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A CHARACTERIZATION OF HYPERBOLIC SPACES

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Abstract. Let M be a complete spacelike hypersurface in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$. Suppose that every unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle \geq -1/r^2$ and there exists a point $p \in M$ such that for every unit speed geodesic $X(s)$ of M through the point p, $\langle X''(s), X''(s)\rangle = -1/r^2$ holds. Then, we show that up to isometries of $\mathbb{L}^{n+1}$, M is the hyperbolic space $H^n(r)$.

1. Introduction

We consider an n-dimensional hypersurface M in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1} = \mathbb{R}^{n+1}_1$ with the canonical flat metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$, where $x = (x_1, \ldots, x_{n+1}) \in \mathbb{L}^{n+1}$. Let us denote by $H^n(r) \subset \mathbb{L}^{n+1}$ the spacelike hyperquadric of radius r defined by $\langle x, x\rangle = -r^2$ with $x_{n+1} > 0$. Then $H^n(r)$ is a Riemannian space form with constant sectional curvature $K = -1/r^2$, which is called the standard imbedding of the hyperbolic space of curvature $K = -1/r^2$, or simply the hyperbolic space ([3]).

We have the following property of the hyperbolic space $H^n(r)$.

Proposition 1.1. *The hyperbolic space $H^n(r)$ satisfies the following condition.*
(C) *Every unit speed curve $X(s)$ on $H^n(r)$ satisfies $\langle X''(s), X''(s)\rangle \geq -1/r^2$.*

Proof. Let $X(s)$ be a unit speed curve in the hyperbolic space $H^n(r)$, so we have $\langle X(s), X(s)\rangle = -r^2$ and $\langle X'(s), X'(s)\rangle = 1$ for all s. For the unit normal $U(s) = X(s)/r$ along $X(s)$, we consider the following orthogonal decomposition:

$$X''(s) = \mathbf{k}_g(s) - \langle X''(s), U(s)\rangle\, U(s), \tag{1.1}$$

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where $\mathbf{k}_g(s)$ is the tangential part of $X''(s)$. Together with $\langle X'(s), X'(s)\rangle = 1$, differentiating $\langle X(s), X(s)\rangle = -r^2$ twice yields $\langle X''(s), X(s)\rangle = -1$. Hence we get from (1.1)

$$X''(s) = \mathbf{k}_g(s) + \frac{1}{r}U(s), \tag{1.2}$$

which shows that

$$\langle X''(s), X''(s)\rangle = \langle \mathbf{k}_g(s), \mathbf{k}_g(s)\rangle - \frac{1}{r^2}. \tag{1.3}$$

Since $\langle \mathbf{k}_g(s), \mathbf{k}_g(s)\rangle \geq 0$, (1.3) completes the proof. □

The hyperbolic spaces of radius exceeding r also satisfy the condition (C). Thus it is natural to ask what conditions should be added to the property (C) in order to characterize the hyperbolic space $H^n(r)$.

In this note, we prove the following:

Theorem 1.2. *Let M be a complete spacelike hypersurface in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$. Then, up to isometries of $\mathbb{L}^{n+1}$, M is the hyperbolic space $H^n(r)$ if and only if it satisfies*

(C) *Every unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle \geq -1/r^2$.*

(G) *There exists a point $p \in M$ such that for every unit speed geodesic $X(s)$ of M through the point p, $\langle X''(s), X''(s)\rangle = -1/r^2$ holds.*

With the help of Lemma 2.1 and (2.4) in Section 2, we obtain a characterization of the hyperbolic space $H^n(r)$ in terms of normal curvatures as follows.

Theorem 1.3. *Let M be a complete spacelike hypersurface in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$. Then, up to isometries of $\mathbb{L}^{n+1}$, M is the hyperbolic space $H^n(r)$ if and only if it satisfies*

(N) *For every unit tangent vector v to M, the normal curvature $\kappa_n(v)$ in the direction of v satisfies $|\kappa_n(v)| \leq 1/r$.*

(G') *There exists a point $p \in M$ such that for every unit speed geodesic $X(s)$ of M through the point p, the normal curvature $\kappa_n(X'(s))$ in the direction of $X'(s)$ satisfies $|\kappa_n(X'(s))| = 1/r$.*

For closed hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, the following characterizations were established ([1]).

Proposition 1.4. *Suppose that M is a closed hypersurface in $\mathbb{E}^{n+1}$ satisfying the following two conditions*:

$(C1)$ *every curve on M has curvature ≥ 1;*

$(C2)$ *on M there exists a curve γ_0 of length π with constant curvature* 1.

Then M is the unit sphere.

In [2], the first two authors and D. W. Yoon provide several characterizations for the standard imbeddings of hyperbolic spaces in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$ under suitable intrinsic and extrinsic assumptions on quantities such as the n-dimensional area of the sections cut off by hyperplanes, the $(n+1)$-dimensional volume of regions between parallel hyperplanes, and the n-dimensional surface area of regions between parallel hyperplanes.

Throughout this note, all objects are smooth and connected, unless otherwise mentioned.

2. Proofs

Suppose that M is a spacelike hypersurface (that is, the induced metric is positive definite) in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$. We consider a timelike unit normal vector U to the hypersurface M. Then we have $\langle U, U\rangle = -1$. For a point $p \in M$, we denote by $S_p : T_pM \to T_pM$ the shape operator defined by

$$S_p(v) = -\nabla_v U, \tag{2.1}$$

where v is a tangent vector to M at p and ∇ the usual connection of $\mathbb{L}^{n+1}$. For a unit tangent v at $p \in M$ the normal curvature $\kappa_n(v)$ in the direction of v is given by

$$\kappa_n(v) = \langle S_p(v), v\rangle\,. \tag{2.2}$$

First, we give a condition that is equivalent to (C).

Lemma 2.1. *For a spacelike hypersurface M in $\mathbb{L}^{n+1}$, the following conditions are equivalent*:

(C) *Every unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle \geq -1/r^2$.*

(N) *For every unit tangent vector v to M, the normal curvature $\kappa_n(v)$ in the direction of v satisfies $|\kappa_n(v)| \leq 1/r$.*

Proof. For any unit tangent vector v to M at p, we consider the geodesic X with $X(0) = p$ and $X'(0) = v$. Then for a given local unit normal vector field U to M around p we have $X''(0) = -\langle X''(0), U\rangle\, U$ because $X''(0)$ is normal to the hypersurface M. Hence the normal curvature satisfies

$$\kappa_n(v) = \langle S_p(v), v\rangle = \langle X''(0), U\rangle\,. \tag{2.3}$$

In the direction of v, we have

$$\kappa_n(v)^2 = -\langle X''(0), X''(0)\rangle\,. \tag{2.4}$$

Thus, (N) follows from (C).

For any unit speed curve $X(s)$ on M, we consider the following orthogonal decomposition:

$$X''(s) = \mathbf{k}_g(s) - \langle X''(s), U\rangle\, U, \tag{2.5}$$

where $\mathbf{k}_g(s)$ is the tangential part of $X''(s)$. Then, it follows from (2.3) that the normal curvature $\kappa_n(X'(s))$ of M in the direction of $X'(s)$ at $X(s)$ satisfies

$$X''(s) = \mathbf{k}_g(s) - \kappa_n(X'(s))U. \tag{2.6}$$

Hence we get

$$\langle X''(s), X''(s)\rangle = \langle \mathbf{k}_g(s), \mathbf{k}_g(s)\rangle - \kappa_n(X'(s))^2 \geq -\kappa_n(X'(s))^2, \tag{2.7}$$

where the inequality follows from $\langle \mathbf{k}_g(s), \mathbf{k}_g(s)\rangle \geq 0$. Thus, (N) together with (2.7) implies (C). □

Next, we show that the curve $X(s)$ in Condition (G) of the main theorem is nothing but a hyperbola as follows.

Lemma 2.2. *Suppose that a spacelike hypersurface M in $\mathbb{L}^{n+1}$ satisfies Condition (C). If a unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle = -1/r^2$, then it is a hyperbola given by*

$$X(s) = a\cosh\frac{s}{r} + b\sinh\frac{s}{r} + c \tag{2.8}$$

for some vectors $a, b, c \in \mathbb{L}^{n+1}$.

Proof. Suppose that a unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle = -1/r^2$. We consider the orthogonal decomposition of $X''(s)$ given by (2.6). Then, we get

$$\begin{aligned}\langle X''(s), X''(s)\rangle &= |\mathbf{k}_g(s)|^2 - \kappa_n(X'(s))^2 \\ &\geq |\mathbf{k}_g(s)|^2 - \frac{1}{r^2},\end{aligned} \tag{2.9}$$

where the inequality follows from Lemma 2.1.

Together with (2.9), the hypothesis $\langle X''(s), X''(s)\rangle = -1/r^2$ implies that $X(s)$ is a geodesic with $\kappa_n(X'(s)) = 1/r$ (after replacing U with $-U$ if necessary). Hence we obtain from (2.6)

$$X''(s) = -\frac{1}{r}U(s). \tag{2.10}$$

Since $\kappa_n(X'(s)) = 1/r$ is a maximum value of normal curvatures at $X(s)$, $X'(s)$ is a principal direction. Hence we get

$$S_{X(s)}(X'(s)) = \frac{1}{r}X'(s). \tag{2.11}$$

By the definition of S, we have $S_{X(s)}(X'(s)) = -U'(s)$. Thus, it follows from (2.10) and (2.11) that

$$X'''(s) = \frac{1}{r^2}X'(s). \tag{2.12}$$

This completes the proof. □

Proof of Theorem 1.2. Suppose that M is a complete spacelike hypersurface in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$ satisfying Conditions (C) and (G). Then, by isometries of $\mathbb{L}^{n+1}$, we may assume that $p = (0, \ldots, 0, r)$ and the unit normal $U(p)$ at p is given by $U(p) = (0, \ldots, 0, -1)$.

For any arbitrary point $q \in M$, the completeness of M implies that there exists a unit speed geodesic $X(s)$ of M connecting p and q with $X(0) = p$.

Together with the hypotheses, Lemma 2.2 shows that for some vectors $a, b, c \in \mathbb{L}^{n+1}$ the geodesic $X(s)$ is given by (2.8). Hence we get

$$X'(s) = \frac{a}{r}\sinh\frac{s}{r} + \frac{b}{r}\cosh\frac{s}{r}. \tag{2.13}$$

Since the geodesic $X(s)$ is of unit speed, it follows from (2.13) that

$$\langle a, b\rangle = 0, \quad \langle a, a\rangle + \langle b, b\rangle = 0, \quad \langle b, b\rangle - \langle a, a\rangle = 2r^2, \tag{2.14}$$

which shows that

$$\langle a, b\rangle = 0, \quad -\langle a, a\rangle = \langle b, b\rangle = r^2. \tag{2.15}$$

It follows from (2.10) and the assumption $U(p) = (0, \ldots, 0, -1)$ that $a = (0, \ldots, 0, r)$. Hence, from (2.15) we see that $b = ru$ for some unit vector $u = (u_1, \ldots, u_n, 0)$. Together with the initial condition $X(0) = p$, this shows that $X(s)$ is given by

$$X(s) = (0, \ldots, 0, r)\cosh\frac{s}{r} + ru\sinh\frac{s}{r}. \tag{2.16}$$

Thus, $X(s)$ satisfies

$$\langle X(s), X(s)\rangle = -r^2. \tag{2.17}$$

This completes the proof of the if part of Theorem 1.2.

The converse follows from Proposition 1.1 and (1.3). □

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