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# DISJOINT SUPERCYCLIC WEIGHTED COMPOSITION OPERATORS

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Abstract. In this paper, we discovered a sufficient condition ensuring the weighted composition operators $C_{w_1,\varphi_1},\dots,C_{w_N,\varphi_N}$ were disjoint supercyclic on $H(\Omega)$ endowed with the compact open topology. Besides, we provided a condition on inducing symbols to guarantee the disjoint supercyclicity of non-constant adjoint multipliers $M^*_{\varphi_1}, M^*_{\varphi_2},\dots,M^*_{\varphi_N}$ on a Hilbert space $\mathcal{H}$.

## 1. Introduction

As usual, $\mathbb{N}$ is the set of all non-negative integers and given an integer $N\in\mathbb{N}$, we always assume that $N\geq 2$. Let $H(\Omega)$ denote the space of holomorphic functions on a simply connected domain $\Omega$ of the complex plane, endowed with the compact open topology. If $K$ is a compact subset of $\Omega$, for $f\in H(\Omega)$, define

$$P_K(f)=\sup_{z\in K}|f(z)|.$$

Then $\{P_K : K\subset\Omega,\ K \text{ is compact}\}$ is a family of seminorms that make $H(\Omega)$ a locally convex space. Indeed, this topology is the topology of uniform convergence on compact subsets of the simply connected domain $\Omega$. In this way, $H(\Omega)$ turns into a Fréchet space. Moreover, by Runge's theorem ([12, p. 359]), $H(\Omega)$ is separable, see [12, Exercise 4.3.1]. Given a holomorphic self-map $\varphi$ of $\Omega$, we can define the composition operator $C_\varphi : H(\Omega)\to H(\Omega)$ with $C_\varphi(f)=f\circ\varphi$. Given $\psi\in H(\Omega)$, then it can induce a pointwise multiplication operator $M_\psi(f)=\psi\cdot f$ for all $f\in H(\Omega)$. Combining the composition operator $C_\varphi$ and the multiplication operator $M_\psi$, we define the weighted composition operator $C_{\psi,\varphi}f(z)=\psi(z)f(\varphi(z))$ for $f\in H(\Omega)$.

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For an integer $n \in \mathbb{N}$ and a Fréchet space $X$, we let $L(X)$ denote all bounded linear operators on $X$, and then the $n$-th iterate of $T \in L(X)$, denoted by $T^n$, is obtained by composing $T$ with itself $n$ times. In general case, given $N$ analytic self-maps $\varphi_1, \dots, \varphi_N$, the $n$-th iterate of $\varphi_i$, is denoted by $\varphi_i^{[n]}$ for $i = 1, \dots, N$; also, let $\varphi_0$ stand for the identity function. Besides, if $\varphi_i$ is invertible, we can define the $n$-th iterate $\varphi_i^{[-n]}$ for $i = 1, \dots N$. Moreover, it's easy to check that

$$C_{w,\varphi}^n(f) = \prod_{j=0}^{n-1} w \circ \varphi^{[j]} (f \circ \varphi^{[n]}) \text{ for all } f \in H(\Omega) \text{ and } n \geq 1.$$

We recall an operator $T \in L(X)$ is *hypercyclic* if there is an $x \in X$ such that the orbit

$$Orb(T, x) = \{T^n x : n = 0, 1, 2, \dots\}$$

is dense in $X$, such a vector $x$ is said to be *hypercyclic* for $T$. Roughly speaking, hypercyclicity means existence of a dense orbit. It's well known that an operator $T$ on a separable Banach space $X$ is *hypercyclic* if and only if it is *topologically transitive* in the sense of dynamical systems, i.e., for every pair of non-empty open subsets $U$ and $V$ of $X$ there is $n \in \mathbb{N}$ so that $T^n(U) \bigcap V \neq \emptyset$. Hypercyclic operators have received considerable attention recently, especially since they arise in familiar classes of operators, such as, weighted shifts [11, 13, 14, 17–19, 24], composition operators [21, 27] and so on. The first example of hypercyclic operator was given by Rolewicz in [23]. The result is $B$ is a backward shift on the Banach space $\ell^p(\mathbb{N})$, then $\lambda B$ is *hypercyclic* for any complex number $|\lambda| > 1$. So the concept of supercyclic operators arises. For $T \in L(X)$, $T$ is said to be *supercyclic* provided that there is some $x \in X$ such that the projective orbit

$$\mathbb{C} \cdot Orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C},\ n = 0, 1, 2, \dots\}$$

is dense in $X$ and $x$ is a supercyclic vector for $T \in L(X)$. For motivation, examples and background about linear dynamics, we refer the interested readers to the two excellent books [1] by Bayart and Matheron, [12] by Grosse-Erdmann and Manguillot.

In 2007, Bès and Peris and, independently, Beral investigated the property of the orbits

$$\{(x, , \dots, x), (T_1 x, T_2 x, \dots, T_N x), (T_1^2 x, T_2^2 x, \dots, T_N^2 x), \dots\} \ (x \in X)$$

on $X^N$ for $N \geq 2$. They studied the condition under which one of these orbits was dense in $X^N$ endowed with the product topology for some $x \in X$. If there is some vector satisfying the above condition, the operators $T_1, \dots, T_N$ are called *disjoint hypercyclic*, i.e., the existence of a common vector with a dense orbit for several operators, such that the approximation of any fixed vectors is also simultaneously performed by using a common subsequence. The interested readers can refer into [2, 4, 7, 8, 20–22] and their references therein for more information about the disjoint hypercyclicity. At the same time, the disjoint supercyclicity also emerges and becomes an active topic in linear dynamics. Observing the following definition, the disjoint supercyclicity is a natural generalization of supercyclicity of a single operator. Here we cite the definitions in relation with the disjoint supercyclicity of $T_1, T_2, \dots, T_N \in L(X)$.

**Definition 1.1.** Recall that $N \geq 2$ operators $T_1, T_2, \dots, T_N \in L(X)$ are disjoint supercyclic or d-supercyclic if the direct sum $T_1 \oplus T_2 \oplus \cdots \oplus T_N$ has a supercyclic vector of the form $(x, x, \dots, x) \in X^N$ endowed with the product topology.

**Definition 1.2.** We say the operators $T_1, T_2, \dots, T_N \in L(X)$ are *d-topologically transitive for supercyclicity* provided for every non-empty open subsets $V_0, V_1, \dots, V_N$ of $X$, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that

$$V_0 \bigcap (\lambda T_1^{-n})(V_1) \bigcap \cdots \bigcap (\lambda T_N^{-n})(V_N) \neq \emptyset.$$

We apply [8, Proposition 2.3] to the sequences $(\lambda T_1^j)_{\lambda \in \mathbb{C},\, j \in \mathbb{N}}, \dots, (\lambda T_N^j)_{\lambda \in \mathbb{C},\, j \in \mathbb{N}}$ to show that the d-topologically transitive for supercyclicity can imply d-supercyclicity.

**Proposition 1.3.** *Given $N \geq 2$ and the operators $T_1, T_2, \dots, T_N \in L(X)$, they are d-topologically transitive for supercyclicity if and only if the set of d-supercyclic vectors for $T_1, T_2, \dots, T_N$ is a dense $G_\delta$ set.*

This paper was organized as below, the d-supercyclicity of $C_{w_1,\varphi_1}, \dots, C_{w_N,\varphi_N}$ on $H(\Omega)$ was discussed in Section 2 and a sufficient condition for the d-supercyclicity of adjoint multipliers on a Hilbert space $\mathcal{H}$ was investigated in Section 3.

## 2. The weighted composition operators

In this section, we found a characterization for the disjointness of compact sets to ensure the d-supercyclicity of $C_{w_1,\varphi_1}, \dots, C_{w_N,\varphi_N}$ on $H(\Omega)$, which is closely related with some known facts, such as [4, Corollary 2.2] and [16, Corollary 2.6].

As we all know that if $\varphi$ is a univalent holomorphic self-map of the unit disk $\mathbb{D}$, then the composition operator $C_\varphi$ is hypercyclic if and only if $\varphi$ has no fixed point in $\mathbb{D}$. Hereafter, some research on the dynamics of weighted composition operators started. There are various papers concerning the hypercyclicity and weakly-supercyclicity of weighted composition operators, see, e.g. [3, 15, 25, 26] and the reference therein. Especially, the paper [5] showed a complete characterization of disjoint supercyclic tuples of linear fractional composition operators and furthermore the d-mixing property of tuples of operators is deeply studied in [6], including weighted composition operators on spaces of $p$-integrable functions. Very recently the paper [16] dealt with the disjoint hypercyclicity and weakly d-supercyclicity of weighted composition operators on $H(\mathbb{D})$ and a Hilbert space $\mathcal{H}$. Inspired by the above interesting results, we turn our attention to the disjoint supercyclicity of weighted composition operators on $H(\Omega)$ and adjoint multipliers on a Hilbert space $\mathcal{H}$. On the one hand, if the operators $T_1, T_2, \ldots, T_N$ are d-supercyclic, then each of them must be supercyclic. On the other hand, an operator that is supercyclic must necessarily be weakly supercyclic. Hence we firstly cite a proposition which severely limits the kinds of maps that can produce supercyclic weighted composition operators.

**Proposition 2.1** ([3, Proposition 2.1])**.** *Suppose $C_{w,\varphi} : H(\Omega) \to H(\Omega)$ is weakly supercyclic, where $\Omega$ is an arbitrary plane domain. Then*
(i) *the weight symbol $w$ is zero-free, and*
(ii) *the compositional symbol $\varphi$ is univalent and without fixed points.*

Hereafter, we always assume that all composition symbols and all weights satisfy the assumptions in Proposition 2.1.

**Theorem 2.2.** *Let $N \geq 2$ and $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ be supercyclic weighted composition operators on $H(\Omega)$. If for each compact set $K \subset \Omega$, there exist $n \geq 1$ and $\lambda \in \mathbb{C} \setminus \{0\}$ so that the sets $\lambda K$, $\lambda\varphi_1^{[n]}(K), \ldots, \lambda\varphi_N^{[n]}(K)$ are pairwise disjoint, then the weighted composition operators $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ are d-supercyclic on $H(\Omega)$.*

*Proof.* We will show that $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ are *d-topologically transitive for supercyclicity* (Definition 1.2). Denote $V_0, V_1, \ldots, V_N$ the non-empty open subsets of $H(\Omega)$. We want to find $n \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ satisfying

$$V_0 \bigcap \left(\lambda C^{[-n]}_{w_1,\varphi_1}(V_1)\right) \bigcap \left(\lambda C^{[-n]}_{w_2,\varphi_2}\right)(V_2) \bigcap \cdots \bigcap \left(\lambda C^{[-n]}_{w_N,\varphi_N}\right)(V_N) \neq \emptyset. \tag{2.1}$$

For a given $\epsilon > 0$, there exist compact subsets $K_0, \ldots, K_N$ of $\Omega$ and functions $f_0, \ldots, f_N \in H(\Omega)$ such that the set

$$\{h \in H(\Omega) : \sup_{z \in K_j} |h(z) - f_j(z)| < \epsilon\} \subset V_j \text{ for } 0 \leq j \leq N.$$

Denote $K = \bigcup_{j=0}^N K_j$ and then we conclude that

$$\{h \in H(\Omega) : \sup_{z \in K} |h(z) - f_j(z)| < \epsilon\} \subset V_j \text{ for } 0 \leq j \leq N.$$

Considering the compactness of $K$, we can find two simply connected closed sets $B_1 \subset \Omega$ and $B_2 \subset \Omega$ satisfying $K \subset B_1 \subset \mathring{B}_2$. For the compact set $B_2$ and using the hypothesis, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ so that $\lambda B_2, \lambda\varphi_1^{[n]}(B_2), \ldots, \lambda\varphi_N^{[n]}(B_2)$ are pairwise disjoint. In particular, the sets $\lambda B_1, \lambda\varphi_1^{[n]}(B_1), \ldots, \lambda\varphi_N^{[n]}(B_1)$ are also pairwise disjoint and all composition symbols are univalent, hence we can define the mapping

$$g\left(\frac{z}{\lambda}\right) = \begin{cases} f_0\left(\frac{z}{\lambda}\right), & z \in \lambda B_1, \\ \prod_{k=1}^n \frac{\lambda}{w_j \circ \varphi_j^{[-k]}\left(\frac{z}{\lambda}\right)} f_j \circ \varphi_j^{[-n]}\left(\frac{z}{\lambda}\right), & z \in \lambda\varphi_j^{[n]}(B_1) \text{ for } 1 \leq j \leq N. \end{cases}$$

Since each of $\lambda B_1$ and $\lambda\varphi_j^{[n]}(B_1)$ $(1 \le j \le N)$ is compact and simply connected, hence the complement of $\lambda B_1 \cup \lambda\varphi_1^{[n]}(B_1) \cup \cdots \cup \lambda\varphi_N^{[n]}(B_1)$ is connected. Employing the Runge's theorem (see, e.g. [12, Theorem A.24]), it turns out that there exists a polynomial function $p$ such that

$$\left|p\left(\frac{z}{\lambda}\right) - g\left(\frac{z}{\lambda}\right)\right| < \min\left\{\epsilon, \frac{\lambda\epsilon}{\|w_1\|_\infty^n}, \ldots, \frac{\lambda\epsilon}{\|w_N\|_\infty^n}\right\}$$

for all $z \in B_1 \cup \lambda\varphi_1^{[n]}(B_1) \cup \cdots \cup \lambda\varphi_N^{[n]}(B_1)$. It's trivial to check that

$$\left|f_0\left(\frac{z}{\lambda}\right) - p\left(\frac{z}{\lambda}\right)\right| < \epsilon$$

for all $z \in \lambda B_1$. That is to say that $\sup_{u\in B_1} |f_0(u) - p(u)| < \epsilon$, which yields that

$$p \in \{h \in H(\Omega) : \sup_{z\in K} |h(z) - f_0(z)| < \epsilon\} \subseteq V_0. \tag{2.2}$$

On the other hand, for all $z \in \lambda\varphi_j^{[n]}(B_1)$, we derive that

$$\left|\prod_{k=1}^{n} \frac{\lambda}{w_j \circ \varphi_j^{[-k]}\left(\frac{z}{\lambda}\right)} f_j \circ \varphi_j^{[-n]}\left(\frac{z}{\lambda}\right) - p\left(\frac{z}{\lambda}\right)\right| < \frac{\lambda\epsilon}{\|w_j\|_\infty^n} \quad \text{for all } 1 \le j \le N.$$

If $v \in \lambda B_1$, then $\lambda\varphi_j^{[n]}\left(\frac{v}{\lambda}\right) \in \lambda\varphi_j^{[n]}(B_1)$. Replacing $z$ by $\lambda\varphi_j^{[n]}\left(\frac{v}{\lambda}\right)$, the above inequality implies that for $v \in \lambda B_1$,

$$\begin{aligned}&\left|\prod_{k=1}^{n} \frac{\lambda}{w_j \circ \varphi_j^{[-k]}\left(\varphi_j^{[n]}\left(\frac{v}{\lambda}\right)\right)} f_j \circ \varphi_j^{[-n]} \circ \varphi_j^{[n]}\left(\frac{v}{\lambda}\right) - p \circ \varphi_j^{[n]}\left(\frac{v}{\lambda}\right)\right| \\ &= \left|\prod_{k=0}^{n-1} \frac{\lambda}{w_j \circ \varphi_j^{[k]}\left(\frac{v}{\lambda}\right)} f_j\left(\frac{v}{\lambda}\right) - p \circ \varphi_j^{[n]}\left(\frac{v}{\lambda}\right)\right| < \frac{\lambda\epsilon}{\|w_j\|_\infty^n}.\end{aligned}$$

Furthermore, we obtain that

$$\left|f_j\left(\frac{v}{\lambda}\right) - \frac{\prod_{k=0}^{n-1} w_j \circ \varphi_j^{[k]}\left(\frac{v}{\lambda}\right)}{\lambda} p \circ \varphi_j^{[n]}\left(\frac{v}{\lambda}\right)\right| < \epsilon \quad \text{for } v \in \lambda B_1.$$

That is to say that

$$\left|f_j\left(\frac{v}{\lambda}\right) - \frac{1}{\lambda} C_{w_j,\varphi_j}^{[n]} p\left(\frac{v}{\lambda}\right)\right| < \epsilon \quad \text{for } v \in \lambda B_1.$$

The above inequality can be formulated into

$$\left|f_j(u) - \frac{1}{\lambda} C_{w_j,\varphi_j}^{[n]} p(u)\right| < \epsilon \quad \text{for } u \in B_1.$$

Hence $\sup_{u\in B_1} \left|f_j(u) - \frac{1}{\lambda} C_{w_j,\varphi_j}^{[n]} p(u)\right| < \epsilon$, from which we deduce that $\frac{1}{\lambda} C_{w_j,\varphi_j}^{[n]} p \in V_j,\ 1 \le j \le N$. By the linearity of $C_{w_j,\varphi_j}$, we induce that

$$p \in \lambda C_{w_j,\varphi_j}^{[-n]} V_j,\ 1 \le j \le N. \tag{2.3}$$

Combining (2.2) with (2.3), there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ fulfilling

$$p \in V_0 \bigcap \lambda C^{[-n]}_{w_1,\varphi_1}(V_1) \bigcap \lambda C^{[-n]}_{w_2,\varphi_2}(V_2) \bigcap \cdots \bigcap \lambda C^{[-n]}_{w_N,\varphi_N}(V_N).$$

That is, the weighted composition operators $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ are *d-topologically transitive for supercyclicity.* Due to Proposition 1.3, they are d-supercyclic on $H(\Omega)$. This ends the proof. □

Analyzing the proof for Theorem 2.2, there may be a typo in the statement of [4, Corollary 2.2]. Hence we give a remark to account for it.

*Remark* 2.3. In [4, Corollary 2.2], $(C_{\varphi_{1,n}})_{n=1}^{\infty}, \ldots, (C_{\varphi_{N,n}})_{n=1}^{\infty}$ are d-supercyclic on $H(\Omega)$ if and only if for each compact set $K \subset \Omega$, there exist $n \geq 1$ and $\lambda \in \mathbb{C} \setminus \{0\}$ so that the sets $\lambda K, \lambda\varphi_{1,n}(K), \ldots, \lambda\varphi_{N,n}(K)$ are pairwise disjoint and each of $\varphi_{1,n}, \ldots, \varphi_{N,n}$ is injective on $K$.

In view of Theorem 2.2, the following corollary holds for the case $\Omega = \mathbb{D}$, which generalizes [16, Corollary 2.6] to some extent. For the attractive fixed point (or Denjoy-Wolff point) for a self-map on $\mathbb{D}$, the readers can refer to [9, Theorem 2.51 and Definition 2.52] for more information.

**Corollary 2.4.** *Suppose $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ are supercyclic weighted composition operators with univalent self-maps $\varphi_1, \ldots, \varphi_N$ having no interior fixed points. If the attractive fixed points (or Denjoy-Wolff points) of $\varphi_1, \ldots, \varphi_N$ are all distinct, then $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ are d-supercyclic on $H(\mathbb{D})$.*

*Proof.* We will prove the sets $\lambda K$, $\lambda\varphi_1^{[n]}(K), \ldots, \lambda\varphi_N^{[n]}(K)$ are pairwise disjoint for each compact set $K \subset \mathbb{D}$, $n \in \mathbb{N}$ large enough and $\lambda = 1 \in \mathbb{C} \setminus \{0\}$. Denote the attractive fixed point (or Denjoy-Wolff point) of $\varphi_j$ by $\alpha_j \notin \mathbb{D}$ for $1 \leq j \leq N$. For each $\xi \in K \subset \mathbb{D}$, by [9, Theorem 2.51], it yields that $\varphi_j^{[n]}(\xi) \to \alpha_j$, $n \to \infty$, for $1 \leq j \leq N$. Due to the compactness of the set $K$, there exist a compact set $\bar{K}_j \ni \alpha_j$ and $l_j \in \mathbb{N}$ such that $\varphi_j^{[n]}(K) \subset \bar{K}_j$ for $n > l_j$ $(1 \leq j \leq N)$. Since $\alpha_i \neq \alpha_j$ for $i \neq j$ and $K \subset \mathbb{D}$, we can further make the sets $\bar{K}_1, \ldots, \bar{K}_N$ small enough and $l_1, l_2, \ldots, l_N$ large enough such that $\bar{K}_i \cap \bar{K}_j = \emptyset$ for $i \neq j$ and $K \cap \bar{K}_j = \emptyset$ for $1 \leq j \leq N$. In the end, we choose $\bar{K}_1, \ldots, \bar{K}_N$ small enough and $l_1, l_2, \ldots, l_N$ large enough so that $\varphi_j^{[n]}(K) \subset \bar{K}_j$ for $n > l_j (1 \leq j \leq N)$, $\bar{K}_i \cap \bar{K}_j = \emptyset$ for $i \neq j$ and $K \cap \bar{K}_j = \emptyset$ for $1 \leq j \leq N$. As a consequence, there exist $n_0 \in \mathbb{N}$ satisfying $n_0 > \max\{l_1, \ldots, l_N\}$ and $\lambda_0 = 1 \in \mathbb{C} \setminus \{0\}$ so that the sets $K$, $\varphi_1^{[n_0]}(K), \ldots, \varphi_N^{[n_0]}(K)$ are pairwise disjoint for each compact set $K \subset \mathbb{D}$. Hence the d-supercyclicity of $C_{w_1,\varphi_1}, \ldots, C_{w_N,\varphi_N}$ follows from Theorem 2.2. This ends the proof. □

## 3. Adjoint multipliers on a Hilbert space

In this section, let $\mathcal{H}$ be an *infinite dimensional separable Hilbert* space of analytic functions defined on $\mathbb{D}$ such that for each $z \in \mathbb{D}$, the linear functional of point evaluation at $z$ given by $f \to f(z)$ is bounded. In what follows, *a*

*Hilbert space of analytic functions $\mathcal{H}$ we mean one satisfies the above conditions.* Moreover, *the constants and the identity function $f(z) = z$* are in the Hilbert space $\mathcal{H}$. The Riesz representation theorem states that $e_\lambda(f) = \langle f, k_\lambda \rangle$ for some $k_\lambda \in \mathcal{H}$, the reproducing kernel of $\mathcal{H}$. The weighted Hardy space is the well-known example of such Hilbert space $\mathcal{H}$. Let $(\beta(n))_n$ be a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^2(\beta)$ is the space of analytic functions $f = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ on $\mathbb{D}$ satisfying

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 |\beta(n)|^2 < \infty.$$

From the book [9] we know that the classical Hardy space, the Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1$, $\beta(n) = (n+1)^{-1/2}$ and $\beta(n) = (n+1)^{1/2}$, respectively.

Recall that a multiplier of $\mathcal{H}$ is an analytic function $\psi$ on $\mathbb{D}$ such that $\psi(\mathcal{H}) \subset \mathcal{H}$. We collect all multipliers of $\mathcal{H}$ and denote by $M(\mathcal{H})$. Given $\psi \in M(\mathcal{H})$, define the multiplication operator $M_\psi$ by $M_\psi f = \psi f$, bounded on $\mathcal{H}$. It's well-known that every multiplier is a bounded holomorphic function on $\mathbb{D}$. In the first step, we describe a necessary condition for the d-supercyclicity of two adjoint multipliers $M_{\psi_1}^*$ and $M_{\psi_2}^*$ on $\mathcal{H}$.

**Proposition 3.1.** *Let $\psi_1, \psi_2$ be non-constant multipliers of $\mathcal{H}$. If $M_{\psi_1}^*$ and $M_{\psi_2}^*$ are d-supercyclic operators, then $\psi_1/\psi_2$ must be non-constant.*

*Proof.* We assume that $\psi_2 = \mu\psi_1$ for some $\mu \in \mathbb{C}$. Denote $f \neq 0$ a d-supercyclic vector for $M_{\psi_1}^*$ and $M_{\mu\psi_1}^*$. For two different vectors $(f, 0) \in \mathcal{H} \times \mathcal{H}$ and $(0, f) \in \mathcal{H} \times \mathcal{H}$, we can respectively find the sequences $\{\lambda_{n_k}\}_k \subset \mathbb{C}$ and $\{\hat{\lambda}_{m_k}\}_k \subset \mathbb{C}$ satisfying

$$(\lambda_{n_k} M_{\psi_1}^{*n_k} f, \lambda_{n_k} M_{\mu\psi_1}^{*n_k} f) \to (f, 0) \text{ and}$$
$$(\hat{\lambda}_{m_k} M_{\psi_1}^{*m_k} f, \hat{\lambda}_{m_k} M_{\mu\psi_1}^{*m_k} f) \to (0, f)$$

as $k \to \infty$. Due to $M_{\mu\psi_1}^{*n} = \bar{\mu}^n M_{\psi_1}^{*n}$ for all $n \in \mathbb{N}$. Therefore, the above formulas entails that

$$(\lambda_{n_k} M_{\psi_1}^{*n_k} f, \bar{\mu}^{n_k} \lambda_{n_k} M_{\psi_1}^{*n_k} f) \to (f, 0) \text{ and}$$
$$(\hat{\lambda}_{m_k} M_{\psi_1}^{*m_k} f, \bar{\mu}^{m_k} \hat{\lambda}_{m_k} M_{\psi_1}^{*m_k} f) \to (0, f)$$

as $k \to \infty$. The first one verifies that $|\mu| < 1$ and the second one yields that $|\mu| > 1$, which is a contradiction. This ends the proof. □

The d-Supercyclicity Criterion is a sufficient condition for d-supercyclicity, which is one of main tools to seek for the d-supercyclic operators. We cite the following version for our further consequences.

**Definition 3.2** (d-Supercyclicity Criterion, [4, Definition 4.1.1])**.** Let $X$ be a Banach space and $\{n_k\}_k$ be a strictly increasing sequence of positive integers. We say that $T_1, \ldots, T_N$ in $L(X)$ satisfy the d-Supercyclicity Criterion with respect to $\{n_k\}_k$ provided there exist dense subsets $X_0, X_1, \dots, X_N$ of $X$ and mappings

$$S_l : X_l \to X, \ (1 \le l \le N)$$

so that for $1 \le i \le N$

(i) $(T_l^{n_k} S_i^{n_k} - \delta_{i,l} I_{X_i}) \underset{k\to\infty}{\to} 0$ pointwise on $X_i$,

(ii) $\lim\limits_{k\to\infty} \|T_l^{n_k} x\| \cdot \left\| \sum\limits_{j=1}^{N} S_j^{n_k} y_j \right\| = 0$ for $x \in X_0$ and $y_j \in X_j$.

Now we concentrate on the disjoint supercyclicity of $M^*_{\psi_1}$ and $M^*_{\psi_2}$ on $\mathcal{H}$.

**Theorem 3.3.** *Let $\psi_1$ and $\psi_2$ be non-constant multipliers of $\mathcal{H}$ and suppose $M^*_{\psi_1}$ and $M^*_{\psi_2}$ are both supercyclic on $\mathcal{H}$. If the following sets*

$$V_0 = \{z \in \mathbb{D} : |\psi_1(z)| < 1, \ |\psi_2(z)| < 1\}, \tag{3.1}$$

$$V_1 = \{z \in \mathbb{D} : |\psi_1(z)| > 1, \ |\psi_1(z)| > |\psi_2(z)|\}, \tag{3.2}$$

$$V_2 = \{z \in \mathbb{D} : |\psi_2(z)| > 1, \ |\psi_2(z)| > |\psi_1(z)|\} \tag{3.3}$$

*are non-empty, then $M^*_{\psi_1}$ and $M^*_{\psi_2}$ are d-supercyclic on $\mathcal{H}$.*

*Proof.* From the assumptions we can verify that the sets $S_{V_0} = \mathrm{span}\{k_z : z \in V_0\}$, $S_{V_1} = \mathrm{span}\{k_z : z \in V_1\}$ and $S_{V_2} = \mathrm{span}\{k_z : z \in V_2\}$ are dense in $\mathcal{H}$, that is, $\overline{S_{V_0}} = \overline{S_{V_1}} = \overline{S_{V_2}} = \mathcal{H}$. We include the details for the readers' convenience. In fact, if $f \in \mathcal{H}$ is orthogonal to $k_z$ for every $z \in V_0$ and $f(z) = \langle f, k_z\rangle$. Since the set $V_0$ defined in (3.1) is nonempty, then the set $V_0$ has a limit point in $\mathbb{D}$, hence the identity theorem for holomorphic functions implies that $f$ vanishes identically on $\mathcal{H}$. Thus $(S_{V_0})^{\perp} = \{0\}$. That is, $\overline{S_{V_0}} = \mathcal{H}$. By the similar argument and employing (3.2) and (3.3), we can formulate that $\overline{S_{V_1}} = \mathcal{H}$ and $\overline{S_{V_2}} = \mathcal{H}$.

Since $M^{*n}_{\psi_i} k_z = \overline{\psi_i(z)}^n k_z$ for $i = 1, 2$. For $z \in V_0$, employing the fact $|\psi_i(z)| < 1$ for $i = 1, 2$, and the linearity of $M^*_{\psi_i}$, we conclude that

$$\|M^{*n}_{\psi_i}\| \to 0, \text{ pointwise on } S_{V_0} \ \text{ as } n \to \infty \tag{3.4}$$

for $i = 1, 2$.

To find the desired right inverse of $M^*_{\psi_i}$ for $i = 1, 2$, we divide the proof into two cases by the fact that the set $G_{V_1} = \{k_z : z \in V_1\}$ is linearly independent or not.

(**Case i**) Suppose the set $G_{V_1} = \{k_z : z \in V_1\}$ is linearly independent. In this case, we define a linear map $S_1 : S_{V_1} \to \mathcal{H}$ by

$$S_1 k_z = \overline{\psi_1(z)}^{-1} k_z, \ z \in V_1.$$

Since $|\psi_1(z)| > 1$ for all $z \in V_1$, then $S_1$ is well-defined and we can extend $S_1$ by linearity on $S_{V_1} = \mathrm{span}\{k_z, \ z \in V_1\}$. Furthermore, we can get that $S_1^n k_z = \overline{\psi_1(z)}^{-n} k_z, \ z \in V_1$ for all $n \ge 1$. Due to $|\psi_1(z)| > 1$, it is clear that

$$S_1^n k_z \to 0 \text{ as } n \to \infty. \tag{3.5}$$

From the definitions, $M^*_{\psi_1} S_1 k_z = k_z$ and $M^{*n}_{\psi_2} S_1^n k_z = \overline{\psi_2(z)}^n \overline{\psi_1(z)}^{-n} k_z \to 0$ as $n \to \infty$ due to $|\psi_2(z)| < |\psi_1(z)|$ for all $z \in V_1$.

(**Case ii**) Now we assume that the set $G_{V_1} = \{k_z : z \in V_1\}$ is not linearly independent. In this case, we use the method which has been used by Godefroy and Shapiro in [10, Theorem 4.5]. For the convenience of the readers, we exhibit this proof in details. Consider a countable dense subset

$$V_{11} = \{w_n \in \mathbb{D} : n \geq 1\}$$

of the set $V_1$. Next we will use induction to choose a sequence $z_n$. Take $z_1 = w_1$, denote

$$V_{12} = V_{11} \setminus \{w \in V_{11} : k_w \in \operatorname{span}\{k_{z_1}\}\}.$$

Denote the first element of $V_{12}$ by $z_2$ and let

$$V_{13} = V_{12} \setminus \{w \in V_{12} : k_w \in \operatorname{span}\{k_{z_1}, k_{z_2}\}\}.$$

The infinite dimensionality of $\mathcal{H}$ insures the process never terminates. Then we can obtain an infinite subset $L_1 = \{z_n \in \mathbb{D} : n \geq 1\}$ of the set $V_1$, for which the corresponding set of kernel functions $G_{L_1} = \{k_z : z \in L_1\}$ is linearly independent and is dense in $\mathcal{H}$. Now the operator $S_1$ can be defined exactly as above, just with $G_{L_1}$ in place of $G_{V_1}$. To sum up, we can define the map $S_1$ under the above two cases. The same process can be done for $G_{V_2} = \{k_z, z \in V_2\}$ to obtain the map $S_2 : S_{V_2} \to \mathcal{H}$, where $G_{V_2}$ is linearly independent or not.

From the definitions, $M^*_{\psi_i} S_i k_z = k_z$ for all $z \in V_i$ and $i = 1, 2$. On the one hand, for $z \in V_1$, then $M^{*n}_{\psi_2} S_1^n k_z = \overline{\psi_2(z)}^n \overline{\psi_1(z)}^{-n} k_z \to 0$ as $n \to \infty$. On the other hand, for $z \in V_2$, it also holds that

$$M^{*n}_{\psi_1} S_2^n k_z = \overline{\psi_1(z)}^n \overline{\psi_2(z)}^{-n} k_z \to 0 \quad \text{as } n \to \infty.$$

The above formulas reflect that (i) $M^{*n}_{\psi_l} S_i^n - \delta_{i,l} I_{S_{V_i}} \underset{n\to\infty}{\to} 0$ pointwise on $S_{V_i}$. At the same time, the displays (3.4), (3.5) and the definition for $S_2 : S_{V_2} \to \mathcal{H}$ lead that

$$\begin{aligned} &\lim_{n\to\infty} \|M^{*n}_{\psi_l} k_y\| \|S_1^n k_{z_1} + S_2^n k_{z_2}\| \\ \leq &\lim_{n\to\infty} \|\overline{\psi_l(y)}^n k_y\| \|\overline{\psi_1(z_1)}^{-n} k_{z_1} + \overline{\psi_2(z_2)}^{-n} k_{z_2}\| = 0 \end{aligned}$$

for $k_y \in S_{V_0}$, $k_{z_1} \in S_{V_1}$ and $k_{z_2} \in S_{V_2}$. That is, (ii) of Definition 3.2 is also true. As a consequence, the operators $M^*_{\psi_1}, M^*_{\psi_2}$ satisfy the d-Supercyclicity Criterion, thus they are d-supercyclic on $\mathcal{H}$. The proof is complete. □

Adapted from the proof of Theorem 3.3, the sufficient condition ensuring the d-supercyclicity of $N \geq 2$ multipliers $M^*_{\psi_1}, M^*_{\psi_2}, \ldots, M^*_{\psi_N}$ on $\mathcal{H}$ follows.

**Corollary 3.4.** *Let $\psi_1, \psi_2, \ldots, \psi_N$ be non-constant multipliers of $\mathcal{H}$ and suppose $M^*_{\psi_1}, M^*_{\psi_2}, \ldots, M^*_{\psi_N}$ are supercyclic. If the following sets $V_0 = \{z \in \mathbb{D} : |\psi_i(z)| < 1,\ i = 1, 2, \ldots, N\}$ and for $i = 1, 2, \ldots, N$,*

$$V_i = \{z \in \mathbb{D} : |\psi_i(z)| > 1,\ \max\{|\psi_1(z)|, \ldots, |\psi_{i-1}(z)|, |\psi_{i+1}(z)|, \ldots, |\psi_N(z)|\}$$

$$< |\psi_i(z)|\}$$

*are non-empty, then* $M^*_{\psi_1}, M^*_{\psi_2}, \ldots, M^*_{\psi_N}$ *are d-supercyclic on* $\mathcal{H}$.

**Example 1.** Let $\psi_1(z) = z + 1$ and $\psi_2(z) = z - \frac{1}{2}$. Then

$$\psi_1\left(-\frac{1}{3}\right) = \frac{2}{3} \text{ and } \psi_2\left(-\frac{1}{3}\right) = -\frac{5}{6}, \tag{3.6}$$

$$\psi_1\left(\frac{1}{3}\right) = \frac{4}{3} \text{ and } \psi_2\left(\frac{1}{3}\right) = -\frac{1}{6}, \tag{3.7}$$

$$\psi_1\left(-\frac{2}{3}\right) = \frac{1}{3} \text{ and } \psi_2\left(-\frac{2}{3}\right) = -\frac{7}{6}. \tag{3.8}$$

From (3.6)-(3.8), it reads that $-1/3 \in V_0$, $1/3 \in V_1$ and $-2/3 \in V_2$. Employing Theorem 3.3, it yields that $M^*_{z+1}$ and $M^*_{z-\frac{1}{2}}$ are d-supercyclic on $\mathcal{H}$.

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