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THICKLY SYNDETIC SENSITIVITY OF SEMIGROUP ACTIONS

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Abstract. We show that for an M-action on a compact Hausdorff uniform space, if it has at least two disjoint compact invariant subsets, then it is thickly syndetically sensitive. Additionally, we point out that for a P-M-action of a discrete abelian group on a compact Hausdorff uniform space, the multi-sensitivity is equivalent to both thick sensitivity and thickly syndetic sensitivity.

1. Introduction

A *topological dynamical system* (or *dynamical system* for short) in the present article is a triple (S, X, ϕ), where S is a topological semigroup, X is a Hausdorff topological space and

$$\phi : S \times X \to X, (s, x) \mapsto sx$$

is a continuous acting map with the property that $t(sx) = (ts)x$ for all $x \in X$, $t, s \in S$. If S has an identity e, then we also require that $ex = x$ for all $x \in X$. Sometimes we write the dynamical system as a pair (S, X). A semigroup S is a *monoid* if it has an identity. A semigroup S is *abelian* if $ab = ba$ for all $a, b \in S$. Sometimes we write the dynamical system as a pair (S, X). If $S = \{T^n : n = 0, 1, 2, \ldots\}$ and $T : X \to X$ is a continuous map, then the classical dynamical system (S, X) is called a *cascade*. We use the standard notation: (X, T). Moreover to avoid uninteresting cases we assume that S and X are *infinite*. In this paper, let $\mathbb{N}$ be the set of natural numbers.

For $a \in S$ and $A, B \subset S$ denote

$$a^{-1}A = \{s \in S : as \in A\}, \ B^{-1}A = \bigcup_{b \in B} b^{-1}A, \text{ and } AB = \{ab : a \in A, b \in B\}.$$

We need the following definitions (e.g., [5, 12, 14]).

Definition 1.1. Let S be a topological semigroup.

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(1) A subset $P \subset S$ is *syndetic* if there is a compact subset $F \subset S$ such that $F^{-1}P = S$.
(2) A subset $P \subset S$ is *thick* if for every compact subset $A \subset S$ there is $t \in S$ such that $P \supset At$.

We also need the following definitions .

Definition 1.2. Let S be a topological semigroup.

(1) A subset P of S is called *piecewise syndetic* if there is a compact subset F of S satisfying that for every compact subset A of S there is s_A of S such that $F^{-1}P \supset As_A$ (cf. [2, 3, 6, 15]).
(2) A subset P of S is called *thickly syndetic* if for every compact subset $A \subset S$ there is a syndetic set $Q_A \subset S$ such that $AQ_A \subset P$, that is $Q_A \subset \bigcap_{a\in A} a^{-1}P$ (cf. [15]).

Let (S, X) be a dynamical system, where $(X, \mathcal{U})$ is a uniform space. Let $U \subset X$ and let $\Theta \in \mathcal{U}$ be an entourage. We define

$$E(U, \Theta) = \{s \in S : \text{there are } x, y \in U \text{ such that } (sx, sy) \notin \Theta\}.$$

A dynamical system (S, X) is *sensitive* (*thickly sensitive, thickly syndetically sensitive*, respectively) if there exists an entourage $\Theta \in \mathcal{U}$ such that the set $E(U, \Theta)$ is nonempty (thick, thickly syndetic, respectively) for every nonempty open subset U of X. Such an entourage Θ is also called a sensitivity constant entourage of the system (S, X). A dynamical system (S, X) is *multi-sensitive* if there exists an entourage $\Theta \in \mathcal{U}$ such that $\bigcap_{i=1}^{n} E(U_i, \Theta) \neq \emptyset$ for any finite collection of nonempty open subsets $U_1, U_2, \ldots, U_n$ of X.

Sensitivity is an important property of chaotic dynamical systems which is related to what was popularized as the butterfly effect. Recently, some stronger versions of sensitivity were studied (e.g., [7, 8, 11–13]). In [11, Theorem 8], it was proved that if a cascade (X, T) is non-minimal M-system, then it is thickly syndetic sensitive, where X is a compact metric space. In [8, Theorem 4.2] it was proved that for an M-system of cascade (X, T), multi-sensitivity is equivalent to both thick sensitivity and thickly syndetic sensitivity, where X is a compact metric space.

The present work is inspired by the results from the papers mentioned above and is organized as follows. In Sect. 2, we introduce some notions and results to be used in the article. For examples, the notions of the M-action and the P-M-action are introduced. An action is called an M-action (a P-M-action, respectively) if it is transitive (point transitive, respectively) and the set of almost periodic points is dense. In Sect. 3, we discuss the properties of thickly syndedic sets. Finally, we study the sensitivity of semigroup actions on uniform space. We show that for an M-action on a compact Hausdorff uniform space, if it has at least two disjoint compact invariant subsets, then is thickly syndetically sensitive (see Theorem 4.4). Additionally, we point out that for a P-M-action of a discrete abelian group on a compact Hausdorff uniform space, the multi-sensitivity is equivalent to both thick sensitivity and thickly syndetic sensitivity (see Corollary 4.9).

2. Preliminaries and some lemmas

We need the following lemmas (cf. [2, 6, 15, 16]).

Lemma 2.1. *Let S be a topological semigroup.*

(1) *If $P \subset S$ is syndetic, then so is aP for every $a \in S$.*
(2) *If $A \subset S$ is syndetic and $B \subset S$ is thick, then $A \cap B \neq \emptyset$.*

Proof. (1) If $P \subset S$ is syndetic, then there is a compact set $F \subset S$ such that $F^{-1}P = S$. Then $(aF)^{-1}aP = S$, so the set aP is also syndetic. (2) Since A is syndetic, there is a compact set $F \subset S$ such that $F^{-1}A = S$ which implies that $A \cap (\bigcap_{s \in S} Fs) \neq \emptyset$. As B is thick, there is $t \in S$ such that $B \supset Ft$. Therefore $A \cap B \supset A \cap Ft \neq \emptyset$. □

Remark 2.2. Every thickly syndetic set is thick and syndetic. In fact, let P be a thickly syndetic and choose $t \in S$. Then there is a syndetic set $Q_t \subset S$ such that $tQ_t \subset P$. By Lemma 2.1, P is syndetic. It is obvious that a thickly syndetic set is thick.

In order to describe the behaviors of dynamical systems, we need some notions as follow. Let (S, X) be a dynamical system.

For $s_0 \in S$, $x_0 \in X$ and $U, V \subset X$ denote

$$s_0^{-1}U = \{x \in X : s_0 x \in U\},$$
$$N(x_0, U) = \{s \in S : sx_0 \in U\},$$
$$N(U, V) = \{s \in S : U \cap s^{-1}V \neq \emptyset\}.$$

By $X \setminus A$ or A' denote the complement of $A \subset X$. By $\overline{U}$ we will denote the closure of a subset $U \subset X$. Let $F \subset S$. The *orbit of the point $x \in X$ with respect to F* is the set $Fx = \{fx : f \in F\}$. Specially, we call Sx the *orbit* of the point x. A subset $Y \subset X$ is called *invariant* if $sy \in Y$ for all $y \in Y$, $s \in S$. If $Y \subset X$ is a closed invariant subset of (S, X), the (S, Y) is called a subsystem of (S, X).

We also need the following definitions (cf. [4, 10]).

Definition 2.3. Let (S, X) be a dynamical system.

(1) (S, X) is called *minimal* if $\overline{Sx} = X$ for every $x \in X$.
(2) A point x is called *minimal* if the subsystem $\overline{Sx}$ is minimal.
(3) A point x is called *almost periodic* if the set $N(x, U)$ is syndetic for every neighborhood U of x.
(4) (S, X) is called *point transitive* if there is a point $x \in X$ such that $\overline{Sx} = X$. Such a point is called a *transitive point* and the set of all transitive points is denoted by $Trans(S, X)$.
(5) (S, X) is called *transitive* if for every pair of nonempty open subsets U, V in X there exists $s \in S$ such $U \cap s^{-1}V \neq \emptyset$.

(6) (S,X) is called *weakly mixing* if for any nonempty open subsets U_1, U_2, V_1, V_2 in X there exists $s \in S$ such $(U_1 \times V_1) \cap s^{-1}(U_2 \times V_2) \neq \emptyset$.

We need the following result (e.g., Proposition 5.21 in [4], or Lemma 4.3 in [10]).

Proposition 2.4. *Let (S,X) be a dynamical system and let $x \in X$, where S is a discrete semigroup and X is a compact Hausdorff space. Then the following conditions are equivalent.*

(1) *x is an almost periodic point of (S,X);*
(2) *x is a minimal point of (S,X).*

Definition 2.5. Let (S,X) be a dynamical system.

(1) (S,X) is called an *M-system* (or S is called an *M-action* on X) if it is transitive and the set of almost periodic points is dense in X.
(2) (S,X) is called a *P-M-system* (or S is called a *P-M-action* on X) if it is point transitive and the set of almost periodic points is dense in X.

In general, the point transitive and the transitive are independent properties (cf. [9]).

Proposition 2.6. *Let (S,X) be a dynamical system, where X is a compact Hausdorff space, S is an abelian semigroup and each $s \in S$ is a surjective action on X. If (S,X) is a P-M-system, then it is an M-system.*

Proof. Let $x \in X$ be a transitive point of (S,X). Then for every $t \in S$, we have
$$\overline{S(tx)} = \overline{tSx} = t(\overline{Sx}) = tX = X.$$
Hence, every tx is also a transitive point of (S,X). Let U, V be two nonempty open subsets of X. Since x is a transitive point of (S,X), there is $s_1 \in S$ such that $s_1x \in U$. And since s_1x is also a transitive point of (S,X), there is $s_2 \in S$ such that $s_2s_1x \in V$. Hence $s_2s_1 \in N(U,V)$, it following that (S,X) is transitive. □

When X is a compact metric space, if (S,X) is transitive, then $\overline{Trans(S,X)} = X$ (cf. Proposition 3.2 in [10]). Hence, the following proposition holds.

Proposition 2.7. *Let (S,X) be a dynamical system, where X is a compact metric space. If (S,X) is an M-system, then it is a P-M-system.*

3. Thickly syndetic sets

Proposition 3.1. *Let S be a topological semigroup, and let $P \subset S$. Then P is syndetic if and only there is a compact subset $F \subset S$ such that for every $s \in S$, we have $Fs \cap P \neq \emptyset$.*

Proof. Let $P \subset S$ be syndetic. Then there is a compact subset $F \subset S$ such that $F^{-1}P = \bigcup_{f \in F} f^{-1}P = S$. So for every $s \in S$, there is $f \in F$ such that $fs \in P$. This implies that $Fs \cap P \neq \emptyset$. Conversely, it is clear. □

Proposition 3.2. *Let S be a topological semigroup, and let $P_1, P_2 \subset S$. If P_1 and P_2 are thickly syndetic, then so is $P_1 \cap P_2$.*

Proof. Assume that P_1 and P_2 are thickly syndetic. If the set $P_1 \cap P_2$ is not thickly syndetic, then there is a compact subset H of S such that the set $\bigcap_{h\in H} h^{-1}(P_1 \cap P_2)$ is not syndetic. By Lemma 3.1, for every compact subset $Q \subset S$ there is $s_Q \in S$ such that

$$(\bigcap_{h\in H} h^{-1}(P_1 \cap P_2)) \cap Qs_Q = (\bigcap_{h\in H} h^{-1}P_1) \cap (\bigcap_{h\in H} h^{-1}P_2) \cap Qs_Q = \emptyset. \tag{3.1}$$

Since P_1 is thickly syndetic, then the set $\bigcap_{h\in H} h^{-1}P_1$ is syndetic. By Lemma 3.1, there is a compact subset $F \subset S$ such that for every $s \in S$ we have

$$(\bigcap_{h\in H} h^{-1}P_1) \cap Fs \neq \emptyset. \tag{3.2}$$

Since P_2 is thickly syndetic, then the set $\bigcap_{t\in HF} t^{-1}P_2$ is syndetic. By Lemma 3.1, there is a compact subset $L \subset S$ such that for every $s \in S$ we have

$$(\bigcap_{t\in HF} t^{-1}P_2) \cap Ls \neq \emptyset. \tag{3.3}$$

That is, for every $s \in S$ there is $l \in L$, we have $hfls \in P_2$ for all $h \in H$ and $f \in F$. Hence, for every $s \in S$ there is $l \in L$, for all $f \in F$ we have

$$fls \in \bigcap_{h\in H} h^{-1}P_2. \tag{3.4}$$

By (3.2), for $ls \in S$ there is $f_1 \in F$ such that $f_1ls \in \bigcap_{h\in H} h^{-1}P_1$. By (3.4), we have $f_1ls \in (\bigcap_{h\in H} h^{-1}P_1) \cap (\bigcap_{h\in H} h^{-1}P_2)$. Hence, we show that $FLs \cap (\bigcap_{h\in H} h^{-1}P_1) \cap (\bigcap_{h\in H} h^{-1}P_2) \neq \emptyset$ for all $s \in S$, this contradicts the (3.1). □

Proposition 3.3. *Let S be a topological semigroup, and let $P \subset S$. Then $S \setminus P$ is not piecewise syndetic if and only P is thickly syndetic.*

Proof. Assume that $S \setminus P$ is not piecewise syndetic. Then for every a compact subset $F \subset S$ there is a compact subset $A \subset S$ such that $F^{-1}(S \setminus P) \not\supseteq As$ for all $s \in S$. This implies that

$$(F^{-1}(S \setminus P))' \cap As = (\bigcup_{f\in F} f^{-1}(S \setminus P))' \cap As = \bigcap_{f\in F} f^{-1}(P) \cap As \neq \emptyset.$$

By Proposition 3.1, the set $\bigcap_{f\in F} f^{-1}(P)$ is syndetic. Let $Q = \bigcap_{f\in F} f^{-1}(P)$. Then $FQ \subset P$, so P is thickly syndetic.

Conversely, assume that P is thickly syndetic. Then for every a compact subset $F \subset S$ there is a syndetic set $Q \subset S$ such that $FQ \subset P$. It follows that $Q \subset \bigcap_{f\in F} f^{-1}(P)$. Since $\bigcap_{f\in F} f^{-1}(P)$ is syndetic, by Proposition 3.1 we have that there is a compact subset $A \subset S$ such that $\bigcap_{f\in F} f^{-1}(P) \cap As \neq \emptyset$ for all

$s \in S$. This implies that $As \not\subset (\bigcap_{f\in F} f^{-1}(P))' = F^{-1}(S \setminus P)$. Hence, the set $S \setminus P$ is not piecewise syndetic. $\square$

In [2, Theorem 2.5], it was proved that when S is a discrete semigroup, if $A \cup B \subset S$ is piecewise syndetic, then either A or B is piecewise syndetic. Of course, it is easy to see that when S is any topological semigroup, the result holds too. Hence, Proposition 3.2 is obtained by Proposition 3.3.

Endowing the semigroup S with the discrete topology, we take the points of the Stone-Čech compactification βS of S to be the ultrafilter on S. Since $(S,\cdot)$ is a semigroup, we extend the operation $\cdot$ to βS such that $(\beta S,\cdot)$ is a compact Hausdorff right topological semigroup. We denote the minimal ideal of $(\beta S,\cdot)$ by $K(\beta S)$. If $A \subset S$, then $\widehat{A} = cl_{\beta S}A = \{p \in \beta S : A \in p\}$ is a basic clopen subset of βS (cf. [6]).

The following result comes from [2, Theorem 2.9(b)].

Proposition 3.4. *Let S be a discrete topological semigroup. Then $A \subset S$ is piecewise syndetic if and only if $K(\beta S) \cap cl_{\beta S}A \neq \emptyset$.*

By Proposition 3.3 and Proposition 3.4, the following result holds (cf. the Lemma 1.9 in [1]).

Proposition 3.5. *Let S be a discrete topological semigroup. Then $A \subset S$ is thickly syndetic if and only if $K(\beta S) \subset cl_{\beta S}A$.*

When S is a discrete topological semigroup, we may obtain Proposition 3.2 by Proposition 3.5. Let $A,\ B \subset S$ be two thickly syndetic subsets. By Proposition 3.5, we have $cl_{\beta S}(A\cap B) = cl_{\beta S}A \cap cl_{\beta S}B \supset K(\beta S)$. By Proposition 3.5 again, the set $A \cap B$ is thickly syndetic.

4. Sensitivity

Let X be a uniform space and let Θ be an entourage of X. Then Θ is called *symmetric* if $\Theta^{-1} = \Theta$. For $x \in X$, let $\Theta(x) = \{y \in X : (x,y) \in \Theta\}$. Regarding a subset $A \subset X$, let $\Theta(A) = \bigcup_{a\in A}\Theta(a)$. The composite $\Theta_1 \circ \Theta_2$ of two entourages Θ_1 and Θ_2 of X is defined as $\Theta_1 \circ \Theta_2 = \{(x,z) :$ there is an element $y \in X$ such that $(x,y) \in \Theta_1$ and $(y,z) \in \Theta_2\}$.

The following lemma comes from [14, Lemma 4.6].

Lemma 4.1. *Let X be a Hausdorff uniform space and let $A, B \subset X$ be compact subsets such that $A \cap B = \emptyset$. Then there exists a symmetric entourage Θ of X such that $\Theta(A) \cap \Theta(B) = \emptyset$.*

The following lemma comes from [14, Lemma 4.4].

Lemma 4.2. *Let X and Y be topological spaces, Z a uniform space, $\varphi : X \times Y \to Z$ a continuous map, $K \subset X$ compact, $y \in Y$, and Θ some entourage of Z. Then there exists an open neighborhood V of y such that $\varphi(k, V) \subset \Theta(\varphi(k, y))$ for all $k \in K$.*

Lemma 4.3. *Let (S, X) be an M-system, where X is a compact uniform space. Let $D \subset X$ be an invariant subset of (S, X), and let U be a nonempty open subset of X. Then the set $N(U, \Theta(D))$ is thickly syndetic for every entourage Θ.*

Proof. Let A be a compact subset of S. Note that X is compact. By Lemma 4.2, for any given entourage Θ of X, there is a symmetric entourage Θ_1 of X such that when $a, b \in X$ and $(a, b) \in \Theta_1$, for every $p \in A$ we have

$$(pa, pb) \in \Theta. \tag{4.1}$$

Let Θ_2 be a symmetric entourage such that $\Theta_2 \circ \Theta_2 \subset \Theta_1$. Let U be a nonempty open subset of X. Since (S, X) is transitive, then $N(U, \Theta_2(D)) \neq \emptyset$. Hence there are $x \in U$ and $t \in S$ such that $tx \in \Theta_2(D)$. Choose an open neighborhood $W \subset U$ of x such that for every $c \in W$, we have $(tc, tx) \in \Theta_2$. Let $y \in W$ be a almost periodic point of (S, X). Then the set $N(y, W)$ is syndetic. For every $s \in N(y, W)$, we have $sy \in W$, hence $(tsy, tx) \in \Theta_2$. Since $tx \in \Theta_2(D)$ and $(tsy, tx) \in \Theta_2$, we have

$$tsy \in \Theta_1(D). \tag{4.2}$$

By (4.1) and (4.2), for every $p \in A$ we have $ptsy \in \Theta(pD)$. Since D is invariant, we have $ptsy \in \Theta(pD) \subset \Theta(D)$. Hence $AtN(y, W) \subset N(W, \Theta(D)) \subset N(U, \Theta(D))$. Since $N(y, W)$ is syndetic, by Lemma 2.1 the set $tN(y, W)$ is also syndetic, therefore $N(U, \Theta(D))$ is thickly syndetic. □

Theorem 4.4. *Let (S, X) be an M-system, where X is a compact Hausdorff uniform space. If (S, X) has at least two disjoint compact invariant subsets $A, B \subset X$, then it is thickly syndetic sensitive.*

Proof. By Lemma 4.1, there exists a symmetric entourage Θ of X such that $\Theta(A) \cap \Theta(B) = \emptyset$. Let Θ_1 be a symmetric entourage such that $\Theta_1 \circ \Theta_1 \circ \Theta_1 \subset \Theta$. Let U be a nonempty open set of X. By Lemma 4.3, $N(U, \Theta_1(A))$ and $N(U, \Theta_1(B))$ are thickly syndetic. By Proposition 3.2, the set

$$N(U, \Theta_1(A)) \cap N(U, \Theta_1(B))$$

is thickly syndetic. We next show that

$$N(U, \Theta_1(A)) \cap N(U, \Theta_1(B)) \subset E(U, \Theta_1). \tag{4.3}$$

Let $t \in N(U, \Theta_1(A)) \cap N(U, \Theta_1(B))$. Then there are $x \in U$ and $a \in A$ such that $(tx, a) \in \Theta_1$. And, there are $y \in U$ and $b \in B$ such that $(ty, b) \in \Theta_1$. If $(tx, ty) \in \Theta_1$, then $(a, b) \in \Theta$. This contradicts to $\Theta(A) \cap \Theta(B) = \emptyset$. By (4.3), the set $E(U, \Theta_1)$ is thickly syndetic. □

Proposition 4.5. *Let (S, X) be a non-minimal M-system, where S is a discrete semigroup, and X is a compact Hausdorff uniform space. Then it is thickly syndetic sensitive.*

Proof. By Proposition 2.4, we know that every almost periodic point of the system (S, X) is a minimal point. Since (S, X) is not minimal, there is $x \in X$ such that $\overline{Sx} \neq X$. Choose a minimal point $y \in X \setminus \overline{Sx}$. Then $\overline{Sx}$ and $\overline{Sy}$ are disjoint compact subsets of X. By Theorem 4.4, the proposition holds. □

Proposition 4.6. *Let (S, X) be a P-M-system, where S is a discrete abelian semigroup, and X is a compact Hausdorff uniform space. Then (S, X) is thickly sensitive if and only it is thickly syndetic sensitive.*

Proof. It suffices to prove that thickly sensitive implies thickly syndetic sensitive. By Proposition 2.4, we know that every almost periodic point of the system (S, X) is a minimal point. For any $n \in \mathbb{N}$, let $X^n = X \times X \times \cdots \times X$ (n times). The action S on X^n is defined by $s(x_1, x_2, \ldots, x_n) = (sx_1, sx_2, \ldots, sx_n)$ for all $s \in S$ and $(x_1, x_2, \ldots, x_n) \in X^n$.

Claim For any $n \in \mathbb{N}$, the system (S, X^n) contains a dense set of minimal points.

Let $u \in X$ be a transitive point of (S, X), and let $W_1, W_2, \ldots, W_n$ be nonempty open subsets of X. There is a subset $\{g_1, g_2, \ldots, g_n\}$ of S such that $g_i u \in W_i$ for $i = 1, 2, \ldots, n$. Since (S, X) is a P-M-system, we may choose a minimal point $v \in X$ such that $g_i v \in W_i$ for $i = 1, 2, \ldots, n$. Next we prove that $(g_1 v, g_2 v, \ldots, g_n v)$ is a minimal point of (S, X^n). Let W_i' be an open neighborhood of $g_i v$ for $i = 1, 2, \ldots, n$. Choose an open neighborhood U' of v such that $g_i(U') \subset W_i'$ for $i = 1, 2, \ldots, n$. Then we have

$$N((g_1 v, g_2 v, \ldots, g_n v), W_1' \times W_2' \times \cdots \times W_n') \supset N(v, U').$$

By Proposition 2.4, $(g_1 v, g_2 v, \ldots, g_n v)$ is a minimal point of (S, X^n). Hence, $W_1 \times W_2 \times \cdots \times W_n$ contains the minimal point $(g_1 v, g_2 v, \ldots, g_n v)$.

Let U be a nonempty open subset of X. Let (S, X) be thickly sensitive with a sensitivity constant entourage Θ. Since the set $E(U, \Theta)$ is thick, for any finite set $F = \{s_1, s_2, \ldots, s_n\}$ of S there is $t \in S$ such that $E(U, \Theta) \supset Ft$. Then for every $s_i t \in Ft$, there are $u_i, v_i \in U$ such that $(s_i t u_i, s_i t v_i) \notin \Theta$. We choose nonempty open subsets $U_i, V_i \subset U$ such that $(s_i t x_i, s_i t y_i) \notin \Theta$ for all $x_i \in U_i$ and $y_i \in V_i$, where $i = 1, 2, \ldots, n$. Since (S, X) is a P-M-system, by the claim the system (S, X^{2n}) contains a minimal point

$$(z_1, z_1', z_2, z_2', \ldots, z_n, z_n') \in U_1 \times V_1 \times U_2 \times V_2 \times \cdots \times U_n \times V_n.$$

Obviously, the set $Q = \bigcap_{i=1}^n (N(z_i, U_i) \cap N(z_i', V_i))$ is syndedic. Let $q \in Q$. Then $q z_i \in U_i$ and $q z_i' \in V_i$, it follows that $(s_i t q z_i, s_i t q z_i') \notin \Theta$ where $i = 1, 2, \ldots, n$. Note that $z_i \in U_i \subset U$, and $z_i' \in V_i \subset U$, where $i = 1, 2, \ldots, n$. So $E(U, \Theta) \supset FtQ$. By Lemma 2.1, tQ is syndedic, hence $E(U, \Theta)$ is thickly syndetic. □

Proposition 4.7. *Let (G, X) be a dynamical system, where G is a discrete abelian group, and X is a uniform space. If (G, X) is multi-sensitive, then (G, X) is thickly sensitive.*

Proof. Assume that (G, X) is multi-sensitive with a sensitivity constant entourage Θ. Take a nonempty open set $U \subset X$. Let $F = \{g_1, g_2, \ldots, g_n\}$ be a finite set of G. For every $g_i \in G$, choose a nonempty open set $U_i \subset X$ such that $U_i \subset g_i U$. By the assumption of Θ we may take $g \in \bigcap_{i=1}^n E(U_i, \Theta) \subset \bigcap_{i=1}^n E(g_i U, \Theta)$. Then for every $i \in \{1, 2, \ldots, n\}$, there are $x_i, y_i \in g_i U$ such that $(gx_i, gy_i) \notin \Theta$. Hence,

$$(gx_i, gy_i) = (gg_i^{-1} g_i x_i, gg_i^{-1} g_i y_i) = (g_i g g_i^{-1} x_i, g_i g g_i^{-1} y_i) \notin \Theta.$$

Note that $g_i^{-1} x_i, g_i^{-1} y_i \in U$. Therefore $E(U, \Theta) \supset Fg$, it follows that the set $E(U, \Theta)$ is thick. □

Proposition 4.8. *Let (S, X) be a point transitive system, where S is an abelian semigroup, and X is a uniform space. If (S, X) is thickly sensitive, then (S, X) is multi-sensitive.*

Proof. Assume that (G, X) is thickly sensitive with a sensitivity constant entourage Θ. Take $n \in \mathbb{N}$. Let $U_1, U_2, \ldots, U_n$ be nonempty open subsets of X. Let $z \in X$ be a transitive point of (S, X). Then there is $s_i \in S$ such that $s_i z \in U_i$ where $i = 1, 2, \ldots, n$. Choose a nonempty open subset $U \subset X$ such that $s_i U \subset U_i$ for all $i \in \{1, 2, \ldots, n\}$. Since (S, X) is thickly sensitive, for any finite $F = \{s_1, s_2, \ldots, s_n\} \subset S$ there is $t \in S$ such that $E(U, \Theta) \supset Ft$. Hence, for every $s_i \in F$ there are $x_i, y_i \in U$ such that $(s_i t x_i, s_i t y_i) = (t s_i x_i, t s_i y_i) \notin \Theta$. This implies that $t \in \bigcap_{i=1}^n E(U_i, \Theta)$, so (S, X) is multi-sensitive. □

By Proposition 4.6, Proposition 4.7 and Proposition 4.8, the following result holds.

Corollary 4.9. *Thickly sensitive, thickly syndetic sensitive, and multi-sensitive are all equivalent properties for a P-M-action of a discrete abelian group on compact Hausdorff uniform space.*

We say that a nonempty collection $\mathcal{F}$ of subsets of S is a *filter base* on S if for any $F_1, F_2 \in \mathcal{F}$ there exists $F \in \mathcal{F}$ such that $F \subset F_1 \cap F_2$, where $\emptyset \notin \mathcal{F}$. It is easy to see that the following result holds.

Lemma 4.10. *Let (S, X) be a dynamical system, where S is an abelian semigroup. Then (S, X) is weakly mixing if and only if $\mathscr{P} = \{N(U, V) : U$ and V are nonempty open subsets of $X\}$ is a filter base on S.*

Proof. Assume that $\mathscr{P}$ is a filter base on S. For any nonempty open subsets U_1, U_2, V_1, V_2 of X, there are nonempty open subsets W_1, W_2 of X such that

$$N(U_1 \times U_2, V_1 \times V_2) = N(U_1, V_1) \cap N(U_2, V_2) \supset N(W_1, W_2) \neq \emptyset.$$

Then $N(U_1 \times U_2, V_1 \times V_2) \neq \emptyset$, so (S, X) is weakly mixing.

Conversely, assume that (S, X) is weakly mixing. Let $N(U_1, V_1), N(U_2, V_2) \in \mathscr{P}$. Then there is $s \in N(U_1, U_2) \cap N(V_1, V_2)$. Let $A = U_1 \cap s^{-1} U_2$, $B =$

$V_1 \cap s^{-1}V_2$. For any $s_0 \in N(A,B)$, we have

$\emptyset \neq A \cap s_0^{-1}B = (U_1 \cap s^{-1}U_2) \cap s_0^{-1}(V_1 \cap s^{-1}V_2) = U_1 \cap s_0^{-1}V_1 \cap s^{-1}(U_2 \cap s_0^{-1}V_2)$.

This implies that

$$U_1 \cap s_0^{-1}V_1 \neq \emptyset, \quad U_2 \cap s_0^{-1}V_2 \neq \emptyset.$$

Then $N(A,B) \subset N(U_1,V_1) \cap N(U_2,V_2)$, hence $\mathscr{P}$ is a filter base. □

Proposition 4.11. *Let (S,X) be a dynamical system, where S is an abelian semigroup, and X is a Hausdorff uniform space. If (S,X) is weakly mixing, then (S,X) is multi-sensitive.*

Proof. Choose $x,y \in X$ such that $x \neq y$. By Lemma 4.1, there exists a symmetric entourage Θ of X such that $\Theta(x) \cap \Theta(y) = \emptyset$. Let Θ_1 be a symmetric entourage such that $\Theta_1 \circ \Theta_1 \circ \Theta_1 \subset \Theta$. Take $n \in \mathbb{N}$. Let $U_1, U_2, \ldots, U_n$ be nonempty open subsets of X. By Lemma 4.10,

$$\bigcap_{i=1}^{n} (N(U_i, \Theta_1(x)) \cap N(U_i, \Theta_1(y))) \neq \emptyset.$$

That is, there are $t \in S$, and $x_i, y_i \in U_i$ such that $tx_i \in \Theta_1(x)$, $ty_i \in \Theta_1(y)$, where $i = 1,2,\ldots,n$. This implies that $(tx_i, ty_i) \notin \Theta_1$ for $i = 1,2,\ldots,n$, so $t \in \bigcap_{i=1}^{n} E(U_i, \Theta_1)$. □

By Proposition 4.7 and Proposition 4.11, the following result holds.

Corollary 4.12. *Let (G,X) be a dynamical system, where G is a discrete abelian group, and X is a Hausdorff uniform space. If (G,X) is weakly mixing, then (G,X) is multi-sensitive and thickly sensitive.*

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