0 = \infty$.

It follows from $\phi \in K_1 \cup \partial K_1$, $E(0) \leq d$ and the proof of Lemma 3.2 that

$$\begin{aligned} \|\phi_x\|^{p+1}_{p+1} &\leq S_*^{p+1}(\|\phi_x\|^2 + \|\phi_{xx}\|^2)^{\frac{p+1}{2}} \\ &\leq S_*^{p+1}(\|\phi_x\|^2 + \|\phi_{xx}\|^2)\left(\frac{2(p+1)d}{p-1}\right)^{\frac{p-1}{2}} \\ &= |r|^{-1}(\|\phi_x\|^2 + \|\phi_{xx}\|^2). \end{aligned}$$

Therefore for any $\lambda > 0$, we obtain

$$J(\lambda u) = \frac{\lambda^2}{2}[\|u_x\|^2 + \|u_{xx}\|^2] + \frac{r\lambda^{p+1}}{p+1}\int_{\mathbb{R}} |u_x|^p u_x dx$$

and

$$\frac{d}{d\lambda}J(\lambda\phi) = \lambda(\|\phi_x\|^2 + \|\phi_{xx}\|^2) + r\lambda^p \int_{\mathbb{R}} |\phi_x|^p \phi_x dx$$
$$\geq \lambda(\|\phi_x\|^2 + \|\phi_{xx}\|^2) - |r|\lambda^p \|\phi_x\|_{p+1}^{p+1}$$
$$\geq \lambda(1-\lambda)(\|\phi_x\|^2 + \|\phi_{xx}\|^2) \geq 0, \forall \lambda \in (0,1).$$

Take a sequence $\{\lambda_m\}$ such that $0 < \lambda_m < 1, m = 1, 2, \ldots$ and $\lambda_m \to 1$ as $m \to \infty$. Let $\phi_m = \lambda_m \phi, \psi_m = \lambda_m \psi$. Consider the problem (1.1) with the initial conditions

$$u(x,0) = \phi_m(x), u_t(x,0) = \psi_m(x). \tag{3.9}$$

Then

$$\|\phi_{mx}\|^2 + \|\phi_{mxx}\|^2 = \lambda_m^2[\|\phi_{mx}\|^2 + \|\phi_{mxx}\|^2] < \frac{2(p+1)d}{p-1}$$

and

$$E_m(0) = \frac{1}{2}[\|\psi_m\|^2 + \|\psi_{mx}\|^2 + \|\psi_{mxx}\|^2$$
$$+ \|\phi_{mx}\|^2 + \|\phi_{mxx}\|^2] + \frac{r}{p+1}\int_{\mathbb{R}} |\phi_{mx}|^p \phi_{mx} dx,$$
$$= \frac{1}{2}[\|\psi_m\|^2 + \|\psi_{mx}\|^2 + \|\psi_{mxx}\|^2] + J(\lambda_m \phi).$$

If $\psi = 0$ and $\phi = 0$, then $E_m(0) = 0 < d$. If $\psi \neq 0$ and $\phi \neq 0$, then

$$E_m(0) < \frac{1}{2}[\|\psi_m\|^2 + \|\psi_{mx}\|^2 + \|\psi_{mxx}\|^2] + J(\phi) = E(0) \leq d.$$

It follows from Lemma 3.4 that for each m, problem (1.1) and (3.9) has a unique global solution $u_m \in C^2([0,\infty); H^s)$ and it satisfies

$$(u_{mt}, v) + (u_{mxt}, v_x) + (u_{mxxt}, v_{xx})$$
$$+ \int_0^t [(u_{mx}, v_x) + (u_{mxx}, v_{xx}) + (r|u_{mx}|^p, v_x)]d\tau$$
$$= (\psi_m, v) + (\psi_{mx}, v_x) + (\psi_{mxx}, v_x), \ \forall v \in H^2, \ t \in [0,\infty) \tag{3.10}$$

and

$$\frac{1}{2}[\|u_{mt}\|^2 + \|u_{mxt}\|^2 + |u_{mxxt}\|^2] + J(u_m) = E_m(0) < d, \tag{3.11}$$

$$I(u_m) > 0, \tag{3.12}$$

$$\|u_{mx}\|^2 + \|u_{mxx}\|^2 < \frac{2(p+1)}{p-1}d. \tag{3.13}$$

By using (3.11)-(3.13), we get

$$\|u_{mx}\|_{p+1}^{p+1} \leq S_*^{p+1}(\|u_{mx}\|^2 + \|u_{mxx}\|^2)^{\frac{p+1}{2}} \leq |r|^{-1}(\|u_{mx}\|^2 + \|u_{mxx}\|^2),$$
$$J(u_m) \geq \frac{1}{2}[\|u_{mx}\|^2 + \|u_{mxx}\|^2] - \frac{|r|}{p+1}\int_{\mathbb{R}} |u_{mx}|^{p+1} dx$$

$$\geq \frac{p-1}{2(p+1)}[\|u_{mx}\|^2 + \|u_{mxx}\|^2] \geq 0.$$

Because of the above inequalities, we have

$$\|u_{mt}\|^2 + \|u_{mxt}\|^2 + \|u_{mxxt}\|^2 \leq 2d, \tag{3.14}$$

$$\|u_m\|_{p+1}^{p+1} \leq |r|^{-1}\frac{2(p+1)}{p-1}d. \tag{3.15}$$

It follows from (3.11), (3.12), (3.14) and (3.15) that there exist a $\bar{u} \in \bar{K_1}$ and a subsequence $\{u_k\}$ such that as $k \to \infty$, $u_{kx} \to \bar{u_x}$ in $L^\infty(0,\infty;H^1)$ weakly star and a.e in $\mathbb{R}\times[0,\infty)$, $u_{kt} \to \bar{u}_t$ in $L^\infty(0,\infty;L^2)$ weakly star, $|u_{kx}|^p \to |\bar{u_x}|^p$ in $L^\infty(0,\infty;L^{\frac{p+1}{p}})$ weakly star.

In equality (3.10), letting $m = k \to \infty$, we obtain

$$(\bar{u}_t, v) + (\bar{u}_{xt}, v_x) + (\bar{u}_{xxt}, v_{xx}) + \int_0^t [(\bar{u}_x, v_x) + \bar{u}_{xx}, v_{xx}) + (r|\bar{u}_x|^p, v_x)]d\tau$$
$$= (\psi, v) + (\psi_x, v_x) + (\psi_{xx}, v_{xx}), \forall t \in [0,\infty),$$

for any $v \in H^2$, which implies that $\bar{u}$ satisfies (1.1). Furthermore, we can get

$$\bar{u}(x,0) = \phi(x), \bar{u}_t(x,0) = \psi(x).$$

Thus $\bar{u}$ is a global solution of the Cauchy problem (1.1) and (1.2). From the uniqueness of the solution of problem (1.1) and (1.2), we get $\bar{u} = u$ on $\mathbb{R}\times[0,T_0)$, and $I(u) = I(\bar{u}) \geq 0$. We obtain $T_0 = \infty$ and $u \in C^2([0,\infty);H^s)$. The theorem is proved. $\square$

Proof of Theorem 1.5. Theorem 1.1 gives the existence of a local weak solution $u \in C^1([0,T_0);H^s)$ satisfying (1.9), where T_0 is the maximal existence time of u. We prove $T_0 < \infty$. We assume that the result is not true, then $T_0 = \infty$. Let

$$L(t) = \|u(t)\|^2 + \|u_x\|^2 + \|u_{xx}(t)\|^2, \tag{3.16}$$

then

$$L'(t) = 2(u,u_t) + 2(u_x,u_{xt}) + 2(u_{xx},u_{xxt}).$$

Using the Schwartz inequality, we have

$$L'(t)^2 \leq 4L(t)[\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2]. \tag{3.17}$$

Similarly to (3.8), from Lemmas 3.2 and 3.3, we obtain

$$\begin{aligned} L''(t) &= (p+3)[\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2] + (p-1)[\|u_x\|^2 + \|u_{xx}\|^2] \\ &\quad - 2(p+1)E(0) \\ &> (p+3)[\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xxt}\|^2] + 2(p+1)[d - E(0)] \\ &\geq 0,\ \forall t \in (0,\infty). \end{aligned} \tag{3.18}$$

It follows that

$$L'(t) > L'(0) + 2(p+1)[d - E(0)]t,\ \forall t \in (0,\infty).$$

This implies that there is $t_1 > 0$ such that for any $t \in [t_1, \infty)$, $L'(t) > 0$. Consequently, $L(t)$ never vanishes on $[t_1, \infty)$. On the other hand, it follows from (3.16)-(3.18) that

$$L(t)L''(t) - (1 + \frac{p-1}{4})L'(t)^2 > 0.$$

Set $N(t) = (L(t))^{-\frac{p-1}{4}}$. Then we get

$$N''(t) = -\frac{p-1}{4}(L(t))^{-\frac{p+7}{4}}[L(t)L''(t) - (1 + \frac{p-1}{4})L'(t)^2] < 0, \forall t \in [t_1, \infty)$$

as well as $N(t_1) > 0$ and $N'(t_1) < 0$. Thus $N(t) \leq N(t_1) + (t - t_1)N'(t_1)$.

So there is a $T_1 \in (t_1, t_1 + \frac{4L(t_1)}{(p-1)L'(t_1)})$ such that

$$\lim_{t \to T_1^-} L(t) = \infty,$$

which contradicts $T_0 = \infty$. The theorem is proved. □

4. Global existence for $E(0) > 0$

For the arbitrary initial energy $E(0) > 0$, using the technique of [4, 24], we derive a sufficient condition on the initial data such that the corresponding local solution of problem (1.1) and (1.2) exists globally.

Lemma 4.1. *Let $2 \leq s < p + 1$, $\phi \in H^s \cap H_0^1$, $\psi \in H^s$ and $u(x,t)$ be the solution of the Cauchy problem* (1.1) *and* (1.2). *Assume that the initial data satisfy* (1.14) *and* (1.16). *Then, the map*

$$\{t \to \|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2\}$$

is strictly decreasing as long as $u(x,t) \in Y_T$.

Proof. From (3.6), (3.7) and (3.8), we have

$$L^{''}(t) = 2\|u_t\|^2 + 2\|u_{xt}\|^2 + 2\|u_{xxt}\|^2 - 2I(t). \tag{4.1}$$

Furthermore, from $u(t) \in Y_T$, we get $L^{''}(t) < 0$ for $t \in [0, T)$. Using (1.16), we obtain

$$2(\phi, \psi) + 2(\phi_x, \psi_x) + 2(\phi_{xx}, \psi_{xx}) < 0,$$

which implies $L'(0) < 0$. So it is easy to see that $L'(t) < L'(0) < 0$, namely $L'(t) < 0$. Therefore, we get the result of the lemma. □

Lemma 4.2. *Let $2 \leq s < p + 1$, $\phi \in H^s \cap H_0^1$, $\psi \in H^s$ and $u(x,t)$ be the weak solution of the Cauchy problem* (1.1) *and* (1.2) *with maximal existence time interval $[0, T)$ such that $u \in C^1((0,T), H^s \cap H_0^1)$ and $u_t \in C((0,T), H^s)$, here $T \leq +\infty$. Assume that the initial data satisfy* (1.14), (1.15) *and* (1.16), *then $u \in Y_T$.*

Proof. Arguing by contradiction, we suppose that there exists a first time $t_2 \in [0, T)$ such that

$$I(u(t_2)) = \|u_t(t_2)\|^2 + \|u_{xt}(t_2)\|^2 + \|u_{xxt}(t_2)\|^2 \tag{4.2}$$

and

$$I(u(t)) > \|u_t(t)\|^2 + \|u_{xt}(t)\|^2 + \|u_{xxt}(t)\|^2, \ \forall t \in [0, t_2).$$

From the (3.6)-(3.8) and Lemma 4.1, we obtain that $L(t)$ and $L'(t)$ are both strictly decreasing on the interval $[0, t_2)$. And by (1.16), for all $t \in (0, t_2)$, we get

$$\begin{aligned} L(u(t)) &< \|\phi\|^2 + \|\phi_x\|^2 + \|\phi_{xx}\|^2 \\ &< -2(\phi, \psi) - 2(\phi_x, \psi_x) - 2(\phi_{xx}, \psi_{xx}) - \frac{2(p+1)}{p+3}E(0). \end{aligned}$$

Moreover, from the continuity of $\|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2$ in t, we obtain

$$L(u(t_2)) < -2(\phi, \psi) - 2(\phi_x, \psi_x) - 2(\phi_{xx}, \psi_{xx}) - \frac{2(p+1)}{p+3}E(0). \tag{4.3}$$

On the other hand, by Theorem 1.2, (1.10) and (1.11), we get

$$\begin{aligned} E(0) = E(t_2) &= \frac{1}{2}(\|u_t(t_2)\|^2 + \|u_{xt}(t_2)\|^2 + \|u_{xxt}(t_2)\|^2) + J(u(t_2)) \\ &= \frac{1}{2}(\|u_t(t_2)\|^2 + \|u_{xt}(t_2)\|^2 + \|u_{xxt}(t_2)\|^2) \\ &\quad + \frac{p-1}{2(p+1)}(\|u_x(t_2)\|^2 + \|u_{xx}(t_2)\|^2) + \frac{1}{p+1}I(u(t_2)). \end{aligned}$$

Using (4.2), we obtain

$$\begin{aligned} E(0) &= (\frac{1}{2} + \frac{1}{p+1})(\|u_t(t_2)\|^2 + \|u_{xt}(t_2)\|^2 + \|u_{xxt}(t_2)\|^2) \\ &\quad + (\frac{1}{2} - \frac{1}{p+1})(\|u_x(t_2)\|^2 + \|u_{xx}(t_2)\|^2) \\ &\geq \frac{p+3}{2(p+1)}(\|u_t(t_2)\|^2 + \|u_{xt}(t_2)\|^2 + \|u_{xxt}(t_2)\|^2). \end{aligned} \tag{4.4}$$

Then from the following equalities

$$\begin{aligned} \|u_t(t_2)\|^2 &= \|u_t(t_2) + u(t_2)\|^2 - \|u(t_2)\|^2 - 2(u(t_2), u_t(t_2)), \\ \|u_{xt}(t_2)\|^2 &= \|u_{xt}(t_2) + u_x(t_2)\|^2 - \|u_x(t_2)\|^2 - 2(u_x(t_2), u_{xt}(t_2)), \\ \|u_{xxt}(t_2)\|^2 &= \|u_{xxt}(t_2) + u_{xx}(t_2)\|^2 - \|u_{xx}(t_2)\|^2 - 2(u_{xx}(t_2), u_{xxt}(t_2)) \end{aligned}$$

and Lemma 4.1, we get

$$\begin{aligned} E(0) \geq \frac{p+3}{2(p+1)}(&\|u_t(t_2) + u(t_2)\|^2 + \|u_{xt}(t_2) + u_x(t_2)\|^2 \\ &+ \|u_{xxt}(t_2) + u_{xx}(t_2)\|^2) \end{aligned}$$

$$
\begin{aligned}
&-\frac{p+3}{2(p+1)}(\|u(t_2)\|^2+\|u_x(t_2)\|^2+\|u_{xx}(t_2)\|^2)\\
&-2\frac{p+3}{2(p+1)}((u(t_2),u_t(t_2))+(u_x(t_2),u_{xt}(t_2))+(u_{xx}(t_2),u_{txx}(t_2)))\\
\geq\ &-\frac{p+3}{2(p+1)}(\|u(t_2)\|^2+\|u_x(t_2)\|^2+\|u_{xx}(t_2)\|^2)\\
&-2\frac{p+3}{2(p+1)}((\phi(t_2),\psi(t_2))+(\phi_x(t_2),\psi_x(t_2))+(\phi_{xx}(t_2),\psi_{xx}(t_2))). \qquad (4.5)
\end{aligned}
$$

So,

$$
L(t_2)\geq -2((\phi(t_2),\psi(t_2))+(\phi_x(t_2),\psi_x(t_2))+(\phi_{xx}(t_2),\psi_{xx}(t_2)))-\frac{2(p+1)}{p+3}E(0). \tag{4.6}
$$

It is obvious that (4.6) contradicts (4.3). This completes the proof. □

Proof of Theorem 1.6. From Theorem 1.1, there exists a unique local solution of problem (1.1) and (1.2) defined on a maximal time interval $[0,T), T<+\infty$. Let $u(t)$ be the weak solution of problem (1.1) and (1.2) such that $u\in C^1((0,T),H^1\cap H_0^1)$ and $u_t\in C((0,T),H^1)$ with (1.14), (1.15) and (1.16). Then from Lemma 4.2, we have $u(x,t)\in Y_T$, namely for $t\in[0,T)$,

$$
I(u(t))>\|u_t(t)\|^2+\|u_{xt}(t)\|^2+\|u_{xxt}\|^2. \tag{4.7}
$$

Therefore from Theorem 1.2, (1.10), (1.11) and (4.7), we obtain

$$
\begin{aligned}
E(0)=E(t)&=\frac{1}{2}(\|u_t(t)\|^2+\|u_{xt}(t)\|^2+\|u_{xxt}(t)\|^2)+J(u)\\
&=\frac{1}{2}(\|u_t(t)\|^2+\|u_{xt}(t)\|^2+\|u_{xxt}(t)\|^2)\\
&\quad+\frac{p-1}{2(p+1)}(\|u_x(t)\|^2+\|u_{xx}(t)\|^2)+\frac{1}{p+1}I(u(t))\\
&>\frac{p+3}{2(p+1)}(\|u_t(t)\|^2+\|u_{xt}(t)\|^2+\|u_{xxt}(t)\|^2)\\
&\quad+\frac{p-1}{2(p+1)}(\|u_x(t)\|^2+\|u_{xx}(t)\|^2).
\end{aligned}
$$

From the Poincaré inequality, that is $\|v\|^2\leq C_0\|v_x\|^2$, $\forall v\in H_0^1$, where C_0 is a positive constant, it follows that $u(x,t)$ is bounded in $C^1((0,T),H^2\cap H_0^1)$, $u_t(x,t)$ is bounded in $C^1((0,T),H^2)$. Hence from Theorem 1.1, it follows that $T=\infty$ and the solution of problem (1.1) and (1.2) exists globally. □

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