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MORPHISMS OF VARIETIES OVER AMPLE FIELDS

Lior Bary-Soroker, Wulf-Dieter Geyer, and Moshe Jarden

Abstract. We strengthen a result of Michiel Kosters by proving the following theorems:
$(*)$ Let ϕ: $W \to V$ be a finite surjective morphism of algebraic varieties over an ample field K. Suppose V has a simple K-rational point $\mathbf{a}$ such that $\mathbf{a} \notin \phi(W(K_{\mathrm{ins}}))$. Then, $\mathrm{card}(V(K) \smallsetminus \phi(W(K)) = \mathrm{card}(K)$.

$(**)$ Let K be an infinite field of positive characteristic and let $f \in K[X]$ be a non-constant monic polynomial. Suppose all zeros of f in $\tilde{K}$ belong to $K_{\mathrm{ins}} \smallsetminus K$. Then, $\mathrm{card}(K \smallsetminus f(K)) = \mathrm{card}(K)$.

Introduction

Recall that a field K is *ample* if it has the following property:

(1) If an absolutely integral variety V over K has a simple K-rational point, then $V(K)$ is Zariski-dense in V.

Here, a *variety over* K is a separated K-scheme V of finite type over $\mathrm{Spec}(K)$. We say that V is *absolutely integral* if its extension $V_{\tilde{K}}$, with $\tilde{K}$ being the algebraic closure of K, is integral.

It turns out that in addition to PAC fields and Henselian fields more families of fields are ample. Among those we find the real closed fields [9, p. 74, Example 5.6.3], fields that satisfy a local-global principle with respect to a family of ample fields [9, p. 75, Example 5.6.4], fields of power series $K_0((X_1, \ldots, X_n))$ over an arbitrary field K_0 [15, Thm. 1.1], and fields with a pro-p absolute Galois group (a result of Jean-Louis Colliot-Thélène [2, p. 360, second paragraph], generalized in [9, p. 83, Thm. 5.8.3]).

If K is ample and an absolutely integral variety V over K of positive dimension has a simple K-rational point, then, by a result of Florian Pop, $\mathrm{card}(V(K)) = \mathrm{card}(K)$ [9, p. 70, Prop. 5.4.3]. As a result, Arno Fehm proved that if in addition ϕ is a non-constant rational function on V, then

$$\mathrm{card}(\phi(V(K))) = \mathrm{card}(K) \text{ [9, p. 71, Cor. 5.4.4].}$$

On the other hand, Philipp Lampe asks the following question in [12]:

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Question A. *Does there exist an infinite field K and a polynomial $f \in K[X]$ such that $K \smallsetminus f(K)$ is a finite non-empty set?*

Pete Clark mentions in [12] that Question A has a negative answer if K is Hilbertian. Indeed, if in this case there exists $a \in K \smallsetminus f(K)$, then $d = \deg(f) \geq 2$ and $f(X) - T$ is irreducible in $K[T, X]$. Hence, there exist infinitely many $a' \in K$ such that $f(X) - a'$ is irreducible in $K[X]$ of degree ≥ 2. In particular, there exists no $x \in K$ with $f(x) = a'$.

On the other hand, Jochen Koenigsmann observed that if K is an ample field, then an application of Krasner's lemma to $K((t))$ implies that if $f \in K[X]$ is an irreducible polynomial of degree > 1, then $K \smallsetminus f(K)$ is an infinite set [1, last paragraph of page 6].

Michiel Kosters [11, Thm. 1.2] strengthens Koenigsmann's negative result to Question A:

Theorem B. *Let K be a perfect ample field, let C, D be normal projective curves over K, and let ϕ: $C \to D$ be a finite morphism. Suppose that the induced map ϕ: $C(K) \to D(K)$ is not surjective. Then,* $\mathrm{card}(D(K) \smallsetminus \phi(C(K))) = \mathrm{card}(K)$.

Theorem B assumes K to be perfect and C, D to be normal curves. It follows that both C, D are smooth curves. Also, the assumption on the map ϕ: $C(K) \to D(K)$ to be non-surjective, implies that $D(K)$ is non-empty. Thus, D has a K-rational simple point $\mathbf{a}$ which is not the image of a K-rational point of C under ϕ.

We strengthen Theorem B in two ways: First, we deal with arbitrary K-varieties rather than K-curves. Second, instead of assuming that K is perfect, we assume only that none of the points of the inverse image of the missing point is purely inseparable over K. To this end we denote the maximal purely inseparable extension of K by K_{ins}.

Theorem C (cf. Thm. 2.6)**.** *Let ϕ: $W \to V$ be a finite surjective morphism of integral algebraic varieties over an ample field K. Suppose V has a simple K-rational point $\mathbf{a}$ such that $\mathbf{a} \notin \phi(V(K_{\mathrm{ins}}))$. Then,* $\mathrm{card}(V(K) \smallsetminus \phi(W(K))) = \mathrm{card}(K)$.

While Kosters' proof tacitely applies the "field crossing argument" [4, p. 562, proof of Lemma 24.1.1], we follow Koenigsman's idea, apply a generalized form of Krasner's lemma and the existential closedness of K in $K((t))$.

In addition, we give a negative answer to Question A in a special case in which the assumptions are in a sense the inverse to those of Theorem C.

Theorem D (cf. Thm. 3.3)**.** *Let K be an infinite field of positive characteristic and let $f \in K[X]$ be a non-constant monic polynomial. Suppose that all zeros of f in $\tilde{K}$ belong to $K_{\mathrm{ins}} \smallsetminus K$. Then,* $\mathrm{card}(K \smallsetminus f(K)) = \mathrm{card}(K)$.

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1. Fibers of morphisms of integral algebraic varieties over Henselian fields

Krasner's lemma and the theorem about the continuity of roots are well known useful tools to deal with roots of polynomials over Henselian fields. Roughly speaking, Krasner's lemma says that if an element y' of the algebraic closure $\tilde{K}$ of a Henselian field K is sufficiently close to an element y of the separable closure K_{sep} of K, then $K(y) \subseteq K(y')$. The theorem about the continuity of roots asserts that if a polynomial $g \in \tilde{K}[X]$ is sufficiently close to a given polynomial $f \in \tilde{K}[X]$, then the roots of g are "respectively" sufficiently close to the roots of f in $\tilde{K}$.

The classical proof of Krasner's lemma [3, p. 91, Thm. 4.1.7] immediately generalizes to tuples of elements rather than elements of $\tilde{K}$. A classical proof of the theorem about the continuity of roots of a separable polynomials appears in [3, p. 53, Thm. 2.4.7]. We follow the model theoretic exposition of the theorem about the continuity of roots of an arbitrary polynomial [8, Prop. 12.2] in order to apply it to inverse images of morphisms of varieties. To this end we need a few concepts from the theory of valuations.

Let (K, v) be a valued field. Whenever it makes sense, we abbreviate n-tuples $(a_1, \ldots, a_n) \in K^n$ by the boldface letter $\mathbf{a}$. Then, we may extend v to a function v from K^n to the value group of K completed by the symbol ∞ in the following way:

$$v(\mathbf{a}) = \min(v(a_1), \ldots, v(a_n)).$$

The extended function satisfies the following rules:

(1a) $v(\mathbf{a}) \neq \infty$ if $\mathbf{a} \neq \mathbf{0}$.

(1b) $v(\mathbf{a} + \mathbf{b}) \geq \min(v(\mathbf{a}), v(\mathbf{b}))$.

(1b') $v(\mathbf{a} + \mathbf{b}) = \min(v(\mathbf{a}), v(\mathbf{b}))$ if $v(\mathbf{a}) \neq v(\mathbf{b})$.

(1c) $v(c\mathbf{a}) = v(c) + v(\mathbf{a})$ for each $c \in K$.

Lemma 1.1 (Generalized Krasner's Lemma)**.** *Let (K, v) be a Henselian valued field. Consider a complete system $\mathbf{y}_1, \ldots, \mathbf{y}_m$ of K-conjugates of a point $\mathbf{y} \in K_{\mathrm{sep}}^r$ with $\mathbf{y}_1 = \mathbf{y}$. If a point $\mathbf{y}' \in \tilde{K}^r$ satisfies*

$$v(\mathbf{y}' - \mathbf{y}) > \max_{i \geq 2} v(\mathbf{y}_i - \mathbf{y}), \tag{2}$$

then $K(\mathbf{y}) \subseteq K(\mathbf{y}')$.

Proof. Assume toward contradiction that $K(\mathbf{y}) \not\subseteq K(\mathbf{y}')$. Then, there exists $\sigma \in \mathrm{Aut}(\tilde{K}/K)$ such that $(\mathbf{y}')^\sigma = \mathbf{y}'$ and $\mathbf{y}^\sigma \neq \mathbf{y}$. In particular, there exists $i \geq 2$ such that $\mathbf{y}^\sigma = \mathbf{y}_i$. Using the identity $\mathbf{y}' - \mathbf{y}_i = (\mathbf{y}' - \mathbf{y}) + (\mathbf{y} - \mathbf{y}_i)$, (2) and (1b'), we get that $v(\mathbf{y}' - \mathbf{y}_i) = v(\mathbf{y} - \mathbf{y}_i)$. Since (K, v) is Henselian, $v(\mathbf{y}' - \mathbf{y}) = v((\mathbf{y}')^\sigma - \mathbf{y}^\sigma)$. By assumption (2), this leads to the the following contradiction:

$$\begin{aligned} v(\mathbf{y}_i - \mathbf{y}) = v(\mathbf{y}' - \mathbf{y}_i) &= v((\mathbf{y}')^\sigma - \mathbf{y}^\sigma) \\ &= v(\mathbf{y}' - \mathbf{y}) > v(\mathbf{y}_i - \mathbf{y}). \end{aligned}$$

It follows that $K(\mathbf{y}) \subseteq K(\mathbf{y}')$. $\square$

Given a valued field (K,v), we write $O_v = \{x \in K \mid v(x) \geq 0\}$ for the valuation ring of v. If w is another valuation of K and $O_v \subseteq O_w$, we say that w is *coarser* than v. If K admits no valuation w of rank 1 which is coarser than v, then we say that v is *unbounded*.

Following [13, p. 55, Def. 2.3.47], we define an *affine variety* over a field K as $\mathrm{Spec}(R)$, where R is a finitely generated algebra over K. In particular, $\mathrm{Spec}(R)$ is separated [13, p. 100, Prop. 3.3.4]. As usual, we say that $V = \mathrm{Spec}(R)$ is *integral* if R is an integral domain. In this case we choose $x_1,\dots,x_m \in R$ such that $R = K[\mathbf{x}]$ and consider V as embedded in the affine space $\mathbb{A}_K^m$. If L is a field extension of K, then an *L-rational point* of V is an m-tuple $\mathbf{a} = (a_1,\dots,a_m) \in L^m$ that satisfies $f(\mathbf{a}) = 0$ for each $f \in K[X_1,\dots,X_m]$ such that $f(\mathbf{x}) = 0$. In other words, there is a K-homomorphism $K[\mathbf{x}] \to L$ that maps $\mathbf{x}$ onto $\mathbf{a}$. In particular, $\mathbf{x}$ is a $K(\mathbf{x})$-rational point of V. Following the classical language of algebraic geometry, we say that $\mathbf{x}$ is a *generic point* of V. (Note that the kernel of the K-homomorphism $R \to K[\mathbf{x}]$ is the zero ideal of R.) We write $V(L)$ for the set of all L-rational points of V.

An *algebraic variety* over K is a separated scheme V over K that can be covered by a finite number of Zariski-open affine varieties over K.

Let V and W be affine varieties over K embedded in $\mathbb{A}_K^m$ and $\mathbb{A}_K^n$, respectively. For every morphism $\phi\colon W \to V$ and for every field extension L of K there exists a map from $W(L)$ into $V(L)$ that we also denote by ϕ. Indeed, there exist polynomials $h_1,\dots,h_m \in K[Y_1,\dots,Y_n]$ independent of L such that $\phi(\mathbf{b}) = (h_1(\mathbf{b}),\dots,h_m(\mathbf{b})) \in V(L)$ for each $\mathbf{b} \in W(L)$ (essentially [13, p. 48, Prop. 2.3.25]).

This classical description of morphisms, together with a description of $V(L)$ and $W(L)$ as the zero sets of finitely many polynomials with coefficients in K allows us to speak about them in the first order language $\mathcal{L}(\mathrm{ring}, K)$ of rings with a constant symbol for each element of K. For this concept as well as for other notions of logic and model theory used in the sequel, the reader may consult [4, Chap. 7].

Lemma 1.2 (Generalized continuity of roots)**.** *Let (K,v) be a valued field and extend v to $\tilde{K}$. Consider a finite morphism $\phi\colon W \to V$ of integral affine algebraic varieties over K. Let $\mathbf{a}$ be a point in $V(\tilde{K})$ and let $\phi^{-1}(\mathbf{a})(\tilde{K}) = \{\mathbf{b}_1,\dots,\mathbf{b}_m\}$. Then, for each $\beta \in v(\tilde{K}^\times)$ there exists $\alpha \in v(K^\times)$ such that if $\mathbf{a}' \in V(\tilde{K})$ satisfies $v(\mathbf{a}'-\mathbf{a}) > \alpha$, then for each $\mathbf{b}' \in \phi^{-1}(\mathbf{a}')(\tilde{K})$ there exists $1 \leq i \leq m$ such that $v(\mathbf{b}'-\mathbf{b}_i) > \beta$.*

Proof. Since $v(K^\times)$ is cofinal in $v(\tilde{K}^\times)$ [8, Cor. 7.2], we may assume that K is algebraically closed. Since the statement of the lemma is elementary in the language $\mathcal{L}_v(\mathrm{ring}, K)$ of valued fields with a constant symbol for each element of K, we may use [8, Lemma 10.3] in order to replace (K,v) by an appropriate non-principal ultra-power of itself and assume that the valuation

v is unbounded. We may also assume that $\mathbf{b}_1, \ldots, \mathbf{b}_m$ are distinct and

$$\beta > \max_{i \neq j}(v(\mathbf{b}_i - \mathbf{b}_j)). \tag{3}$$

The assumption $\phi^{-1}(\mathbf{a}) = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ is equivalent to a sentence in the language $\mathcal{L}(\mathrm{ring}, K)$. Therefore, the theorem about the elimination of quantifiers for the theory of algebraically closed fields [4, p. 167, Cor. 9.2.2] gives $c \in K^\times$ such that if a valuation w of K satisfies $w(c) = 0$ and we denote reduction modulo w by a bar, we have

(4) $\bar{\varphi}\colon \bar{V} \to \bar{W}$ is a finite morphism and $\bar{\varphi}^{-1}(\bar{\mathbf{a}}) = \{\bar{\mathbf{b}}_1, \ldots, \bar{\mathbf{b}}_m\}$ (in particular each of the objects appearing in this statement is well defined).

Lemma 10.1 of [8] supplies a valuation w of K which is coarser than v such that

(5) $w(\mathbf{a}), w(c), w(c^{-1}), w(\mathbf{b}_1), \ldots, w(\mathbf{b}_m) \geq 0$

and for each $z \in K$,

(6) $w(z) > 0$ implies $v(z) > \beta$.

In the other direction, [8, Lemma 3.2] gives $\alpha \in v(K^\times)$ such that

(7) $v(z) > \alpha$ implies $w(z) > 0$ for each $z \in K$.

We assume without loss of generality that $\alpha > 0$ and consider $\mathbf{a}' \in V(K)$ such that $v(\mathbf{a}' - \mathbf{a}) > \alpha$. By (7), $w(\mathbf{a}' - \mathbf{a}) > 0$. Hence, by (5), $w(\mathbf{a}') \geq 0$. Therefore, $\overline{\mathbf{a}'} = \bar{\mathbf{a}}$ in the residue field $\bar{K}_w$ of K at w.

Now we consider a point $\mathbf{b}' \in W(K)$ that satisfies $\phi(\mathbf{b}') = \mathbf{a}'$. By (4) and (5), there exists $1 \leq i \leq m$ with $\overline{\mathbf{b}'} = \overline{\mathbf{b}_i}$. Hence, $w(\mathbf{b}' - \mathbf{b}_i) > 0$. It follows from (6) that $v(\mathbf{b}' - \mathbf{b}_i) > \beta$, as desired. □

Lemma 1.3. *Let (K, v) be a Henselian valued field and $\phi\colon W \to V$ a finite morphism of integral algebraic varieties over K. For each point $\mathbf{a} \in V(K)$ and for each $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{K})$ let $b_{\mathbf{c}}$ be an element of $K_{\mathrm{sep}} \cap K(\mathbf{c})$. Then, $\mathbf{a}$ has a v-open neighborhood U in $V(K)$ such that for every $\mathbf{a}' \in U$ and for each $\mathbf{c}' \in \phi^{-1}(\mathbf{a}')(\tilde{K})$ there exists $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{K})$ such that $K[b_{\mathbf{c}}] \subseteq K[\mathbf{c}']$.*

Proof. Since ϕ is finite and the problem is Zariski-local, we may assume that both V and W are affine and W is embedded in $\mathbb{A}^n_K$ for some positive integer n. Let $\phi^{-1}(\mathbf{a})(\tilde{K}) = \{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$. For each $1 \leq i \leq r$ we choose $b_i \in K_{\mathrm{sep}}$ with $K[b_i] = K_{\mathrm{sep}} \cap K(\mathbf{c}_i)$. By assumption, there exists $f_i \in K[X_1, \ldots, X_n]$ such that $b_i = f_i(\mathbf{c}_i)$. Let $b_{i1}, \ldots, b_{is}$ be the distinct conjugates of b_i over K.

We set $\beta = \max_{1 \leq i \leq r} \max_{j \neq j'} v(b_{ij} - b_{ij'})$. If $b' \in \tilde{K}$ satisfies $v(b' - b_i) > \beta$, then by Krasner's Lemma (e.g. Lemma 1.1), $K[b_i] \subseteq K[b']$. Since the polynomials f_i are v-continuous, there exists $\gamma \in v(K^\times)$ such that for all $1 \leq i \leq r$ and $\mathbf{c}' \in \tilde{K}^n$, the inequality $v(\mathbf{c}' - \mathbf{c}_i) > \gamma$ implies $v(f_i(\mathbf{c}') - b_i) = v(f_i(\mathbf{c}') - f_i(\mathbf{c}_i)) > \beta$.

By Lemma 1.2, there exists $\alpha \in v(K^\times)$ such that if $\mathbf{a}' \in V(K)$ satisfies $v(\mathbf{a}' - \mathbf{a}) > \alpha$, then for each $\mathbf{c}' \in \phi^{-1}(\mathbf{a}')(\tilde{K})$ there exists i between 1 and r such that $v(\mathbf{c}' - \mathbf{c}_i) > \gamma$. It follows from the preceding paragraph that $v(f_i(\mathbf{c}') - b_i) > \beta$ and then $K[b_i] \subseteq K[f_i(\mathbf{c}')] \subseteq K[\mathbf{c}']$, as desired. □

2. Ample fields

Starting from an ample field K of cardinality κ we consider the field $K((t))$ of formal power series in t over K. Under the assumptions of Theorem C, we first find a generic point $\mathbf{x}$ of V in $V(K((t))) \smallsetminus \phi(W(K((t))))$. Then, we specialize $\mathbf{x}$ to κ points in $V(K) \smallsetminus \phi(W(K))$.

Lemma 2.1. *Let Ω/K be a regular extension of fields of infinite transcendental degree and let v be a non-trivial Henselian valuation on Ω. Let V be an absolutely integral algebraic variety over K with a simple Ω-rational point $\mathbf{a}$ and let U be a v-open neighborhood of $\mathbf{a}$ in $V(\Omega)$. Then, V has a generic Ω-rational point that lies in U.*

Proof. Let $r = \dim(V)$. By [6, p. 71, Thm. 9.2], there exist an affine neighborhood V_0 of $\mathbf{a}$ in V and a morphism $\phi\colon V_0 \to \mathbb{A}^r_K$ that induces a homeomorphism of a v-open neighborhood U of $\mathbf{a}$ in $V_0(\Omega)$ onto a v-open neighborhood U_0 of the origin $\mathbf{o}$ of Ω^r. Let y be an element of $\Omega^\times$ such that $\{\mathbf{b} \in \Omega^r \mid v(\mathbf{b}) > v(y)\} \subseteq U_0$.

By assumption, Ω contains elements $z_0', z_1', \ldots, z_r'$ that are algebraically independent over K. Let y' be an element of $\Omega^\times$ such that $v(z_i') + v(y') > v(y)$ for $i = 0, \ldots, r$ and set $z_i = z_i' y'$ for $i = 0, \ldots, r$. Then,

$$\mathrm{trans.deg}(K(y', z_0', \ldots, z_r')/K) \geq r + 1.$$

Since $K(y', z_0, \ldots, z_r) = K(y', z_0', \ldots, z_r')$, we have that

$$\mathrm{trans.deg}(K(y', z_0, \ldots, z_r)/K) \geq r + 1.$$

Hence, $\mathrm{trans.deg}(K(z_0, \ldots, z_r)/K) \geq r$. It follows that at least r elements of the set $\{z_0, \ldots, z_r\}$ are algebraically independent over K. We assume without loss of generality that $z_1, \ldots, z_r$ are algebraically independent over K. By construction, $v(z_i) > v(y)$ for $i = 1, \ldots, r$. Hence, by the preceding paragraph, the point $\mathbf{z} = (z_1, \ldots, z_r)$ lies in U_0.

By the first paragraph of the proof, there exists a point $\mathbf{x}$ of U, hence of $V_0(\Omega)$ such that $\phi(\mathbf{x}) = \mathbf{z}$. Since $K(\mathbf{z}) \subseteq K(\mathbf{x})$, we have

$$r = \mathrm{trans.deg}(K(\mathbf{z})/K) \leq \mathrm{trans.deg}(K(\mathbf{x})/K) \leq r.$$

Therefore, $\mathrm{trans.deg}(K(\mathbf{x})/K) = r$, so $\mathbf{x}$ is a generic point of V, as desired. $\square$

Lemma 2.2. *Let $\pi\colon V' \to V$ be a non-constant rational map of algebraic varieties over an ample field K. Suppose that V is integral and affine. Suppose in addition that V' is absolutely integral of positive dimension and has a K-rational simple point. Then,* $\mathrm{card}(\pi(V'(K))) = \mathrm{card}(K)$.

Proof. We may assume that V is embedded in $\mathbb{A}^n_K$. Let π_i be the projection of V on the ith coordinate. Then, one of the rational functions $\phi_i = \pi_i \circ \pi\colon V' \to \mathbb{A}^1_K$ is non-constant. It follows from a result of Fehm [9, p. 71, Cor. 5.4.4] that $\mathrm{card}(\phi_i(V'(K))) = \mathrm{card}(K)$. Hence, $\mathrm{card}(K) \leq \mathrm{card}(\pi(V'(K)))$. Since

$\mathrm{card}(\pi(V'(K)) \le \mathrm{card}(V(K)) \le \mathrm{card}(K)$, we have $\mathrm{card}(\pi(V'(K))) = \mathrm{card}(K)$, as claimed. □

Lemma 2.3. *Let K be a field, $R \subseteq S$ finitely generated integral domains over K, and w a non-zero element of S. We set $E = \mathrm{Quot}(R)$, $F = \mathrm{Quot}(S)$, and assume that F/E is a finite extension. Then, there exists a non-zero element $v \in E$ such that $\psi(w) \neq 0$ for every K-homomorphism $\psi\colon S[v] \to \tilde{K}$.*

Proof. We choose an irreducible polynomial $f(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$ with coefficients in R such that $f(w) = 0$. Since $w \neq 0$, we have $a_0 \neq 0$. So, $v = a_0^{-1}$ satisfies the conclusion of the lemma. □

Given an algebraic variety V over a field K we write V_{simp} for the Zariski-open subset of V that consists of all simple points of V. In addition we note that for elements $a_1, \ldots, a_r$ of $\tilde{K}$ the ring $K[a_1, \ldots, a_r]$ coincides with its quotient field $K(a_1, \ldots, a_r)$ and use either of these notations as it better fits in the context.

Remark 2.4. The assumption on the existence of a K-rational simple point $\mathbf{a}$ on the integral algebraic variety V over K implies that $V_{\tilde{K}}$ is integral, i.e., V is *absolutely integral*. Indeed, we may assume that V is affine, let $\mathbf{x}$ be a generic point of V and set $F = K(\mathbf{x})$. By [10, p. 457, Cor. 3], F has a K-rational place. By [4, p. 42, Lemma 2.6.9], F/K is a regular extension. Hence, by [4, p. 175, Cor. 10.2.2], V is absolutely integral. See also [14, Lemmas 4 and 5].

Alternatively, the simplicity and the rationality of $\mathbf{a}$ imply that the local ring $A = \mathcal{O}_{V,\mathbf{a}}$ of $\mathbf{a}$ is of dimension n and "formellement lisse" over K. By [7, p. 102, Thm. 19.6.4], the $\mathfrak{m}_{V,x}$-adic completion $\hat{A}$ of A is K-isomorphic to the ring $K[[T_1, \ldots, T_n]]$ of formal power series in n-variables. Hence, for every finite extension L of K, the ring $\widehat{A \otimes_K L} = \hat{A} \otimes_K L = L[[T_1, \ldots, T_n]]$ is a domain. This implies again that V is absolutely integral.

Lemma 2.5. *Let $\phi\colon W \to V$ be a finite surjective morphism of integral algebraic varieties over an ample field K of cardinality κ with $\dim(V) \geq 1$. Let $\mathbf{a} \in V_{\mathrm{simp}}(K)$ and for each $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{K})$ let $b_{\mathbf{c}}$ be an element of $K_{\mathrm{sep}} \cap K(\mathbf{c})$. Then, $V(K)$ has κ points $\mathbf{a}'$ such that for each $\mathbf{c}' \in \phi^{-1}(\mathbf{a}')(\tilde{K})$ there exist $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{K})$ with $K(b_{\mathbf{c}}) \subseteq K(\mathbf{c}')$.*

Proof. Since the statement of the lemma has a Zariski-local nature and ϕ is finite, we may assume that both V and W are affine and break up the rest of the proof into six parts.

Part A: *System of representatives.* For each $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{K})$, we have $K(b_{\mathbf{c}}) \subseteq K_{\mathrm{sep}} \cap K(\mathbf{c})$. Replacing $b_{\mathbf{c}}$ with a primitive element of $K_{\mathrm{sep}} \cap K_{\mathbf{c}}$, we may assume that $K(b_{\mathbf{c}}) = K_{\mathrm{sep}} \cap K(\mathbf{c})$.

Now let $\mathbf{c}_1, \ldots, \mathbf{c}_k$ be representatives for the K-conjugacy classes of the points in $\phi^{-1}(\mathbf{a})(\tilde{K})$. Let $\mathbf{d} \in \phi^{-1}(\mathbf{a})(\tilde{K})$. By the choice of the $\mathbf{c}_i$'s there exist an element $\sigma \in \mathrm{Aut}(\tilde{K}/K)$ and a unique $i \in \{1, \ldots, k\}$ such that $\sigma\mathbf{d} = \mathbf{c}_i$. If τ

is another element of $\mathrm{Aut}(\tilde{K}/K)$ with $\tau\mathbf{d} = \mathbf{c}_i$, then $\tau^{-1}\sigma\mathbf{d} = \mathbf{d}$, so $\sigma b_{\mathbf{d}} = \tau b_{\mathbf{d}}$. Hence, we may replace $b_{\mathbf{d}}$ by $\sigma^{-1}b_{\mathbf{c}_i}$, if necessary, in order to assume that

(1) if $\mathbf{c}, \mathbf{d} \in \phi^{-1}(\mathbf{a})(\tilde{K})$ and $\sigma \in \mathrm{Aut}(\tilde{K}/K)$ satisfy $\sigma\mathbf{d} = \mathbf{c}$, then $\sigma b_{\mathbf{d}} = b_{\mathbf{c}}$.

Part B: *Base change.* Consider the field $\Omega = K((t))$ of power series in t over K. Let ϕ_Ω: $W_\Omega \to V_\Omega$ be the morphism obtained from ϕ by base change from K to Ω. By [5, p. 325, Prop. 12.11(2) and p. 108, Prop. 4.32(2)], ϕ_Ω is a finite surjective morphism. Moreover, $\mathbf{a} \in V_{\mathrm{simp}}(\Omega) = V_{\Omega,\mathrm{simp}}(\Omega)$ and for each $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{\Omega})$ we have $\Omega[b_{\mathbf{c}}] \subseteq \Omega[\mathbf{c}]$.

The t-adic valuation on Ω is discrete and complete, hence Henselian [3, p. 20, Thm. 1.3.1]. By Lemma 1.3, $\mathbf{a}$ has a t-adically open neighborhood U in $V(\Omega)$ such that for each $\mathbf{a}' \in U$ and for each $\mathbf{c}' \in \phi_\Omega^{-1}(\mathbf{a}')(\tilde{\Omega})$ there exists $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{\Omega})$ with $\Omega[b_{\mathbf{c}}] \subseteq \Omega[\mathbf{c}']$.

By [9, p. 159, Prop. 8.5.2], trans.deg$(\Omega/K) = \infty$. By Remark 2.4, V is absolutely integral. Hence, by Lemma 2.1, V has an Ω-rational generic point $\mathbf{x}$ that lies in U. By the previous paragraph, for each $\mathbf{y}' \in \phi_\Omega^{-1}(\mathbf{x})(\tilde{\Omega})$ there exists $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{\Omega})$ with $\Omega[b_{\mathbf{c}}] \subseteq \Omega[\mathbf{y}']$.

Part C: *A finitely generated extension of* K. Since $\phi^{-1}(\mathbf{a})(\tilde{\Omega})$ and $\phi_\Omega^{-1}(\mathbf{x})(\tilde{\Omega})$ are finite sets, there exist $u_1, \ldots, u_l \in \Omega$ such that for each $\mathbf{y}' \in \phi_\Omega^{-1}(\mathbf{x})(\tilde{\Omega})$ there exists $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{K})$ with $K(\mathbf{x}, \mathbf{u})[b_{\mathbf{c}}] \subseteq K(\mathbf{x}, \mathbf{u})[\mathbf{y}']$. In particular, there exists nonzero $z_{\mathbf{c}} \in K[\mathbf{x}, \mathbf{u}]$ with $b_{\mathbf{c}} \in z_{\mathbf{c}}^{-1}K[\mathbf{x}, \mathbf{u}, \mathbf{y}']$. Replacing $\mathbf{u}$ by $(\mathbf{u}, z_{\mathbf{c}}^{-1})_{\mathbf{c}\in\phi^{-1}(\mathbf{a})(\tilde{K})}$, if necessary, we may assume that

(2) for each $\mathbf{y}' \in \phi_\Omega^{-1}(\mathbf{x})(\tilde{\Omega})$ there exists $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{K})$ such that $K[\mathbf{x}, \mathbf{u}, b_{\mathbf{c}}] \subseteq K[\mathbf{x}, \mathbf{u}, \mathbf{y}']$.

Part D: *An elementary statement.* Let $\mathbf{y}_1, \ldots, \mathbf{y}_m$ be the finitely many points of $\phi_\Omega^{-1}(\mathbf{x})(\tilde{\Omega}) = \phi_\Omega^{-1}(\mathbf{x})(\widetilde{K(\mathbf{x})})$ and set $S = K[\mathbf{y}_1, \ldots, \mathbf{y}_m]$. Then, $K[\mathbf{x}] \subseteq S$ (because $\phi(\mathbf{y}_i) = \mathbf{x}$ for $i = 1, \ldots, m$) and the following statement holds in $\widetilde{K(\mathbf{x})}$:

$$(\forall \mathbf{Y} \in W(\widetilde{K(\mathbf{x})}))[\phi(\mathbf{Y}) = \mathbf{x} \leftrightarrow \bigvee_{i=1}^{m} \mathbf{Y} = \mathbf{y}_i]. \tag{3}$$

In this statement $\mathbf{Y}$ is a tuple of variables for the elements of $W(\widetilde{K(\mathbf{x})})$. Observe that (3) is equivalent to an elementary statement on $\widetilde{K(\mathbf{x})}$ in the language of rings $\mathcal{L}(\mathrm{ring}, S)$. By [4, p. 167, Cor. 9.2.2], there exists a non-zero $w \in S$ such that if ψ: $S \to \tilde{K}$ is a K-homomorphism with $\psi(w) \neq 0$, then the following statement on $\tilde{K}$ holds:

$$(\forall \mathbf{Y} \in W(\tilde{K}))[\phi(\mathbf{Y}) = \psi(\mathbf{x}) \leftrightarrow \bigvee_{i=1}^{m} \mathbf{Y} = \psi(\mathbf{y}_i)]. \tag{4}$$

Note that the quotient field $K(\mathbf{y}_1, \ldots, \mathbf{y}_m)$ of S is a finite extension of $E = K(\mathbf{x})$. By Lemma 2.3, there exists $v \in K(\mathbf{x})$ such that $\psi(w) \neq 0$ for every K-homomorphism ψ: $S[v] \to \tilde{K}$.

We add v to $u_1, \ldots, u_l$, if necessary, to assume that $v \in \{u_1, \ldots, u_l\}$. Then, every K-homomorphism ψ: $S[\mathbf{u}] \to \tilde{K}$ satisfies $\psi(w) \neq 0$, hence (4) is true.

Since ϕ: $W \to V$ is finite, each of the ring extensions $K[\mathbf{y}_i]/K[\mathbf{x}]$ is integral, hence

(5) the ring $S' = S[\mathbf{u}]$ is an integral extension of $K[\mathbf{x}, \mathbf{u}]$.

Part E: *The variety V'.* By [4, p. 61, Example 3.5.1], Ω is a regular extension of K, hence so is $K(\mathbf{x}, \mathbf{u})$. Therefore, by [4, p. 175, Cor. 10.2.2], $V' = \mathrm{Spec}(K[\mathbf{x}, \mathbf{u}])$ is an absolutely integral variety over K and $(\mathbf{x}, \mathbf{u})$ is an Ω-rational generic point of V'. In particular, $(\mathbf{x}, \mathbf{u})$ is an Ω-rational simple point of V'. Since K is existentially closed in Ω [9, p. 68, Def. 5.3.2], $V'_{\mathrm{simp}}(K) \neq \emptyset$.

Let π: $V' \to V$ be the rational map defined by $\pi(\mathbf{x}, \mathbf{u}) = \mathbf{x}$. Then, π is dominating, hence it follows from $\dim(V) \geq 1$ that π is non-constant. Therefore, by Lemma 2.2, $\mathrm{card}(\pi(V'(K))) = \mathrm{card}(K) = \kappa$.

Part F: *Conclusion of the proof.* For each of the κ points $\mathbf{a}'$ of $\pi(V'(K))$ there is a $\mathbf{u}' \in K^l$ such that $(\mathbf{a}', \mathbf{u}') \in V'(K)$. Let ψ be the K-homomorphism from $K[\mathbf{x}, \mathbf{u}]$ to K defined by $\psi(\mathbf{x}, \mathbf{u}) = (\mathbf{a}', \mathbf{u}')$. Let $\mathbf{c}'$ be an element of $W(\tilde{K})$ such that $\phi(\mathbf{c}') = \mathbf{a}' = \psi(\mathbf{x})$. We use that S' is an integral extension of $K[\mathbf{x}, \mathbf{u}]$ (by (5)) in order to extend ψ to a K-homomorphism ψ: $S' \to \tilde{K}$. By (4), there exists $1 \leq i \leq m$ such that $\psi(\mathbf{y}_i) = \mathbf{c}'$. By (2), there exists $\mathbf{c}'' \in \phi^{-1}(\mathbf{a})(\tilde{K})$ such that

(6) $K[\mathbf{x}, \mathbf{u}, b_{\mathbf{c}''}] \subseteq K[\mathbf{x}, \mathbf{u}, \mathbf{y}_i]$.

Since $\mathbf{c}''$ is algebraic over K, the ring $S'[\mathbf{c}'']$ is integral over S', so we may extend ψ to a homomorphism ψ: $S'[\mathbf{c}''] \to \tilde{K}$. In particular, since $b_{\mathbf{c}''} \in K(\mathbf{c}'')$, the element $\psi(b_{\mathbf{c}''})$ is well defined. Since ψ maps K identically onto itself, ψ maps $K[b_{\mathbf{c}''}]$ isomorphically onto $K[\psi(b_{\mathbf{c}''})]$. Since ϕ is defined over K and $\mathbf{a}$ is K-rational, $\mathbf{c} = \psi(\mathbf{c}'') \in \phi^{-1}(\mathbf{a})(\tilde{K})$. By (1), $b_{\mathbf{c}} = b_{\psi(\mathbf{c}'')} = \psi(b_{\mathbf{c}''})$. Applying ψ on both sides of (6), we get $K[b_{\mathbf{c}}] \subseteq K[\mathbf{c}']$, as desired. $\square$

Lemma 2.5 implies a stronger version of Theorem B.

Theorem 2.6. *Let ϕ: $W \to V$ be a finite surjective morphism of integral algebraic varieties over an ample field K of cardinality κ. Suppose* $\dim(V) \geq 1$ *and there exists* $\mathbf{a} \in V_{\mathrm{simp}}(K) \smallsetminus \phi(W(K_{\mathrm{ins}}))$. *Then,* $\mathrm{card}(V(K) \smallsetminus \phi(W(K_{\mathrm{ins}}))) = \kappa$.

Proof. Let $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{K})$. By assumption $K(\mathbf{c})$ is not a purely inseparable extension of K. Hence, $K(\mathbf{c})$ contains an element $b_{\mathbf{c}}$ which lies in $K_{\mathrm{sep}} \smallsetminus K$. By Lemma 2.5, $V(K)$ has κ points $\mathbf{a}'$ such that for each $\mathbf{c}' \in \phi^{-1}(\mathbf{a}')(\tilde{K})$ there exists $\mathbf{c} \in \phi^{-1}(\mathbf{a})(\tilde{K})$ with $K(b_{\mathbf{c}}) \subseteq K(\mathbf{c}')$. In particular, $\mathbf{c}' \notin W(K_{\mathrm{ins}})$. In other words, $\mathbf{a}' \in V(K) \smallsetminus \phi(W(K_{\mathrm{ins}}))$, as desired. $\square$

Example 2.7. It is impossible to weaken the assumption $\mathbf{a} \notin \phi(W(K_{\mathrm{ins}}))$ in Theorem 2.6 to $\mathbf{a} \notin \phi(W(K))$.

Indeed, let p be a prime number and t a transcendental element over $\mathbb{F}_p$. Then, the field $K = \mathbb{F}_p((t))$ of formal power series over $\mathbb{F}_p$ is ample [9, p. 73, Example 5.6.2]. Let $V = \mathbb{A}^1_K$. By Eisenstein's criterion, the polynomial $Y^p -$

$tX^p - t$ is irreducible over $K(X)$. Hence, the K-curve W defined in $\mathbb{A}^2_K$ by the equation $Y^p - tX^p - t = 0$ is integral (but not absolutely integral). Since the latter polynomial is monic in Y, the morphism ϕ: $W \to V$ defined by projection $(X, Y) \mapsto X$ is finite. Finally note that there is no $y \in K$ such that $y^p = t$. Hence, ϕ maps no point of $W(K)$ onto the point 0 of $V(K)$. However, every K-rational point of $V(K)$ has a (single) preimage in $W(K_{\rm ins})$, so $\phi(W(K_{\rm ins})) = V(K)$.

Question 2.8. Is it possible to replace $K_{\rm ins}$ by K in the assumption and the conclusion of Theorem 2.6?

Remark 2.9. The assumption that ϕ is finite in Theorem 2.6 is essential.

Indeed, the field $\mathbb{R}$ is ample [9, p. 73, Example 5.6.3]. Let $V = \mathrm{Spec}([\mathbb{R}[X^3])$ $\cong \mathbb{A}^1_{\mathbb{R}}$ and $W = \mathrm{Spec}(\mathbb{R}[X, (X-1)^{-1}]) \cong \mathbb{A}^1_R \smallsetminus \{1\}$. The extension

$$\mathbb{R}[X, (X-1)^{-1}]/\mathbb{R}[X^3]$$

is not finite, so the morphism ϕ: $W \to V$ associated to the inclusion of rings $\mathbb{R}[X^3] \subseteq \mathbb{R}[X, (X-1)^3]$ is not finite. The identity $1 = \omega^3$ with ω being a primitive root of order 3 shows that ϕ is nevertheless surjective. In addition, ϕ has finite fibers.

However, $\phi(W(\mathbb{R})) = V(\mathbb{R}) \smallsetminus \{1\}$ (because $\omega \notin \mathbb{R}$), so $\mathrm{card}(V(\mathbb{R}) \smallsetminus \phi(V(\mathbb{R})))$ $= 1$.

Question 2.10. Let K be an ample field and ϕ: $C \to D$ a morphism of integral curves over K. Suppose that $\kappa = \mathrm{card}(D(K) \smallsetminus \phi(C(K)))$ is infinite. Is $\kappa = \mathrm{card}(K)$?

We note without proof that the question has an affirmative answer if ϕ is separable or ϕ is purely inseparable.

3. Polynomial maps

In this section we give a negative answer to Question A of the introduction in two special cases. The first one is a consequence of Theorem 2.6.

Corollary 3.1. *Let K be an ample field and let f be a polynomial with coefficients in K that has no roots in $K_{\rm ins}$. Then,* $\mathrm{card}(K \smallsetminus f(K_{\rm ins})) = \mathrm{card}(K)$.

Proof. We consider f as a morphism f: $\mathbb{A}^1_K \to \mathbb{A}^1_K$. By assumption $f \neq 0$, so f is a finite morphism. Again, by assumption, $0 \in K \smallsetminus f(K_{\rm ins})$. Hence, by Theorem 2.6, $\mathrm{card}(K \smallsetminus f(K_{\rm ins})) = \mathrm{card}(K)$, as claimed. ☐

The other extreme case occurs when all of the roots of the polynomial belong to the purely inseparable extension of K.

Lemma 3.2. *Let K be a field of positive characteristic p. Then, the following statements hold*:

(a) $\mathrm{card}(K^p) = \mathrm{card}(K)$.

(b) *If $K \neq K_{\rm ins}$, then* $\mathrm{card}(K \smallsetminus K^p) = \mathrm{card}(K)$.

(c) *If K is separably closed and there exists a monic polynomial $f \in K[X]$ of positive degree such that $f(K) \neq K$, then* $\mathrm{card}(K \smallsetminus K^p) = \mathrm{card}(K)$.

Proof. (a) The map $x \mapsto x^p$ is an isomorphism of K onto K^p, so $\mathrm{card}(K) = \mathrm{card}(K^p)$.

(b) By assumption there exists $x \in K$ which is not a p-power in K. Hence, $x - a^p$ is also not a p-power in K for all $a \in K$. It follows from (a) that $\mathrm{card}(K \smallsetminus K^p) = \mathrm{card}(K)$.

(c) Since K is separably closed, $K_{\mathrm{ins}} = \tilde{K}$. The existence of f implies that $K \neq \tilde{K}$. Hence, $K \neq K_{\mathrm{ins}}$. It follows from (b) that $\mathrm{card}(K \smallsetminus K^p) = \mathrm{card}(K)$. □

Theorem 3.3. *Let K be a field of positive characteristic p and let $f \in K[X]$ be a non-constant monic polynomial. Suppose that all zeros of f in $\tilde{K}$ belong to $K_{\mathrm{ins}} \smallsetminus K$. Then,* $\mathrm{card}(K \smallsetminus f(K)) = \mathrm{card}(K)$.

Proof. Let g be a monic irreducible factor of f in $K[X]$. By assumption, g has a unique root x in $K_{\mathrm{ins}} \smallsetminus K$. It follows that there exists a positive integer i such that $x^{p^i} \in K$. Let k be the smallest positive integer with this property and let $b = x^{p^k}$. Then, $g(X) = X^{p^k} - b$. Therefore, $f(X) = \prod_{i \in I} \prod_{j \in J_i} (X^{p^i} - b_{ij})$, where I is a non-empty finite set of positive integers, J_i is a finite set, and $b_{ij} \in K \smallsetminus K^p$ for all $i \in I$ and $j \in J_i$. It follows that

$$f \in K[X^p] \smallsetminus K. \tag{1}$$

If $f \in K^p[X^p]$, then $f(K) \subseteq K^p$. Hence, $K \smallsetminus K^p \subseteq K \smallsetminus f(K)$. It follows from Lemma 3.2(b) that $\mathrm{card}(K \smallsetminus f(K)) = \mathrm{card}(K)$.

It remains to consider the case where $f \in K[X^p] \smallsetminus K^p[X^p]$. In other words (2) $f(X) = \sum_{i=0}^n a_i X^{pi}$, where $a_0, \ldots, a_n \in K$, $a_n = 1$, and there exists j between 0 and $n-1$ such that $a_j \notin K^p$.

We consider the hyperplane defined in $\mathbb{A}_K^{n+1}$ by the equation $\sum_{i=0}^n a_i X_i = 0$. Thus, $W_0 = \{(x_0, \ldots, x_n) \in (K^p)^{n+1} \mid \sum_{i=0}^n a_i x_i = 0\}$ is a vector space over K^p. (Note that the superscript p in K^p means raising to the pth power whereas the superscript $n+1$ over K^p means taking all $(n+1)$-tuples with coordinates in K^p.)

Now we consider the vector subspace

$$W = \{(x_0, \ldots, x_n) \in (K^p)^{n+1} \mid \sum_{i=0}^n a_i x_i \in K^p\} \tag{3}$$

of $(K^p)^{n+1}$. We note that W contains W_0 and $\dim_{K^p}(W_0) \leq \dim_{K^p}(W)$. Then, there exists a linear subvariety W^* in $\mathbb{A}_{K^p}^{n+1}$ such that $W^*(K^p) = W$. Indeed, W is a sub vector space of $(K^p)^{n+1}$ over K^p. As such, W is defined by a system of linear equations $\sum_{i=0}^n b_{ik} X_i = 0$ with $b_{ik} \in K^p$ for $i = 0, \ldots, n$ and $k = 1, \ldots, n$. The same system also defines W^*.

Let D be the closed subscheme of $\mathbb{A}^{n+1}_{K^p}$ defined by the equations $X_0 = 1$ and $X_i X_{i+2} = X_i^2$ with $i = 0, \dots, n-2$. Let C be the irreducible component of D such that $C(K^p) = \{(1, t, \dots, t^n) \mid t \in K^p\}$.
Claim: $C(K^p) \not\subseteq W^*(K^p)$ Assume toward contradiction that

$$C(K^p) \subseteq W^*(K^p).$$

We use that K is infinite in order to choose $n+1$ distinct elements $t_0, \dots, t_n$ of K^p. For each $0 \le k \le n$ we have $(1, t_k, \dots, t_k^n) \in C(K^p)$, so $(1, t_k, \dots, t_k^n) \in W$. By (3), there exists $c_k \in K^p$ such that

$$\sum_{i=0}^{n} a_i t_k^i = c_k, \qquad k = 0, \dots, n. \tag{4}$$

We consider (4) as a system of $n+1$ inhomogeneous linear equations in the variables $a_0, \dots, a_n$. The coefficients matrix on the left hand side of (4) is the Van der Monde matrix of $t_0, \dots, t_n$. Since the latter elements are distinct, the determinant of that matrix is non-zero. Hence, by Cramer's rule, each of the elements $a_0, \dots, a_n$ is a quotient of two elements of K^p with a non-zero denominator, so each of the a_i's is in K^p. In particular, $a_j \in K^p$, contradicting (2).

Since C is a curve, the Claim implies that $C \cap W^*$ is a finite set. Hence, $C(K^p) \cap W^*(K^p)$ is a finite set, so,
(5) $C(K^p) \cap W$ is a finite set.

Now, an element $x \in K$ satisfies $f(x) \in K^p$ if and only if $\sum_{i=0}^n a_i x^{pi} \in K^p$, that is if and only if $(1, x^p, \dots, x^{pn}) \in C(K^p) \cap W$. It follows from (5) that there exist only finitely many $x \in K$ with $f(x) \in K^p$. Let $x_1, \dots, x_m$ be all of these elements. Then,

$$K^p \smallsetminus \{f(x_1), \dots, f(x_m)\} \subseteq K \smallsetminus f(K). \tag{6}$$

Since K is infinite, the cardinality of the left hand side of (6) is $\mathrm{card}(K)$ (Lemma 3.2(a)). We conclude that $\mathrm{card}(K \smallsetminus f(K)) = \mathrm{card}(K)$, as claimed. □

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Lior Bary-Soroker
School of Mathematics
Tel Aviv University
Ramat Aviv, Tel Aviv, Israel
Email address: barylior@gmail.com

Wulf-Dieter Geyer
Departement Mathematik
Universität Erlangen
Erlangen, Germany
Email address: geyer@mi.uni-erlangen.de

Moshe Jarden
School of Mathematics
Tel Aviv University
Ramat Aviv, Tel Aviv, Israel
Email address: jarden@post.tau.ac.il