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η-PARALLEL H-CONTACT 3-MANIFOLDS

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Abstract. In this paper, we give a local and global classification of 3-dimensional H-contact manifolds whose Ricci tensor is η-parallel.

1. Introduction

Boeckx and the present author [3] proved that a locally symmetric contact Riemannian manifold is either Sasakian and of constant curvature 1 or locally isometric to the unit tangent sphere bundle (with its standard contact metric structure) of a Euclidean space. We may also refer to [2] for the 3-dimensional case. This result says that Cartan's local symmetry ($\nabla R = 0$) is quite a strong condition in contact Riemannian geometry, where R denotes the Riemannian curvature tensor. In this context, a weaker condition called *η-parallelism* is introduced. For a contact Riemannian manifold $M = (M; \eta, g, \phi, \xi)$, the *contact form* η determines the *contact distribution* D which is given by the kernel of η. We say that the Ricci tensor S is η-parallel if S satisfies $g((\nabla_X S)Y, Z) = 0$ for any $X, Y, Z \in D$. In this paper, we shall study 3-dimensional contact Riemannian manifolds whose Ricci tensor is η-parallel. In a previous paper [4], Lee and the present author gave a classification of such contact 3-manifolds under the condition $\nabla_\xi h = \mu h\phi$, where $\mu \in \mathbb{R}$ and $h = \frac{1}{2} L_\xi \phi$.

On the other hand, Perrone ([10]) introduced a so-called *H-contact structure*, which means that the Reeb vector field ξ is a harmonic vector field. In the same paper, it was proved that a contact Riemannian manifold is H-contact if and only if ξ is an eigenvector of the Ricci operator S, that is, $S\xi = \alpha\xi$ for some function α. In the present paper, we give a local and global classification of H-contact 3-manifolds whose Ricci tensor is η-parallel. More precisely, we prove that:

Theorem A (local classification)**.** *Let M be a 3-dimensional H-contact manifold. Then the Ricci tensor S is η-parallel if and only if M is locally isometric to one of the following*:

(1) *a Sasakian ϕ-symmetric space*;

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(2) $SU(2)$ *(or* $SO(3)$*),* $SL(2,\mathbb{R})$ *(or* $O(1,2)$*),* $E(2)$ *(the group of rigid motions of Euclidean* 2*-space) including a flat manifold,* $E(1,1)$ *(the group of rigid motions of Minkowski* 2*-space) with a left invariant contact metric structure, respectively.*

In [11] the authors gave a classification of a 3-dimensional Sasakian ϕ-symmetric space (complete and simply connected Sasakian locally ϕ-symmetric space). Together with this we have:

Theorem B (global classification)**.** *Let* M *be a complete and simply connected* 3*-dimensional H-contact manifold. Then* S *is* η*-parallel if and only if* M *is isometric to one of the following*:

(1) *the standard unit sphere* S^3; $SU(2)$*,* $\widetilde{SL(2,\mathbb{R})}$ *(the universal covering of* $SL(2,\mathbb{R})$*) or the Heisenberg group* H *with a left invariant Sasakian metric, respectively*;

(2) $SU(2)$*,* $\widetilde{SL(2,\mathbb{R})}$*,* $\widetilde{E(2)}$*,* $E(1,1)$ *with a left invariant contact metric structure, respectively.*

At the end of Section 4, we remark the relationship between the H-contact condition and the condition $\nabla_\xi h = \mu h\phi$ ($\mu \in \mathbb{R}$).

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2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. First, we give a brief review of some fundamental facts and formulas on contact manifolds, which will be used later. We may also refer to [4]. A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, which is called the *Reeb vector field*, satisfying $\eta(\xi) = 1$ and $L_\xi \eta = 0$ (or $i_\xi d\eta = 0$), where L_ξ denotes Lie differentiation for ξ and i_ξ denotes the interior product operator by ξ. Then we have a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$\eta(X) = g(X,\xi), \quad d\eta(X,Y) = g(X,\phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. A Riemannian manifold M equipped with structure tensors (η, g, ϕ, ξ) satisfying (2.1) is said to be a *contact Riemannian manifold* and is denoted by $M = (M; \eta, g, \phi, \xi)$. For a contact Riemannian manifold M, we define a $(1,1)$-tensor field h by $h = \frac{1}{2} L_\xi \phi$. Then we may observe that h is self-adjoint and satisfies

$$h\xi = 0 \quad \text{and} \quad h\phi = -\phi h, \tag{2.2}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.3}$$

where ∇ is Levi-Civita connection. From (2.2) and (2.3) we see that each trajectory of ξ is a geodesic. Along a trajectory of ξ, the Jacobi operator

$\ell = R(\cdot, \xi)\xi$ is a symmetric $(1,1)$-tensor field, where R denotes the Riemannian curvature tensor. We have

$$\text{trace } \ell = g(S\xi, \xi) = 2n - \text{trace } (h^2), \tag{2.4}$$

$$\nabla_\xi h = \phi - \phi\ell - \phi h^2, \tag{2.5}$$

$$g(R(X,Y)\xi, Z) = g((\nabla_Z\phi)X, Y) + g((\nabla_Y\phi h)X - (\nabla_X\phi h)Y, Z) \tag{2.6}$$

for all vector fields X, Y, Z on M, where S is the Ricci tensor field of type (1,1).

A contact Riemannian manifold for which ξ is Killing is called a K-contact manifold. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. Moreover, we find that for a K-contact manifold $S\xi = 2n\xi$, from which we see that a K-contact manifold admits already an H-contact structure, which means that the Reeb vector field ξ is a harmonic vector field (see, [10]). For a contact Riemannian manifold M one may define naturally an almost complex structure J on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be normal or Sasakian. A Sasakian manifold is characterized by a condition

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.7}$$

for all vector fields X and Y. For more details about contact Riemannian manifolds, we refer to [1].

For a contact Riemannian manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$(direct sum), where we denote $D_p = \{v \in T_pM \,|\, \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ, which is called the *contact distribution*.

Definition 1 ([4])**.** A contact Riemannian manifold $M = (M; \eta, g, \phi, \xi)$ is said to have an η-parallel Ricci tensor if $g((\nabla_U S)V, W) = 0$ for all vector fields $U, V, W \in D$.

A Sasakian manifold $M = (M; \eta, g, \phi, \xi)$ is said to be locally ϕ-symmetric if M satisfies

$$\phi^2(\nabla_V R)(X,Y)Z = 0$$

for all vector fields $V, X, Y, Z \in D$ ([11]). Then we have:

Proposition 1. *Let $M = (M; \eta, g, \phi, \xi)$ be a Sasakian manifold. If M is locally ϕ-symmetric, then M has an η-parallel Ricci tensor. In dimension* 3*, the converse also holds.*

Proof. For a Sasakian manifold M, we compute

$$\begin{aligned}(\nabla_U\rho)(V,W) = g((\nabla_U R)(\xi, V)W, \xi) + \sum_{i=1}^n g((\nabla_U R)(e_i, V)W, e_i) \\ + \sum_{i=1}^n g((\nabla_U R)(\phi e_i, V)W, \phi e_i)\end{aligned} \tag{2.8}$$

for any adapted orthonormal basis $\{\xi, e_i, \phi e_i\}$ $(i = 1, 2, \ldots, n)$. From (2.3) and (2.7), we obtain $g((\nabla_U R)(\xi, V)W, \xi) = 0$ for $U, V, W \in D$. Thus, from (2.8) we find that a Sasakian locally ϕ-symmetric space has an η-parallel Ricci tensor. For the dimension 3, it is well-known that the curvature tensor R is expressed only in terms of the Ricci tensor S, the metric tensor g and the scalar curvature τ (see (3.3)). Then, since $S\xi = 2\xi$, we see that an η-parallelism of Ricci tensor implies a local ϕ-symmetry. □

3. η-parallel contact 3-manifolds

In this section, we prove Theorem A. For a 3-dimensional contact Riemannian manifold M, it is known that the associated almost CR-structure is integrable. Then we have (cf. [12])

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{3.1}$$

Lemma 2 ([4]). *A 3-dimensional contact Riemannian manifold is Sasakian if and only if $h = 0$.*

From (2.6) and (3.1) we compute

$$R(X, Y)\xi = \eta(Y)(X + hX) - \eta(X)(Y + hY) + \phi((\nabla_Y h)X - (\nabla_X h)Y) \tag{3.2}$$

for all vector fields X and Y.

Proof of Theorem A. First of all, we recall that the curvature tensor R of a 3-dimensional Riemannian manifold is expressed by

$$\begin{aligned} R(X, Y)Z = \rho(Y, Z)X - \rho(X, Z)Y + g(Y, Z)SX - g(X, Z)SY \\ - \frac{\tau}{2}\{g(Y, Z)X - g(X, Z)Y\} \end{aligned} \tag{3.3}$$

for all vector fields X, Y, Z, where $\rho(X, Y) = g(SX, Y)$ and τ is the scalar curvature of the manifold. Let $M = (M^3; \eta, g, \phi, \xi)$ be a 3-dimensional H-contact manifold whose Ricci tensor S is η-parallel. If $h = 0$ on M, then from Lemma 2 we see that M is Sasakian. Moreover, by Proposition 1 we see that M is locally ϕ-symmetric. From now we suppose that M is non-Sasakian, that is, h is not identically zero on M. Let W be the subset of M on which the number of distinct eigenvalues of h is constant. Then W is an open and dense subset of M. We fix any point q in W. Then from (2.2) there exist a positive function λ and a local orthonormal frame field $\{e_1, e_2 = \phi e_1, e_3 = \xi\}$ on a neighborhood $N(q)(\subset W)$ containing q such that $he_1 = \lambda e_1$, $he_2 = -\lambda e_2$, $h\xi = 0$. We denote $\Gamma_{ijk} = g(\nabla_{e_i} e_j, e_k)$, $\rho_{ij} = \rho(e_i, e_j)$, $\nabla_i \rho_{jk} = (\nabla_{e_i}\rho)(e_j, e_k)$ and $\nabla_h R_{ijkl} = g((\nabla_{e_h} R)(e_i, e_j)e_k, e_l)$ for $h, i, j, k, l = 1, 2, 3$. Then from (2.3) we get

$$\Gamma_{132} = -\Gamma_{123} = -(1 + \lambda), \quad \Gamma_{231} = -\Gamma_{213} = 1 - \lambda \tag{3.4}$$

and

$$\Gamma_{131} = \Gamma_{113} = \Gamma_{232} = \Gamma_{223} = 0. \tag{3.5}$$

Also, from (2.5) and taking account of (2.4) and (3.3), we have

$$\xi(\lambda) = \rho_{12} \tag{3.6}$$

and

$$4\lambda\Gamma_{312} = \rho_{22} - \rho_{11}. \tag{3.7}$$

The above general setting and general formulas (3.4)-(3.7) for a 3-dimensional contact metric manifold are referred to [4]. Since $S\xi = \sigma\xi$, we have

$$\rho_{13} = \rho_{31} = 0,\ \ \rho_{23} = \rho_{32} = 0. \tag{3.8}$$

From (3.5) and (3.8) we have

$$\nabla_3\rho_{13} = \nabla_3\rho_{31} = 0,\ \ \nabla_3\rho_{23} = \nabla_3\rho_{32} = 0. \tag{3.9}$$

Since M has η-parallel Ricci tensor, we have

$$\nabla_a\rho_{bc} = 0 \tag{3.10}$$

for $a, b, c = \{1, 2\}$.

On the other hand, applying the second Bianchi identity in (3.3), we have

$$2\nabla_2\rho_{12} + 2\nabla_3\rho_{13} + \nabla_1\rho_{11} - \nabla_1\rho_{22} - \nabla_1\rho_{33} = 0, \tag{3.11}$$

$$2\nabla_1\rho_{21} + 2\nabla_3\rho_{23} - \nabla_2\rho_{11} + \nabla_2\rho_{22} - \nabla_2\rho_{33} = 0. \tag{3.12}$$

From (3.9), (3.10), (3.11) and (3.12), we have

$$\nabla_1\rho_{33} = \nabla_2\rho_{33} = 0.$$

Since $\rho_{33} = 2-2\lambda^2$, together with (3.5) and (3.8) we see that $e_1(\lambda) = e_2(\lambda) = 0$. So,

$$\begin{aligned} 0 &= [e_1, e_2](\lambda) \\ &= \eta([e_1, e_2])(\xi\lambda) \\ &= 2d\eta(e_1, e_2)(\xi\lambda), \end{aligned} \tag{3.13}$$

which yields that $\xi(\lambda) = 0$. Hence, since M is connected, we find that λ is a constant on M. And from (3.6) we obtain

$$\rho_{12} = \rho_{21} = 0. \tag{3.14}$$

From (3.14) by using (3.8) and (3.10) we obtain

$$\nabla_1\rho_{12} = \Gamma_{112}(\rho_{11} - \rho_{22}) = 0 \tag{3.15}$$

and

$$\nabla_2\rho_{12} = \Gamma_{221}(\rho_{22} - \rho_{11}) = 0. \tag{3.16}$$

We now set $N(q) = N^0(q) \cup N^1(q)$, where $N^0 = \{p \in N(q) \,|\, \rho_{11}(p) = \rho_{22}(p)\}$ and $N^1 = \{p \in N(q) \,|\, \rho_{11}(p) \neq \rho_{22}(p)\}$. Here we divide our arguments into three cases: (I) $N = N^0$, (II) $N = N^1$, or (III) N^0 and N^1 respectively have interior points.

(I) $N = N^0$; Then $\rho_{11} = \rho_{22}$ on N. Then since $\lambda \neq 0$ from (3.7) we get

$$\Gamma_{312} = \Gamma_{321} = 0. \tag{3.17}$$

Taking account of (3.8) we get $R(e_1, e_2)\xi = 0$ in (3.3). Hence, by using (3.2) we have

$$\Gamma_{212}e_2 - \Gamma_{121}e_1 = 0. \tag{3.18}$$

From (3.18) we get

$$\Gamma_{212} = \Gamma_{221} = \Gamma_{121} = \Gamma_{112} = 0. \tag{3.19}$$

Thus, together with (3.4), (3.5), (3.17) and (3.19), we have

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = (1 - \lambda)e_1,\ [e_3, e_1] = (1 + \lambda)e_2. \tag{3.20}$$

Then due to J. Milnor's classification for 3-dimensional Lie groups admitting unimodular Lie algebra with left invariant metric ([7]), we see that M is locally isometric to one of the following:

(i) $SU(2)$ (or $SO(3)$) with a left invariant metric when $0 < \lambda < 1$;
(ii) $SL(2, \mathbb{R})$ (or $O(1, 2)$) with a left invariant metric when $\lambda > 1$;
(iii) the group $E(2)$ of rigid motions of the Euclidean 2-space when $\lambda = 1$.

In fact, if $\lambda = 1$, then from (3.4), (3.5), (3.17), (3.19) and (3.20) we find that $R = 0$.

(II) $N = N^1$; First, we show $\rho_{22} - \rho_{11}$ is constant. From (3.15) and (3.16) we get

$$\Gamma_{212} = \Gamma_{221} = \Gamma_{121} = \Gamma_{112} = 0.$$

Since S is η-parallel, from (3.14) and (3.10) we see that $e_1(\rho_{22} - \rho_{11}) = e_2(\rho_{22} - \rho_{11}) = 0$, and further from (3.7) we find that

$$e_1(\Gamma_{312}) = e_2(\Gamma_{312}) = 0. \tag{3.21}$$

Together with (3.4), (3.5) and (3.21) we calculate

$$\begin{aligned} R(e_1, e_2)e_1 &= \nabla_{e_1}(\nabla_{e_2}e_1) - \nabla_{e_2}(\nabla_{e_1}e_1) - \nabla_{[e_1,e_2]}e_1 \\ &= (1 + \lambda)(1 - \lambda)e_2 - 2\Gamma_{312}e_2, \end{aligned} \tag{3.22}$$

$$\begin{aligned} R(e_1, e_2)e_2 &= \nabla_{e_1}(\nabla_{e_2}e_2) - \nabla_{e_2}(\nabla_{e_1}e_2) - \nabla_{[e_1,e_2]}e_2 \\ &= -(1 + \lambda)(1 - \lambda)e_1 + 2\Gamma_{312}e_1, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} R(e_3, e_1)e_1 &= \nabla_{e_3}(\nabla_{e_1}e_1) - \nabla_{e_1}(\nabla_{e_3}e_1) - \nabla_{[e_3,e_1]}e_1 \\ &= -\Gamma_{312}(1 + \lambda)e_3 + (\Gamma_{312} + 1 + \lambda)(1 - \lambda)e_3, \end{aligned} \tag{3.24}$$

$$\begin{aligned} R(e_3, e_1)e_3 &= \nabla_{e_3}(\nabla_{e_1}e_3) - \nabla_{e_1}(\nabla_{e_3}e_3) - \nabla_{[e_3,e_1]}e_3 \\ &= \Gamma_{312}(1 + \lambda)e_1 - (\Gamma_{312} + 1 + \lambda)(1 - \lambda)e_1, \end{aligned} \tag{3.25}$$

and similarly we obtain

$$R(e_1, e_2)e_3 = R(e_2, e_3)e_1 = R(e_3, e_1)e_2 = 0. \tag{3.26}$$

Moreover, together with (3.22), (3.23), (3.24), (3.25) and (3.26) we have

$$(\nabla_{e_3}R)(e_1,e_2)e_1 = -2e_3(\Gamma_{312})e_2,$$
$$(\nabla_{e_1}R)(e_2,e_3)e_1 = (\nabla_{e_2}R)(e_3,e_1)e_1 = 0.$$

Thus, by the 2nd Bianchi identity, we have $e_3(\Gamma_{312}) = 0$, and hence with (3.21) we see that Γ_{312} is constant. From (3.7), we further see that $\rho_{22} - \rho_{11}$ is constant. After all, together with (3.4), (3.5) we have

$$(3.27)\quad [e_1,e_2] = 2e_3,\ [e_2,e_3] = (1-\lambda+\Gamma_{312})e_1,\ [e_3,e_1] = (1+\lambda+\Gamma_{312})e_2.$$

By similar arguments as in the first case, we see that M is locally isometric to one of the following:

(i) $SU(2)$ (or $SO(3)$) with a left invariant metric when $1+\Gamma_{312} > \lambda$;

(ii) $SL(2,\mathbb{R})$ (or $O(1,2)$) with a left invariant metric when $-\lambda < 1+\Gamma_{312} < \lambda$ or $1+\Gamma_{312} < -\lambda$;

(iii) $E(2)$ with a left invariant metric when $1+\Gamma_{312} = \lambda$;

(iv) the group $E(1,1)$ of rigid motions of Minkowski 2-space with a left invariant metric when $1+\Gamma_{312} = -\lambda$.

(III) N^0 and N^1 respectively have interior points; In view of the above cases (I) and (II), by the continuity of $\rho_{22} - \rho_{11}$ we see that this case cannot occur.

Conversely, since a Sasakian manifold is H-contact, due to Proposition 1 we see that a Sasakian locally ϕ-symmetric space is H-contact and it has η-parallel Ricci tensor. Now, we consider a 3-dimensional Lie group with the Lie algebra structure

$$(3.28)\qquad [e_1,e_2] = c_1e_3,\ [e_2,e_3] = c_2e_1,\ [e_3,e_1] = c_3e_2$$

for some constants $c_1(\neq 0), c_2, c_3$. Let $\{\omega_i\}$ be the dual 1-forms to the vector fields $\{e_i\}$. By using (3.28) we get $d\omega_3(e_1,e_2) = -d\omega_3(e_2,e_1) = -\frac{c_1}{2}$ and $d\omega_3(e_i,e_j) = 0$ for $(i,j) \neq (1,2),(2,1)$. Further we easily check that $\omega_3 \wedge d\omega_3(e_1,e_2,e_3) = -\frac{c_1}{6}(\neq 0)$, and hence ω_3 is a contact form and e_3 is the Reeb vector field. Define a Riemannian metric g and a (1,1)-tensor field ϕ by

$$g(e_i,e_j) = \delta_{ij},\ d\omega_3(e_i,e_j) = g(e_i,\phi e_j)$$

for $i,j = 1,2,3$. Then, for (ω_3, g, ϕ, e_3) to be a contact Riemannian structure, it must follow that $g(\phi e_i, \phi e_j) = g(e_i,e_j) - \omega_3(e_i)\omega_3(e_j)$ for $i,j = 1,2,3$, and hence we have $c_1 = 2$.

Recall the Koszul formula

$$\begin{aligned}2g(\nabla_XY,Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)\\ &\quad + g(Y,[Z,X]) + g(Z,[X,Y]) - g(X,[Y,Z])\end{aligned}$$

for X,Y,Z are smooth vector fields on the manifold. We put

$$\Gamma_{ijk} = g\,(\nabla_{e_i}e_j,\ e_k)\quad \text{for}\quad i,\ j,\ k = 1,\ 2,\ 3.$$

Then together with (3.28) we obtain

$$\begin{cases} \Gamma_{123} = \frac{1}{2}(c_3 - c_2 + 2), \\ \Gamma_{213} = \frac{1}{2}(c_3 - c_2 - 2), \\ \Gamma_{312} = \frac{1}{2}(c_3 + c_2 - 2), \\ \text{all others are zero.} \end{cases} \tag{3.29}$$

From (3.29) we have

$$\begin{aligned} Se_1 &= \big(\frac{1}{2}(c_2^2 - c_3^2) - 2 + 2c_3\big)e_1, \\ Se_2 &= \big(\frac{1}{2}(c_3^2 - c_2^2) - 2 + 2c_2\big)e_2, \\ Se_3 &= \big(-\frac{1}{2}(c_2 - c_3)^2 + 2\big)e_3. \end{aligned} \tag{3.30}$$

That is, a 3-dimensional unimodular Lie group with left invariant contact metric structure is a H-contact manifold. Moreover, from (3.29) and (3.30) we can show that S is η-parallel, that is, $\nabla_a \rho_{bc} = 0$ for $a, b = 1, 2$.

Therefore, summing up all the arguments so far and using the continuity argument of λ, then we have our Theorem A. □

Perrone [8] classified all simply connected homogeneous contact Riemannian 3-manifolds. Recall that M is called unimodular if its left invariant Haar measure is also right invariant. In terms of the Lie algebra $\mathfrak{m}$, M is unimodular if and only if the adjoint transformation ad_X has trace zero for every $X \in \mathfrak{m}$.

4. Remarks

We remark that there are lots of H-contact 3-manifolds which do not satisfy $\nabla_\xi h = \mu h\phi$ ($\mu \in \mathbb{R}$) (cf. [6]). Actually, there is no inclusion relation between the class of H-contact 3-manifolds and the class of contact 3-manifolds which satisfy $\nabla_\xi h = \mu h\phi$. In this section, we show a homogeneous example and a non-homogeneous example which satisfy the condition $\nabla_\xi h = \mu h\phi$ ($\mu \in \mathbb{R}$), but they are not H-contact (cf. [5]).

Example 1. Let M be a 3-dimensional non-unimodular Lie group with left invariant contact metric structure. Then we know that there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\} \in \mathfrak{m}$ such that

$$[e_1, e_2] = \alpha e_2 + 2e_3,\ [e_2, e_3] = 0,\ [e_3, e_1] = \gamma e_2, \tag{4.1}$$

where $\alpha \neq 0$. Moreover, M is Sasakian if and only if $\gamma = 0$ (cf. [8]). From (4.1), by using the Koszul formula we have

$$\begin{cases} \Gamma_{123} = \dfrac{\gamma+2}{2}, \\ \Gamma_{212} = -\alpha, \\ \Gamma_{213} = \dfrac{\gamma-2}{2}, \\ \Gamma_{312} = \dfrac{\gamma-2}{2}, \\ \text{all others are zero.} \end{cases} \tag{4.2}$$

From (4.2), we have the Ricci tensor

$$\begin{aligned} Se_1 &= \left(-\alpha^2 - 2 + 2\gamma - \frac{\gamma^2}{2}\right) e_1, \\ Se_2 &= \left(-\alpha^2 - 2 + \frac{\gamma^2}{2}\right) e_2 + \alpha\gamma e_3, \\ Se_3 &= \alpha\gamma e_2 + \left(2 - \frac{\gamma^2}{2}\right) e_3. \end{aligned} \tag{4.3}$$

Moreover, from (4.2) we have

$$he_1 = \gamma/2e_1, \quad he_2 = -\gamma/2e_2. \tag{4.4}$$

From (4.2) and (4.4), we see that $\nabla_\xi h = (2-\gamma)h\phi$.

Modifying Perrone's example in [9], then we have a following example.

Example 2. Let M be the open submanifold $\{(x,y,z) \in \mathbf{R}^3 \mid x \neq 0\}$ of Cartesian 3-space together with a contact form $\eta = xydx + dz$. The characteristic vector field of this contact 3-manifold is $\xi = \partial/\partial z$. For $\beta \in \mathbb{R}$, take a global frame field

$$e_1 = -\frac{2}{x}\frac{\partial}{\partial y}, \; e_2 = \frac{\partial}{\partial x} - \frac{\beta z}{x}\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}, \; e_3 = \xi.$$

Then $\{e_1, e_2\}$ generates the contact distribution, and the vector fields e_1, e_2, e_3 satisfy

$$[e_3, e_1] = 0, \quad [e_2, e_3] = -\frac{\beta}{2}e_1, \quad [e_1, e_2] = 2e_3 + \frac{1}{x}e_1. \tag{4.5}$$

Define a Riemannian metric g with respect to $\{e_1, e_2, e_3\}$ to be an orthonormal frame. Moreover, define an endomorphism field ϕ by $\phi e_1 = e_2$, $\phi e_2 = -e_1$ and $\phi\xi = 0$. (g, ϕ, ξ) is an associated almost contact metric structure for η. The endomorphism field h satisfies $he_1 = \frac{\beta}{4}e_1$, $he_2 = -\frac{\beta}{4}e_2$. Hence, M is Sasakian if and only if $\beta = 0$. Perrone's example in [9] is just the case $\beta = 4$. We compute that

$$\nabla_\xi h = (2 + \frac{\beta}{2})h\phi.$$

From the straightforward computations we have $S\xi = -\frac{\beta}{2x}e_1 + 2(1 - \frac{\beta^2}{16})\xi$. From this, we see at once that M is H-contact if and only if $\beta = 0$ (M is Sasakian). Furthermore, we note that M has η-parallel curvature tensor, that is, M is weakly ϕ-symmetric in the sense of [9] if and only if $\beta = 4$.

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