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APPROXIMATE AND CHARACTER AMENABILITY OF VECTOR-VALUED LIPSCHITZ ALGEBRAS

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Abstract. For a metric space (X, d) and $\alpha > 0$. We study the structure and properties of vector-valued Lipschitz algebra $Lip_\alpha(X, E)$ and $lip_\alpha(X, E)$ of order α. We investigate the approximate and Character amenability of vector-valued Lipschitz algebras.

1. Introduction and preliminaries

Let (X, d) be a metric space and $B(X)$ (resp. $C_b(X)$) indicates the Banach space consisting of all bounded complex valued functions on X, endowed with the norm

$$\|f\|_{\sup} = \sup_{x\in X} |f(x)| \qquad (f \in B(X)).$$

Take $\alpha \in \mathbb{R}$ with $\alpha > 0$. Then $Lip_\alpha X$ is the subspace of $B(X)$, consisting of all bounded complex-valued functions f on X such that

$$p_\alpha(f) := \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X, \ \ x \neq y\right\} < \infty.$$

It is known that $Lip_\alpha X$ endowed with the norm $\|\cdot\|_\alpha$ given by

$$\|f\|_\alpha = p_\alpha(f) + \|f\|_{\sup};$$

and pointwise product is a unital commutative Banach algebra, called Lipschitz algebra.

Let (X, d) be a metric space with at least two elements and $(E, \|\cdot\|)$ be a Banach space over the scalar field $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$ for a constant $\alpha > 0$ and a function $f : X \longrightarrow E$, set

$$p_{\alpha,E}(f) := \sup_{x\neq y} \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha},$$

which is called the Lipschitz constant of f. Define

$$Lip_\alpha(X, E) = \{f : X \longrightarrow E : f \text{ is bounded and } p_\alpha(f) < \infty\}.$$

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The Lipschitz algebra $lip_\alpha(X,E)$ is the subalgebra of $Lip_\alpha(X,E)$ defined by

$$lip_\alpha(X,E) = \{f : X \longrightarrow E : \frac{\|f(x)-f(y)\|}{d(x,y)^\alpha} \longrightarrow 0 \text{ as } d(x,y) \longrightarrow 0\}.$$

Finally, if X is a locally compact metric space, then $lip^0_\alpha(X,E)$ is the subalgebra of $lip_\alpha(X,E)$ consisting of those functions whose norm tend to zero at infinity. The elements of $Lip_\alpha(X,E)$ and $lip_\alpha(X,E)$ are called big and little Lipschitz operators and for each element f of $Lip_\alpha(X,E)$, define

$$\|f\|_{\alpha,E} = p_{\alpha,E}(f) + \|f\|_{\infty,E}.$$

Let $C_b(X,E)$ be the set of all bounded continuous functions from X into E and for each $f \in C_b(X,E)$, define

$$\|f\|_{\infty,E} = \sup_{x\in X} \|f(x)\|,$$

and for $f, g \in C_b(X,E)$ and $\lambda \in \mathbb{F}$, define

$$(f+g)(x) = f(x) + g(x),\ (\lambda f)(x) = \lambda f(x),\ (x \in X).$$

It is easy to see that $(C_b(X,E), \|\cdot\|_{\infty,E})$ becomes a Banach space over $\mathbb{F}$ and $Lip_\alpha(X,E)$ is a linear subspace of $C_b(X,E)$ [6]. In their papers [6, 7] Cao, Zhang and Xu proved that $(Lip_\alpha(X,E), \|\cdot\|_{\alpha,E})$ is a Banach space over $\mathbb{F}$ and $lip_\alpha(X,E)$ is a closed linear subspace of $(Lip_\alpha(X,E), \|\cdot\|_{\alpha,E})$. If E is a Banach algebra, then $(Lip_\alpha(X,E), \|\cdot\|_{\alpha,E})$ is a Banach algebra under pointwise and scalar multiplication and $lip_\alpha(X,E)$ is a closed linear subalgebra of $(Lip_\alpha(X,E), \|\cdot\|_{\alpha,E})$. The spaces $Lip_\alpha(X,E)$ and $lip_\alpha(X,E)$ are called big and little Lipschitz operators algebra. It is clear that the Lipschitz algebras $Lip_\alpha(X,E)$ contains the space $Cons(X,E)$ consisting of all constant E-valued functions on X. The Lipschitz algebras were first considered by Sherbert [22]; see also Bishop [5]. There are valuable works related to some notions of amenability of Lipschitz algebras. Gourdeau [11] discussed amenability of vector-valued Lipschitz algebras. Also he proved that if a Banach algebra A is amenable, then $\Delta(A)$ is uniformly discrete with respect to norm topology induced by A^*; see also Bade, Curtis, and Dales [3], Gourdeau [10] and Zhang [23]. Moreover Hu, Monfared and Traynor investigated character amenability of Lipschitz algebras, see [13]. They showed that if X is an infinite compact metric space and $0 < \alpha < 1$, then $Lip_\alpha X$ is not character amenable. Moreover, recently C-character amenability of Lipschitz algebras was studied by Dashti, Nasr Isfahani and Soltani for each $\alpha > 0$ see [9]. In fact as a generalization of [20], they showed that for $\alpha > 0$ and any locally compact metric space X, the algebra $Lip_\alpha X$ is C-character amenable, for some $C > 0$, if and only if X is uniformly discrete. Also a necessary and sufficient condition for amenability and character amenability of Lipschitz algebras was provided; see [9]. Sherbert [22], Honary, Nikou and Sanatpour [12] and Bade [3], studied some properties of Lipschitz algebras.

In this paper, we study some structure and properties of vector-valued Lipschitz algebra $Lip_\alpha(X,E), lip_\alpha(X,E)$ and $lip^0_\alpha(X,E)$ of order α. Furthermore we obtain some necessary and sufficient conditions for amenability, C-character amenability and approximately amenability of vector-valued Lipschitz operators algebras.

2. Preliminaries and some basic results

Let (X,d) be a metric space and $\alpha > 0$. It is easy to show that $Lip_\alpha(X,E)$, $lip_\alpha(X,E)$ and $lip^0_\alpha(X,E)$ is a vector space, Banach space and Banach algebra, whenever E is so, respectively. Also it is easy to see that if (X,d) is a metric space and E be a Banach algebra. Then $Lip_\alpha(X,E)$ is a commutative (unital) Banach algebra if and only if E is a commutative (unital) Banach algebra. Let E be a $*$-Banach algebra, $f^*(x) = (f(x))^*$ for $x \in X$ and $f \in Lip_\alpha(X,E)$. Then $p_\alpha(f^*) = p_\alpha(f)$ and $\|f^*\|_{\infty,E} = \|f\|_{\infty,E}$, so $Lip_\alpha(X,E)$ is a $*$-Banach algebra. We first bring some preliminary result and definitions. The first one, which appeared in [12] for compact metric space and $0 < \alpha \leqslant 1$. We generalized for arbitrary metric space and for each $\alpha > 0$.

Lemma 2.1. *Let (X,d) be a metric space, $\alpha > 0$ and E be a Banach algebra. Then the following statements are equivalent.*

(i) *$f \in Lip_\alpha(X,E)$.*
(ii) *$\sigma \circ f \in Lip_\alpha X$ for all $\sigma \in E^*$.*

Proof. Suppose that $f \in Lip_\alpha(X,E)$ and $\sigma \in E^*$. Then

$$|\sigma \circ f(x) - \sigma \circ f(y)| \leqslant \|\sigma\|\|f(x) - f(y)\| \leqslant \|\sigma\| p_{\alpha,E}(f) d(x,y)^\alpha,\ (x,y \in X).$$

Hence $p_\alpha(\sigma \circ f) \leqslant \|\sigma\| p_{\alpha,E}(f) < \infty$ and $\|\sigma \circ f\|_\infty \leqslant \|\sigma\|\|f\|_{\infty,E} < \infty$. Thus $\sigma \circ f \in Lip_\alpha X$. Conversely, suppose that $T_\sigma : Lip_\alpha(X,E) \longrightarrow Lip_\alpha X$, where $f \mapsto \sigma \circ f$ for all $\sigma \in E^*$ and $f \in Lip_\alpha(X,E)$. Then $\{T_\sigma\}_{\sigma \in E^*}$ is a family of continuous linear functions such that

$$\sup_{\|\sigma\| \leqslant 1} \|T_\sigma(f)\|_\alpha \leqslant \|f\|_{\alpha,E} < \infty.$$

So by the Principle of Uniform Boundedness, we have $\sup_{\|\sigma\| \leqslant 1} \|T_\sigma\|_\alpha < M$ for some $M > 0$. So $\|\sigma \circ f\|_\alpha \leqslant M$. Hence, $p_\alpha(\sigma \circ f) \leqslant M$ and $\|\sigma \circ f\|_\infty \leqslant M$ for all $f \in Lip_\alpha(X,E)$. Thus,

$$\|f(x) - f(y)\| = \sup_{\|\sigma\| \leqslant 1} \{|\sigma(f(x) - f(y))| \leqslant M d(x,y)^\alpha.$$

Also, $\|f(x)\| = \sup_{\|\sigma\| \leqslant 1} |\sigma(f(x))| \leqslant M$ and $f \in Lip_\alpha(X,E)$. □

Remark 2.2. Let (X,d) be a metric space, $\alpha > 0$ and E be a Banach algebra. Then,

$$\|f\|_{\alpha,E} = \sup\{\|\sigma \circ f\|_\alpha : \sigma \in E^*,\ \|\sigma\| \leqslant 1\}.$$

Corollary 2.3. *Let (X,d) be a metric space, $0 < \alpha \leqslant \beta$ and E be a Banach algebra. Then*

(i) $Lip_\beta(X,E) \subseteq Lip_\alpha(X,E)$.
(ii) $lip_\beta(X,E) \subset Lip_\beta(X,E) \subset lip_\alpha(X,E) \subset Lip_\alpha(X,E)$.

Proof. Suppose that $f \in Lip_\beta(X,E)$, then $\sigma \circ f \in Lip_\beta X$ and since $Lip_\beta X \subseteq Lip_\alpha X$ and so $\sigma \circ f \in Lip_\beta X$, then by Lemma 2.1 $f \in Lip_\alpha(X,E)$ for each $\sigma \in E^*$.

Suppose that $f \in Lip_\beta(X,E)$, then for all $\sigma \in E^*$ we have $\sigma \circ f \in Lip_\beta X \subset lip_\alpha X$. Hence $f \in lip_\alpha(X,E)$. □

In this section we study the structure of Lipschitz algebra $Lip_\alpha(X,E)$. The following examples show that the algebraic and topological properties of $Lip_\alpha(X,E)$ depend the metric space (X,d) order α and the structure of E. By using Lemma 2.1, for Banach space, $C_0(Y)$, Hilbert space H and $L_p(Y,\mu)$ for $1 \leqslant p < \infty$ and σ-finite measure μ. The following is immediate.

Example 2.4. Let (X,d) be a metric space and $\alpha > 0$.

(1) Let Y be a locally compact Hausdorff space and $E = C_0(Y)$, $E^* = M_b(Y)$, the space of all complex-valued bounded, regular measures of Y. Thus $f \in Lip_\alpha(X,E)$ if and only if for each $\mu \in M_b(Y)$ there exists $M > 0$ such that

$$|\int_Y (f(a) - f(b))(y)d\mu(y)| \leqslant Md(a,b)^\alpha, \ (a,b \in X).$$

(2) Let H be a Hilbert space and $E = H$. Then $E^* = E$, so $f \in Lip_\alpha(X,E)$ if and only if for each $h \in H$, there exist $M > 0$ and $k \in H$ such that

$$|\langle f(x) - f(y), k\rangle| \leqslant Md(x,y)^\alpha, \ (x,y \in X).$$

(3) Let $E = L_p(Y,\mu)$. Then $E^* = L_q(Y,\mu)$ for which $\frac{1}{p} + \frac{1}{q} = 1$. Thus $f \in Lip_\alpha(X,E)$ if and only if for each $g \in L_q(Y,\mu)$ there exists $M > 0$ such that

$$|\int_Y (f(a) - f(b))(y)g(y)d\mu(y)| \leqslant Md(a,b)^\alpha, \ (a,b \in X).$$

(4) Let G be a locally compact Hausdorff topological group and $E = (L^1(G), *)$. Then $E^* = L^\infty(G)$, so $f \in Lip_\alpha(X,E)$ if and only if $g \in L^\infty(G)$ there exist $g \in L^\infty(G)$ and $M > 0$ such that

$$|\int_G g(x)(f(a)(x) - f(b)(x))d\lambda(x)| \leqslant Md(a,b)^\alpha, \ (a,b \in G).$$

Where λ is the left Haar measure of G.

(5) Let G be a locally compact Hausdorff topological group and $E = A(G)$, the Fourier algebra of G. Then $E^* = VN(G)$, the Von-Neumann algebra of G. Thus $f \in Lip_\alpha(X,E)$ if and only if $\sigma \in VN(G)$ there exists $M > 0$ such that

$$|\sigma(f(a)) - \sigma(f(b))| \leqslant Md(x,y)^\alpha, \ (a,b \in G).$$

Let $\sigma \in VN(G)$, $g = g_1 * \tilde{g_2} \in A(G)$ for $g_1, g_2 \in L^2(G)$. Then

$$\langle \sigma, g_1 * \tilde{g_2}\rangle := \langle \sigma(g_2), g_1\rangle = \int_G \sigma(g_2)\overline{g_1(x)}d\lambda(x).$$

Example 2.5. Let (X,d) be a normed space, $\alpha > 1$ and E be a Banach algebra. Then $Lip_\alpha(X,E) = cons(X,E)$.

Proof. Suppose that $f \in Lip_\alpha(X,E)$ so by Lemma 2.1 $\sigma \circ f \in Lip_\alpha X$ for all $\sigma \in E^*$ and for $a, b \in X$, define

$$F(t) := \sigma \circ f(ta + (1-t)b),\ t \in \mathbb{R}.$$

Then, $F : \mathbb{R} \longrightarrow \mathbb{C}$ such that there is $M > 0$ such that

$$\begin{aligned}|F(t_1) - F(t_2)| &= |\sigma \circ f(t_1 a + (1-t_1)b) - \sigma \circ f(t_2 a + (1-t_2)b)| \\ &\leqslant M\|t_1 a + (1-t_1)b - t_2 a - (1-t_2)b\|^\alpha \\ &\leqslant M|t_1 - t_2|^\alpha \|a-b\|^\alpha.\end{aligned}$$

So $F \in Lip_\alpha(\mathbb{R})$. But $Lip_\alpha(\mathbb{R}) = cons(\mathbb{R})$. Hence $F(1) = F(0)$, so for all $\sigma \in E^*$ $\sigma \circ f(a) = \sigma \circ f(b)$. Thus f is a constant. □

Recall that (X,d) is called uniformly discrete if there exists $\varepsilon > 0$ such that $d(x,y) \geq \varepsilon$ for all $x, y \in X$ with $x \neq y$.

Example 2.6. If (X,d) is not uniformly discrete, and $0 < \alpha \leqslant 1$, then $cons(X,E) \subset Lip_\alpha(X,E) \subset l_\infty(X,E)$.

Proof. Since (X,d) is not uniformly discrete, then $Lip_\alpha(X,E) \subset l_\infty(X,E)$. Also, define

$$f_\alpha(x) = \frac{d^\alpha(x,x_1)}{d^\alpha(x,x_1) + d^\alpha(x,x_2)},\ (x_1, x_2 \in X,\ x_1 \neq x_2).$$

Then $f_\alpha \in Lip_\alpha(X,E) \setminus cons(X,E)$. □

Let (X,d) be a compact metric space and $0 < \alpha \leqslant 1$ and E be a Banach algebra. Then $\Delta(C(X,E)) = \{\Delta_{x,\sigma} : x \in X,\ \sigma \in \Delta(E)$, where

$$\Delta_{x,\sigma}(f) = \sigma(f(x)),\ (f \in Lip_\beta(X,E),\ x \in X).$$

Define $\varphi : X \times \Delta(E) \to \Delta(C(X,E))$ where $(x,\sigma) \to \Delta_{x,\sigma}$. Then φ is a bijection and we denote, $\Delta(C(X,E)) = X \times \Delta(E)$.

Let $(A, \|\cdot\|_A)$ in $(B, \|\cdot\|_B)$ be a Banach algebras such that $B \subset A$ and B in A be dense and the inclusion map $i : B \longrightarrow A$ is continuous. Then, $\Delta(B) = \{\varphi|_B : \varphi \in \Delta(A)\}$. In particular, let (X,d) be a compact metric space and $0 < \alpha \leqslant 1$ and E be a Banach algebra. Then $Lip_\alpha(X,E)$ and $lip_\alpha(X,E)$ are dense in $C(X,E)$ and for all $f \in Lip_\alpha(X,E)$, $\|f\|_{\infty,E} \leqslant \|f\|_{\alpha,E}$, see [4]. Thus, $lip_\alpha(X,E)$ and $Lip_\alpha(X,E)$ are Segal algebras in $C(X,E)$. Thus,

$$\Delta(lip_\alpha(X,E)) = \{\varphi|_{lip_\alpha(X,E)} : \varphi \in X \times \Delta(E)\} = \{\Delta^l_{x,\sigma} : x \in X,\ \sigma \in \Delta(E)\}.$$

Also

$$\Delta(Lip_\alpha(X,E)) = \{\varphi|_{Lip_\alpha(X,E)} : \varphi \in X \times \Delta(E)\} = \{\Delta^L_{x,\sigma} : x \in X,\ \sigma \in \Delta(E)\}.$$

Let A be a commutative Banach algebra. Then the radical of A, denoted by $Rad(A)$, is defined by

$$Rad(A) = \cap_{\varphi \in \Delta(A)} \ker \varphi.$$

Clearly, $Rad(A)$ is a closed ideal of A. Also A is called semisimple if

$$Rad(A) = \{0\}.$$

Theorem 2.7. *Let (X, d) be a metric space, E be a commutative Banach algebra and $0 < \alpha \leqslant 1$. Then the following statements are equivalent.*

(i) *$C_b(X, E)$ is semisimple.*
(ii) *$Lip_\alpha(X, E)$ is semisimple.*
(iii) *$lip_\alpha(X, E)$ is semisimple.*
(iv) *E is semisimple.*

Proof. (iv)$\Longrightarrow$(i) Let $x \in X$ and $\theta_x : C_b(X, E) \to E$, define by $\theta_x(f) = f(x)$. Then θ_x is a linear, continuous and epimorphism. Thus,

$$\theta_x(Rad(C_b(X, E))) \subseteq Rad(E) = \{0\}.$$

Thus

$$Rad(C_b(X, E)) \subseteq \ker(\theta_x) = \{f : f(x) = 0\}.$$

Hence $Rad(C_b(X, E)) \subseteq \cap_{x\in X} \ker(\theta_x) = \{0\}$. So $C_b(X, E)$ is semisimple.

(i)$\Longrightarrow$(iv) Let $\varphi : E \to C_b(X, E)$ define by $\varphi(z) = \varphi_z$ where $\varphi_z(x) = z$ for $x \in X$. Then φ is a linear, isometric and homomorphism. Hence

$$\varphi(Rad(E)) \subseteq Rad(C_b(X, E)) = \{0\}.$$

But φ is one to one, so $Rad(E) = \{0\}$.

(ii)$\Longrightarrow$(iv) Let $\varphi : E \to Lip_\alpha(X, E)$ define by $\varphi(z) = f_z$ where $f_z(x) = z$ for $x \in X$. Thus $\|f_z\|_{\alpha,E} = \|z\| = \|f_z\|_{\infty,E}$ for each $z \in E$. so

$$\varphi(Rad(E)) \subseteq Rad(Lip_\alpha(X, E)) = \{0\}.$$

Then $Rad(E) = \{0\}$.

(iv)$\Longrightarrow$(ii) Let $\sigma \in \Delta(E)$ and $\varphi_\sigma : Lip_\alpha(X, E) \longrightarrow Lip_\alpha X$ define by $\varphi_\sigma(f) = \sigma \circ f$. Then φ_σ is a linear, continuous and epimorphism. Thus,

$$\varphi_\sigma(Rad(Lip_\alpha(X, E)) \subseteq Rad(Lip_\alpha X) \subseteq \cap_{x\in X}\delta_x = \{0\},$$

where $\delta_x(g) = g(x)$ for $g \in Lip_\alpha X$. Hence

$$\begin{aligned} Rad(Lip_\alpha(X, E)) \subseteq \cap_{\sigma\in\Delta(E)} \ker \varphi_\sigma &= \{f : \sigma \circ f(x) = 0,\ \sigma \in \Delta(E),\ x \in X\} \\ &= \{f : f(x) \in \cap_{\sigma\in\Delta(E)} \ker \sigma,\ x \in X\} \\ &= \{f : f(x) \in Rad(E),\ x \in X\} = \{0\}. \end{aligned}$$

(i)$\Longrightarrow$(iv) Let $\varphi : E \to Lip_\alpha(X, E)$ define by $\varphi(z) = f_z$, where $f_z(x) = z$ for $x \in X$. Then $\|f_z\|_{\alpha,E} = \|z\| = \|f\|_{\infty,E}$ for each $z \in E$. Thus φ is well-defined. Also;

$$\varphi(Rad(E)) \subseteq Rad(Lip_\alpha(X, E)) = \{0\},$$

and φ is $1 - 1$, so $Rad(E) = \{0\}$.

(iii)$\Longleftrightarrow$(iv) is similar to (ii)$\Longleftrightarrow$(iv). □

Suppose that $x, y \in X$ and $x \neq y$. Let $f(w) = \min\{1, d(w, y)^\alpha\}$. Then

$$f(x) \neq f(y).$$

Let $0 \neq z \in E$ and $g(x) = f(x)z$. Then $g(x) \in E$ and $g(x) \neq 0$, $g(y) = 0$. Hence,

$$p_\alpha(g) = \sup_{a \neq b} \frac{\|f(a)z - f(b)z\|}{d(a,b)^\alpha} = \|z\| p_\alpha(f) < \infty,$$

so $g \in Lip_\alpha(X, E)$. Thus $Lip_\alpha(X, E)$ is a $*$-Banach algebra such that separates the points of X.

Lemma 2.8. *Let (X, d) be a metric space and E be a Banach algebra. Then the followings holds.*

(i) $E^* \circ B(X, E) = B(X)$.
(ii) $E^* \circ Lip_\alpha(X, E) = Lip_\alpha X$.
(iii) $E^* \circ lip_\alpha(X, E) = lip_\alpha X$.
(iv) *If X is a locally compact metric space, then* $E^* \circ lip^0_\alpha(X, E) = lip^0_\alpha X$.
(v) $E^* \circ l^\infty(X, E) = l^\infty(X)$.
(vi) $E^* \circ C_b(X, E) = C_b(X)$.

Proof. i) Suppose that $\sigma \in E^*$ and $f \in B(X, E)$. Then

$$\|\sigma \circ f(x)\| \leqslant \|\sigma\| \|f(x)\| \leqslant \|\sigma\| \|f\|_\infty, \ (x \in X).$$

Thus $\sigma \circ f \in B(X)$.

ii) Let $g \in Lip_\alpha X$, $0 \neq z \in E$. Then there exists $\sigma \in E^*$ such that $\sigma(z) = 1$. Put $f(x) := g(x)z$ for $x \in X$. Then $f \in Lip_\alpha(X, E)$ and $g = \sigma \circ f$.

iii) Suppose that $\sigma \in E^*$ and $f \in lip_\alpha(X, E)$. Then

$$\frac{|\sigma \circ f(x) - \sigma \circ f(y)|}{d(x,y)^\alpha} \leqslant \frac{\|\sigma\| \|f(x) - f(y)\|}{d(x,y)^\alpha} \longrightarrow 0.$$

Therefore $\sigma \circ f \in lip_\alpha X$. Conversely, if $g \in lip_\alpha X$, $0 \neq \sigma \in E^*$ and $0 \neq z \in E$. Define $f : X \longrightarrow E$ with $f(x) = g(x)z$. Then $f \in lip_\alpha(X, E)$ such that $g = \sigma \circ f$. Hence $lip_\alpha X \subseteq E^* \circ lip_\alpha(X, E)$.

iv) Suppose that $\sigma \in E^*$ and $f \in lip^0_\alpha(X, E)$ so, $f \in lip_\alpha(X, E)$ and $f \in C_0(X, E)$. Hence $\sigma \circ f \in lip_\alpha X \cap C_0(X) = lip^0_\alpha X$. Conversely, if $g \in lip^0_\alpha X$ and $0 \neq z \in E$. Then there exists $0 \neq \sigma \in E^*$ such that $\sigma(z) = 1$. Define $f(x) := g(x)z$. Then $f \in lip^0_\alpha(X, E)$ and $\|f(x)\| = \|g(x)z\| = |g(x)| \|z\| < \varepsilon$, $x \in X \setminus K$ for some compact space $K \subseteq X$. Hence $f \in C_0(X, E) \cap lip_\alpha(X, E) = lip^0_\alpha(X, E)$. These complete the proof of lemma. □

Corollary 2.9. *Let (X, d) be a metric space, $0 < \alpha \leqslant 1$ and E be a Banach algebra. Then the following statements are equivalent.*

(i) $Lip_\alpha(X, E) = Cons(X, E)$.
(ii) $Lip_\alpha X = Cons(X)$.

Proof. (i)$\Longrightarrow$(ii) By Lemma 2.8 we have

$$Lip_\alpha X = E^* \circ Lip_\alpha(X,E) = E^* \circ Cons(X,E) = Cons(X).$$

(ii)$\Longrightarrow$(i) Let $f \in Lip_\alpha(X,E)$ and $\sigma \in E^*$. Then by Lemma 2.1, $\sigma \circ f \in Lip_\alpha X = Cons(X)$, so $f \in Cons(X,E)$. □

Theorem 2.10. *Let (X,d) be a metric space, $\alpha > 0$ and E be a Banach algebra. Then the following statements are equivalent.*

(i) *$Lip_\alpha(X,E) = B(X,E)$ with equivalent norms.*
(ii) *$Lip_\alpha X = B(X)$ with equivalent norms.*
(iii) *(X,d) is uniformly discrete.*
(iv) *$lip_\alpha(X,E) = B(X,E)$ with equivalent norms.*
(v) *$lip_\alpha X = B(X)$ with equivalent norms.*

Proof. (i)$\Longrightarrow$(ii) This is clear by Lemma 2.8,

$$Lip_\alpha X = E^* \circ Lip_\alpha(X,E) = E^* \circ B(X,E) = B(X).$$

(ii)$\Longrightarrow$(iii) By using [15] and [1, Proposition 1.1] is immediate.

(iii)$\Longrightarrow$(i) Suppose that (X,d) is uniformly discrete. Thus there exists $\varepsilon > 0$ such that for all $x, y \in X$ with $x \neq y$ we have

$$d(x,y) \geq \varepsilon.$$

Suppose that $f \in B(X,E)$. We have

$$p_\alpha(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)^\alpha} \leq \frac{1}{\varepsilon^\alpha} \sup_{x \neq y} \|f(x) - f(y)\| \leq \frac{2}{\varepsilon^\alpha} \|f\|_\infty < \infty.$$

It follows that $f \in Lip_\alpha(X,E)$. Moreover,

$$\|f\|_\infty \leq \|f\|_\alpha \leq (1 + \frac{2}{\varepsilon^\alpha})\|f\|_\infty,$$

and consequently $B(X,E) = Lip_\alpha(X,E)$ with equivalent norms. By a similar argument an above one can show that the statements (iii), (iv) and (v) are equivalent. □

Theorem 2.11. *Let (X,d) be a metric space, E be a Banach algebra and $\alpha, \beta > 0$ and $\Delta(E) \neq \emptyset$. Then the following statements are equivalent.*

(i) *$Lip_\alpha(X,E) = Lip_\beta(X,E)$ with equivalent norms.*
(ii) *$Lip_\alpha X = Lip_\beta X$ with equivalent norms.*
(iii) *(X,d) is uniformly discrete or $\alpha = \beta$.*
(iv) *$lip_\alpha(X,E) = lip_\beta(X,E)$ with equivalent norms.*
(v) *$lip_\alpha X = lip_\beta X$ with equivalent norms.*

Proof. (i)$\Longrightarrow$(ii) This is obtained by Lemma 2.1. We have

$$Lip_\alpha X = E^* \circ Lip_\alpha(X,E) = E^* \circ Lip_\beta(X,E) = Lip_\beta X.$$

(ii)$\Longleftrightarrow$(iii). This is clear by [2, Lemma 1.5].

(iii)$\Longrightarrow$(i) This is clear, since if (X,d) is uniformly discrete, then by Lemma 2.8 we have $Lip_\alpha(X,E) = B(X,E) = Lip_\beta(X,E)$.

By a similar argument is used in above, one can show that the statements (iii), (iv) and (v) are equivalent. □

Theorem 2.12. *Let (X,d) be a metric space, E be a Banach algebra, where $\Delta(E) \neq \emptyset$ and $\alpha, \beta > 0$. Then the following statements are equivalent.*

(i) *$Lip_\alpha(X,E)$ is a $Lip_\beta(X,E)$-module Banach.*
(ii) *$Lip_\alpha X$ is a $Lip_\beta X$-module Banach.*
(iii) *$lip_\alpha(X,E)$ is a $lip_\beta(X,E)$-module Banach.*
(iv) *$lip_\alpha X$ is a $lip_\beta X$-module Banach.*
(v) *(X,d) is uniformly discrete or $\alpha \leqslant \beta$.*

Proof. (i)$\Longrightarrow$(ii) Suppose that $Lip_\alpha(X,E)$ is a $Lip_\beta(X,E)$-module Banach. Then

$$Lip_\alpha(X,E)Lip_\beta(X,E) \subset Lip_\alpha(X,E),$$

and

$$Lip_\beta(X,E)Lip_\alpha(X,E) \subset Lip_\alpha(X,E).$$

Let $\sigma \in \Delta(E)$ and $z \in E$ such that $\sigma(z) = 1$, $g \in Lip_\beta X$ and $f \in Lip_\alpha X$. Put $\bar{f}(x) := f(x)z$ and $\bar{g}(x) := g(x)z$. Then $\bar{f} \in Lip_\alpha(X,E)$ and $\bar{g} \in Lip_\beta(X,E)$ and so

$$\bar{f}.\bar{g} \in Lip_\alpha(X,E) \text{ and } \bar{g}.\bar{f} \in Lip_\alpha(X,E).$$

Hence,

$$\sigma \circ \bar{f}.\bar{g} \in Lip_\alpha X \text{ and } \sigma \circ \bar{g}.\bar{f} \in Lip_\alpha X.$$

So, for all $x \in X$ we have

$$\sigma \circ \bar{f}.\bar{g}(x) = \sigma(\bar{f}(x).\bar{g}(x)) = \sigma(f(x)z.g(x)z) = f(x)\sigma(z)g(x)\sigma(z) = f.g(x).$$

Therefore $f.g = \sigma \circ \bar{f}.\bar{g} \in Lip_\alpha X$. Similarly, $g.f = \sigma \circ \bar{g}.\bar{f} \in Lip_\alpha X$. Let $f \in Lip_\alpha X$ and $g_n \longrightarrow g$ in $Lip_\beta X$. Then

$$\|f.g_n - f.g\|_\alpha = \|\sigma \circ \bar{f}.\bar{g_n} - \sigma \circ \bar{f}.\bar{g}\|_\alpha \leqslant \|\sigma\|\|\bar{f}.\bar{g_n} - \bar{f}.\bar{g}\|_{\alpha,E} \longrightarrow 0.$$

Thus $Lip_\alpha X$ is a $Lip_\beta X$-module Banach.

The implication, (iii)$\Longrightarrow$(iv) is similar to (i)$\Longrightarrow$(ii).

By using [2, Proposition 1.6] the implication (ii)$\leftrightarrow$(v) and (iv)$\Longrightarrow$(v) is immediate.

(v)$\Longrightarrow$(i) Suppose that (X,d) is uniformly discrete. Then

$$Lip_\alpha(X,E) = B(X,E) = Lip_\beta(X,E),$$

since

$$B(X,E)B(X,E) \subset B(X,E).$$

Thus $Lip_\alpha(X,E)$ is a $Lip_\beta(X,E)$-module Banach. If $\alpha \leqslant \beta$, clearly

$$Lip_\beta(X,E) \subseteq Lip_\alpha(X,E).$$

Hence

$$Lip_\alpha(X,E)Lip_\beta(X,E) \subseteq Lip_\alpha(X,E)Lip_\alpha(X,E) \subseteq Lip_\alpha(X,E).$$

Therefore, $Lip_\alpha(X,E)$ is a $Lip_\beta(X,E)$-module Banach.

The implication (v)$\Longrightarrow$(iii) is similar to (v)$\Longrightarrow$(i). □

3. Character ameanablity of vector-valued Lipschitz algebras

Let A be a Banach algebra and $\Delta(A)$ denotes the spectrum of A consisting of all nonzero multiplicative linear functionals on A. In [16, 17], Kaniuth, Lau and Pym introduced and studied the concept of φ-amenability for Banach algebras, where $\varphi \in \Delta(A)$. In fact, a Banach algebra A is called φ-amenable if there exists a bounded linear functional m on A^*, satisfying $m(\varphi) = 1$ and $m(fa) = m(f).\varphi(a)$ for all $a \in A$ and $f \in A^*$ where $fa \in A^*$ is defined by $(fa)(b) = f(ab)$, $(b \in A)$. Let φ be the identity map, then φ-amenability is the same as amenability, see Lau [18]. Moreover, for some $C > 0$, A is called C-φ-amenable if m is bounded by C; see Hu, Monfared and Traynor [13]. The notion of (right) character amenability was introduced and studied by Monfared [20]. Character amenability of A is equivalent to A being φ-amenable for all $\varphi \in \Delta(A)$, and A having a bounded right approximate identity. The concept of character amenability is defined similarly see [17] for more details in this field.

There are valuable works related to some notions of amenable of Lipschitz algebras. Gourdeau [11], discussed amenability of Lipschitz algebras and proved that if a Banach algebra A is amenable, then $\Delta(A)$ is uniformly discrete with respect to norm topology induced by A^*; see also Bade, Curtis and Dales [3], Gourdeau [10] and Zhang [23]. Hu, Monfared and Traynor [13] investigated character amenability of Lipschitz algebras. They showed that if X is an infinite compact metric space and $0 < \alpha < 1$, then $Lip_\alpha X$ is not character amenable. Moreover, recently C-character amenability of Lipschitz algebras was studied by Dashti, Nasr Isfahani and Soltani for each $\alpha > 0$. In fact as a generalization of [9], they showed that for $\alpha > 0$ and any locally compact metric space X, $Lip_\alpha X$ is C-character amenable for some $C > 0$, if and only if X is uniformly discrete. In this section, we characterized C-character amenability of vector valued Lipschitz algebras.

Lemma 3.1. *Let (X, d) be a metric space and $\alpha > 0$ such that $Lip_\alpha(X,E)$ (resp. $lip_\alpha(X,E)$) separates the points of X and E be a Banach algebra with $\Delta(E) \neq \emptyset$. If $Lip_\alpha(X,E)$ (resp. $lip_\alpha(X,E)$) is C-character amenable for some $C > 0$. Then (X, d) is uniformly discrete.*

Proof. Let A be any of the Lipschitz algebras above. Define for all $f \in A$, $x \in X$ and $\psi \in \Delta(E)$. $\phi_x : A \longrightarrow \mathbb{C}$ where $\phi_x(f) = \psi \circ f(x)$. Then ϕ_x is a character of A. Since A is C-character amenable for some $C > 0$. It follows from [9, Corollary 2.2] and [10] that $\triangle(A)$ is uniformly discrete. Thus there is $\varepsilon > 0$ such that for all distinct elements ϕ_x, ϕ_y in $\triangle(A)$,

$$\|\phi_x - \phi_y\|_{A^*} > \varepsilon.$$

We have,

$$\begin{aligned}\varepsilon < \|\phi_x - \phi_y\|_{A^*} &= \sup_{\|f\|_{\alpha,E}\leqslant 1} \|\phi_x(f) - \phi_y(y)\|_{A^*} \\ &= \sup_{\|f\|_{\alpha,E}\leqslant 1} |\psi\circ f(x) - \psi\circ f(y)| \\ &= \|\psi\| \sup_{\|f\|_{\alpha,E}\leqslant 1} \frac{\|f(x)-f(y)\|}{d(x,y)^\alpha} d(x,y)^\alpha \leqslant \|\psi\| d(x,y)^\alpha\end{aligned}$$

for all $x, y \in X$. Therefore $d(x,y)^\alpha > \frac{\varepsilon}{\|\psi\|}$ and so $d(x,y) > (\frac{\varepsilon}{\|\psi\|})^{\frac{1}{\alpha}}$ and (X,d) is uniformly discrete. □

It should be noted that $Lip_\alpha(X,E)$ does not separate the points of X in general, whenever $\alpha > 1$. For example consider $\mathbb{R}^n$ endowed with the usual Euclidean. Let $f \in Lip_\alpha X$. Then

$$\frac{\|f(x)-f(y)-0(x-y)\|}{\|x-y\|} \leqslant M\|x-y\|^{\alpha-1},\ (x,y\in X)$$

for some $M > 0$. Thus $Df(z) = 0$ for each $z \in X = \mathbb{R}^n$. Also,

$$\|f(x)-f(y)\| \leqslant \|Df(z)\|.\|x-y\|$$

for some $z \in X$, so f is constant. Therefore $Lip_\alpha(X) = Cons(X)$. Thus $Lip_\alpha(\mathbb{R}^n, E) = Cons(\mathbb{R}^n, E)$ for all $\alpha > 1$ by Corollary 2.9. In this situation, $Lip_\alpha(\mathbb{R}^n, E)$ dose not separate the points of X.

Corollary 3.2. *Let (X,d) be a metric space, $0 < \alpha \leqslant 1$ and E be a commutative Banach algebra. If $Lip_\alpha(X,E)$ (resp. $lip_\alpha(X,E)$) is C-character amenable for some $C > 0$, then (X,d) is uniformly discrete.*

Let (X,d) be a metric space and $C_c(X,E)$ be $C_b(X,E)$ with compact open topology. In [19, Theorem 1.1], the author showed that $C_c(X,E) \cong C_c(X)\check{\otimes}E$ is an algebra isomorphism and isometric. If (X,d) is discrete, then compact open topology is the same with discrete topology. Thus $C_c(X,E) = l_\infty(X,E)$ and $C_c(X) = l_\infty(X)$. Hence $l_\infty(X,E) = l_\infty(X)\check{\otimes}E$ is an algebra isomorphism with equivalent norm. Let $\psi : l_\infty(X)\hat{\otimes}E \to l_\infty(X,E)$ defined by $\psi(f\otimes a) = f.a$. Then ψ is a linear, one to one, homomorphism with dense range. If E is amenable, then $l_\infty(X)\hat{\otimes}E$ is amenable [21] . Thus $l_\infty(X,E) = Lip_\alpha(X,E)$ is amenable.

Theorem 3.3. *Let (X,d) be a metric space, $\alpha > 0$ and $Lip_\alpha(X,E)$, $lip_\alpha(X,E)$ separates the points of X and E is a Banach algebra. Then the following statements are equivalent.*

(i) *$Lip_\alpha(X,E)$ (resp. $lip_\alpha(X,E)$) is amenable.*
(ii) *(X,d) is uniformly discrete and E is amenable.*

Proof. (i)⟹(ii) If $Lip_\alpha(X,E)$ is amenable, then (X,d) is uniformly discrete by [11, Theorem, 6]. Let $x_0 \in X$ and $\varphi : Lip_\alpha(X,E) \to E$ defined by $\varphi(f) =$

$f(x_0)$. Then φ is a linear, homomorphism and onto. Thus E is amenable by [21].

(ii)$\Longrightarrow$(i) Let (X,d) be uniformly discrete. Then $l_\infty(X,E) = Lip_\alpha(X,E)$ is a Banach algebra with equivalent norm and $l_\infty(X,E) = \overline{l_\infty(X)\hat{\otimes}E}$. Thus $Lip_\alpha(X,E)$ is amenable by [21]. Similarly for $lip_\alpha(X,E)$. □

Theorem 3.4. *Let (X,d) be a metric space, $\alpha > 0$, $Lip_\alpha(X,E)$ and $lip_\alpha(X,E)$ separates the points of X, E is a amenable Banach algebra, and $\Delta(E)$ is non-empty. Then the following statements are equivalent.*

(i) *$Lip_\alpha(X,E)$ (resp. $lip_\alpha(X,E)$) is C-character amenable for some $C > 0$.*
(ii) *$Lip_\alpha(X,E)$ (resp. $Lip_\alpha(X,E)$) is amenable.*
(iii) *(X,d) is uniformly discrete.*
(iv) *$Lip_\alpha X$ (resp. $lip_\alpha X$) is C-character amenable for some $C > 0$.*
(v) *$Lip_\alpha(X,E)$ (resp. $Lip_\alpha(X,E)$) is amenable.*

Proof. (i)$\Longrightarrow$(iii) By Lemma 3.1.

(iii)$\Longrightarrow$(ii) This follows from [10, Theorem 6] and Theorem 3.3.

(ii)$\Longrightarrow$(i) Since $Lip_\alpha(X,E)$ is amenable, it has an approximate diagonal bounded by some $C > 0$; see [21]. So, $Lip_\alpha(X,E)$ is C-character amenable for some by [13, Theorem 2.9].

Also, (iii), (iv) and (v) are equivalent by [9, Theorem 3.1]. □

Corollary 3.5. *Let (X,d) be a metric space, $0 < \alpha \leqslant 1$ and E be a commutative amenable Banach algebra. Then the following statements are equivalent.*

(i) *$Lip_\alpha(X,E)$ (resp. $lip_\alpha(X,E)$) is C-character amenable for some $C > 0$.*
(ii) *$Lip_\alpha(X,E)$ (resp. $lip_\alpha(X,E)$) is amenable.*
(iii) *(X,d) is uniformly discrete.*
(iv) *$Lip_\alpha X$ (resp. $lip_\alpha X$) is C-character amenable for some $C > 0$.*
(v) *$Lip_\alpha X$ (resp. $lip_\alpha X$) is amenable.*

Remark 3.6. Let (X,d) be a locally compact metric space. Then Lemma 3.1, Theorem 3.4 and Corollary 3.5 holds for $lip^0_\alpha(X,E)$.

4. Approximately amenability of vector-valued Lipschitz algebras

Let A be a Banach algebra and let X be a Banach A-bimodule. A derivation is a bounded linear map $D : A \longrightarrow X$ such that $D(ab) = aD(b) + D(a)b$ $(a; b \in A)$. For $x \in X$, the map $adx : A \longrightarrow X$ defined as $ad_x(a) = ax - xa$ $(a \in A)$ is clearly a derivation on A called an inner derivation. A derivation D is called approximately inner if there is a net (x_α) in X such that

$$D(a) = \lim_\alpha ad_{x_\alpha}(a) \ (a \in A).$$

The groundwork for amenability of Banach algebras was laid by Johnson [14]. The concept of amenability has occupied an important place in research in Banach algebras. In classic memoir [14] Johnson, initiated the theory of amenable Banach algebras. In fact a Banach algebra A is called approximately amenable if for each Banach A-bimodule X every continuous derivation $D : A \longrightarrow X^*$ is approximately inner. For (X, d) compact metric space and $0 < \alpha \leqslant 1$ Choi, Ghahramani [8, Theorem, 3.4] shows that $lip_\alpha(X, d)$ is not approximately amenable. Now if (X, d) is a metric space, then we study approximately amenability of vector-valued Lipschitz Algebras.

Definition 4.1. A separated, unbounded, multiplier-bounded configuration (or SUM configuration for short) in A consists of two sequences $(u_n), (p_n) \subseteq A$ which satisfy the following properties.

(i) (Separated) $u_n p_n = p_n u_n = u_n$ for all n and $u_j p_k = p_k u_j = 0$ whenever $j \neq k$.

(ii) (Unbounded) $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$.

(iii) (Multiplier-bounded) The sequences $(\|u_n\|_{mul})$ and $(\|p_n\|_{mul})$ are bounded.

Define the multiplier norm on A by $\|a\|_{mul} := \max(\|\lambda_a\|, \|\rho_a\|)$ where $\lambda_a : A \to A,\ x \longmapsto ax$ is left multiplication by a and $\rho_a : A \to A,\ x \longmapsto xa$ is right multiplication by a.

In [8] they proved that, if A is a Banach algebra, there exists an unbounded but multiplier-bounded sequence $(E_n)_{n\geqslant 1} \subseteq A$ such that $E_n E_{n+1} = E_n = E_{n+1}E_n$ for all n. Then A contains a SUM configuration. Also if A is a Banach algebra which contains a SUM configuration, then A is not approximately amenable. We now state the main result of this section.

Theorem 4.2. *Let (X, d) be a compact metric space, E be a unital Banach algebra and $0 < \alpha \leqslant 1$. Then $lip_\alpha(X, E)$ and $Lip_\alpha(X, E)$ are not approximately amenable.*

Proof. (i) We show that $lip_\alpha(X, E)$ contains a SUM configuration. Suppose that E has a unit e. By [8, Theorem, 3.4] $lip_\alpha(X, d)$ contains a SUM configuration, so there exists an unbounded but multiplier-bounded sequence $(E_n)_{n\geqslant 1} \subseteq lip_\alpha(X, d)$ such that $E_n E_{n+1} = E_n = E_{n+1}E_n$ for all n. For any $x \in X$ define $F_n := E_n.e$, hence

$$F_n.F_{n+1}(x) = E_n(x).eE_{n+1}(x).e = E_n(x)E_{n+1}(x).e = E_n(x).e = F_n(x).$$

Thus $F_n.F_{n+1} = F_n$. Similarly, $F_{n+1}.F_n = F_n$. Now we shows that $(F_n) \subseteq lip_\alpha(X, E)$ is unbounded, we have

$$\begin{aligned} p_\alpha(F_n) = p_\alpha(E_n.e) &= \sup_{x\neq y} \frac{\|E_n.e(x) - E_n.e(y)\|}{d(x, y)^\alpha} \\ &= \sup_{x\neq y} \frac{\|E_n(x).e - E_n(y).e\|}{d(x, y)^\alpha} \end{aligned}$$

$$= p_\alpha(E_n)\|e\| < \infty$$

and

$$\|E_n\|_\infty = \|E_n.e\|_\infty = \sup_{x\in X}\|E_n.e(x)\| = \sup_{x\in X}\|E_n(x).e\| \leqslant \|E_n\|_\infty\|e\|.$$

Hence,

$$\begin{aligned}\|F_n\|_{\alpha,E} &= \|E_n.e\|_{\alpha,E}\\ &= \|E_n.e\|_\infty + p_\alpha(E_n.e)\\ &= \|E_n\|_\infty\|e\| + p_\alpha(E_n)\|e\|\\ &= \|E_n\|_\alpha\|e\| \longrightarrow \infty.\end{aligned}$$

Also, $(F_n) \subseteq Lip_\alpha(X,E)$ and

$$\begin{aligned}\frac{\|F_n(x)-F_n(y)\|}{d(x,y)^\alpha} &= \frac{\|E_n.e(x)-E_n.e(y)\|}{d(x,y)^\alpha}\\ &= \frac{\|E_n(x).e-E_n(y).e\|}{d(x,y)^\alpha}\\ &= \|e\|\frac{\|E_n(x)-E_n(y)\|}{d(x,y)^\alpha} \longrightarrow 0.\end{aligned}$$

Therefore $(F_n) \subseteq lip_\alpha(X,E)$ Also $\|F_n\|_{\alpha,E} \longrightarrow \infty$. Now we define

$$\|F_n\|_{mul} = \max\{\|\lambda_{F_n}\|, \|\rho_{F_n}\|\},$$

where $\lambda_{F_n} : lip_\alpha(X,E) \longrightarrow lip_\alpha(X,E)$ with $\lambda_{F_n}(f) = F_n \circ f$. Therefore

$$\begin{aligned}\|\lambda_{F_n}\| &= \sup_{\|f\|_{\alpha,E}\leqslant 1}\|\lambda_{F_n}(f)\|_{\alpha,E}\\ &= \sup_{\|f\|_{\alpha,E}\leqslant 1}\|F_n\circ f\|_{\alpha,E}\\ &= \sup_{\|f\|_{\alpha,E}\leqslant 1}\|(E_n.e)\circ f\|_{\alpha,E}\\ &\leqslant \sup_{\|f\|_{\alpha,E}\leqslant 1}\|\lambda_{E_n}(f).e\|_{\alpha,E}\\ &\leqslant \|e\|\sup_{\|f\|_{\alpha,E}\leqslant 1}\|\lambda_{E_n}(f)\|_{\alpha,E}\\ &= \|e\|\|\lambda_{E_n}\| < \infty.\end{aligned}$$

Similarly, $\|\rho_{F_n}\| \leqslant \|\rho_{E_n}\|\|e\| < \infty$. Thus $\|F_n\|_{mul} < \infty$. Therefore $lip_\alpha(X,E)$ contains a SUM configuration. By [8, Theorem, 2.5] $lip_\alpha(X,E)$ is not approximately amenable.

(ii) Since $lip_\alpha(X,E) \subseteq Lip_\alpha(X,E)$. Thus $(F_n) \subseteq Lip_\alpha(X,E)$ is a SUM configuration. Therefore $Lip_\alpha(X,E)$ is not approximately amenable. □

Question 4.3. When are the vector-valued Lipschitz algebras $lip_\alpha(X,E)$ and $Lip_\alpha(X,E)$ Arens regular?

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