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Analysis and Probability of Overestimation by an Imperfect Inspector with Errors of Triangular Distributions

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삼각 과오 분포를 가진 불완전한 검사원의 과대 추정 확률과 분석

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There always exist nonzero inspection errors whether inspectors are humans or automatic inspection machines. Inspection errors can be categorized by two types, type I error and type II error, and they can be regarded as either a constant or a random variable. Under the assumption that two types of random inspection errors are distributed with the "uniform" distribution on a half-open interval starting from zero, it was proved that inspectors overestimate any given fraction defective with the probability more than 50%, if and only if the given fraction defective is smaller than a critical value, which depends upon only the ratio of a type II error over a type I error. In addition, it was also proved that the probability of overestimation approaches one hundred percent as a given fraction defective approaches zero. If these critical phenomena hold true for any error distribution, then it might have great economic impact on commercial inspection plans due to the unfair overestimation and the recent trend of decreasing fraction defectives in industry. In this paper, we deal with the same overestimation problem, but assume a "symmetrical triangular" distribution expecting better results since our triangular distribution is closer to a normal distribution than the uniform distribution. It turns out that the overestimation phenomenon still holds true even for the triangular error distribution.

Keywords : Imperfect Inspection, Overestimation, Triangular Error Distribution

1. Introduction

In the quality inspection, since we do not know whether the fraction defective (or FD) judged by an inspector is equal to a given FD or not, we are likely to be suspicious of inspectors. Whenever automatic inspection machines replaced manual inspection operations, many people thought that the probability of either overestimation or underestimation by inspection machines could not exist at all. However, it turned out immediately that they were wrong. There always exist nonzero errors, i.e., type I and II errors, in even a well-automated inspection machine. The assumption of nonzero errors from unavoidable events, for example due to hardware, software, and deterioration, could not be denied. According to Handbook of biometrics [2], the ranges of type I and type II errors have been measured for biometric inspections respectively : for fingerprints (optical scanners) as (0.0005%, 0.0015%) and (1%, 3%), for hand geometry (whole hand) as (0.05%, 0.15%) and (1%, 3%), for voice

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as (1%, 3%) and (7.5%, 22.5%), for iris as (0.0005%, 0.0015%) and (6%, 18%), and so on.

Sylla and Drury [4] studied the FD judged by an imperfect inspector, calling it the "apparent fraction nonconforming", i.e., $q_e = (1-q)\alpha + q(1-\beta)$, where q is a true FD, α is the probability of a type I error, and β is the probability of a type II error. They found the sample size and the cut-off value for single sampling by attributes, and error-related pay-offs, and proposed the concept of liability, which is an inspector's ability to respond to information. Burk et al. [1] studied the true FD, $q(=\dfrac{q_e-\alpha}{1-\alpha-\beta})$), and found that for very good process, q_e is actually a type I error. They suggested a procedure for estimating the type I and II errors, and gave an industrial example. Ko [3] analyzed a special inspection process that allows up to two times of consecutive testing for each product to decrease type I inspection errors. Using a Markov chain to model the steps of the inspection process and a product unit's quality states during inspection, he demonstrated that his inspection process could help reduce unnecessary rejects and consequently decrease material and production costs.

There are, however, a few papers dealing with the probability of overestimating a true FD, q, by an imperfect inspector, which we are interested in. The research article by Yang and Cho [6] is partially related with our paper. In order to attain a pre-specified quality rate at the end of an assembly line, he suggested a K-stage inspection-rework (K-IR) system, which was composed of a series of K stages, each of which included an inspection process and a rework process. He suspected the effectiveness of the K-IR system, and proved mathematically that FDA (FD after inspection) is always greater than FDB (FD before inspection), if FDB is less than a value that depends on an FD of rework and inspection errors. Based on his assumptions, he suggested a necessary condition for inspection effectiveness: the sum of two errors must be smaller than one.

Yang and Chang [5] studied the probability of overestimation (or PO) by an imperfect inspector with nonzero inspection errors, under the assumptions that (1) the inspector classified one-by-one an infinite sequence of items with a given FD, but which was unknown to the inspector, (2) inspection error is considered as either a constant or a random variable distributed with a "uniform" distribution. They proved that (1) the inspector overestimates the given FD with PO being more than 50%, if and only if the FD is smaller than a value called "critical FD", which gives the 50-50 chance of overestimation and underestimation, and (2) PO increases to 100% as the given FD decreases to zero. However, no one knows whether their result still holds true or not, regardless of any distribution. If so, their result might have great economic impact on commercial inspection plans. Since proving it for the case of any distribution may not be mathematically tractable, we are interested in proving it in the case of a "symmetrical triangular" distribution.

Assuming that inspection error is regarded as either a constant or a random variable distributed with a triangular distribution (except the case that two types of errors are constant at the same time), we construct and describe three statistical models in Section 2. In Section 3 through 5, we derive formulas for PO and a critical FD satisfying PO = 50% for each model. Furthermore, we compare our results from the symmetrical triangular distribution with the previous results from the uniform distribution. In Section 6, we integrate our findings to one fundamental theorem.

2. Problem Statement

For the convenience of the readers, we summarize the paper of Yang and Chang [5] as follows. Assuming that an imperfect inspector with nonzero inspection errors classifies one-by-one an infinite sequence of items with a true FD, denoted by a constant q, which is unknown to the inspector, they derived the FD, denoted by Q, of an infinite number of items judged by an imperfect inspector and the probability of overestimation, denoted by PO(q), by an imperfect inspector respectively as follows :

$$Q = q + \lim_{n\to\infty}\frac{1}{n}\left\{(1-q)\Sigma_{i=1}^{n}A_i - q\Sigma_{i=1}^{n}B_i\right\} \qquad (1)$$

$$\begin{aligned} PO(q) &= \Pr\{Q > q\} \\ &= \Pr \lim_{n\to\infty}\frac{1}{n}\left\{(1-q)\Sigma_{i=1}^{n}A_i - q\Sigma_{i=1}^{n}B_i > 0\right\} \qquad (2)\end{aligned}$$

where

n = n-th item inspected,

A_i = a constant or a random variable representing the type I error, the probability that the i-th conforming item is misclassified as nonconforming and falsely rejected by the inspector, and

B_i = a constant or a random variable representing the type II error, the probability that the i-th nonconforming item is misclassified as conforming and falsely accepted by the inspector.

Note that Q can be regarded as a random variable if and only if both A_i and B_i are not constants at the same time. Also note that PO(q) can be applied for any type of distribution. Since PO(q) depends ultimately upon an error distribution, for convenience, we will use the notation POU(q) instead of PO(q) when errors follows the "uniform" distribution. They defined CFU to be a critical FD satisfying POU(CFU) = 50%, i.e., the fraction defective that the probability of overestimation is exactly equal to the probability of underestimation.

Assuming that

(1) A_i is either a constant α for all i or i.i.d (independently identically distributed) with the uniform density function, $g_A(a) = (\frac{1}{\alpha_u}) I_{(0,\alpha_u)}(a)$, where $I_{(0,x_u]}(x)$ is an indicator function with one for $0 < x \le x_u$, and zero otherwise,

(2) B_i is either a constant β for all i or i.i.d with the uniform density function $g_B(b) = (\frac{1}{\beta_u}) I_{(0,\beta_u)}(b)$,

(3) $E[A] = \alpha$, $E[B] = \beta$, where E[X] is the expectation of a random variable X,

they constructed four statistical models and proved the theorem as follows.

(1) An imperfect inspector with $\rho (= \frac{E[B]}{E[A]})$ has his/her/its own POU curve and CFU,

(2) POU is a function of two variables q and ρ, denoted by POU(q, ρ),

(3) POU is a decreasing function of q with POU(0, ρ) = 1 and POU(1, ρ) = 0.

(4) POU is a decreasing function of ρ with POU(q, 0) = 1 and POU(q, 1) = 0.

(5) There always exists a unique $CFU = \frac{1}{1+\rho}$, which depends on only inspection errors and not q.

(6) The inspector overestimates q with PO > 0.5 for $0 \le q < CFU$, estimates q with PO = 0.5 with PO = 0.5 for q = CFU, and underestimates q with PO < 0.5 for $CFU < q \le 1$.

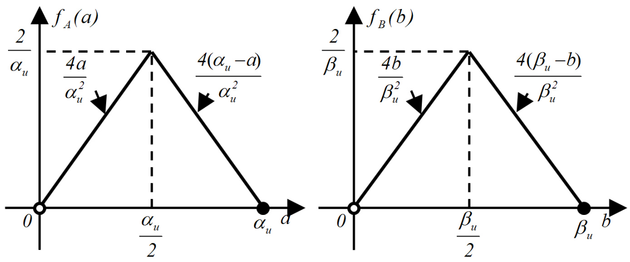
It can be observed from the third statement above that POU increases to 100% as q decreases to 0% regardless of any value of ρ. Based on their findings, they conjectured that their theorem could still be true, regardless of any distribution. Since their theorem is restricted only to the uniform distribution, we are interested in proving whether or not their conjecture still holds true even for the symmetrical triangular distribution as shown in <Figure 1>. For $0 < \alpha_u, \beta_u \le 1$, we define,

$$f_A(a) = \frac{4a}{\alpha_u^2} I_{(0,\alpha_u/2]}(a) + \frac{4(\alpha_u - a)}{\alpha_u^2} I_{[\alpha_u/2,\, \alpha_u]}(a)$$

for $0 < a \le \alpha_u$, and

$$f_A(b) = \frac{4b}{\beta_u^2} I_{(0,\beta_u/2]}(a) + \frac{4(\beta_u - a)}{\beta_u^2} I_{[\beta_u/2,\, \beta_u]}(b)$$

for $0 < b \le \beta_u$



<Figure 1> Symmetrical Triangular Density Functions of $f_A(a)$ and $f_B(a)$ Used in this Paper

In order to compare the results of Yang and Chang [5], we will use the notation POT(q) and CFT instead of POU(q) and CFU respectively, which were derived from the "uniform" distribution. Since inspection error can be regarded as either a constant or a random variable, we construct three statistical models as shown in <Table 1>. Since the case that two types of errors are constants at the same time was discussed in Yang and Chang [4], we exclude this case. Hence, our problem can be described as "Derive POT(q) and CFT for each model, and compare our results with the theorem."

<Table 1> Three Models for POT and CFT

Models		Model T (R, C)	Model T (C, R)	Model T (R, R)
Input	Known FD	constant, q	constant, q	constant, q
	Type I error	random variable, $A \sim f_A(a)$	constant, α	random variable, $A \sim f_A(a)$
	Type II error	constant, β	random variable, $B \sim f_B(b)$	random variable, $B \sim f_B(b)$
Output	POT(q)	$POT_{rc}(q)$	$POT_{cr}(q)$	$POT_{rr}(q)$
	CFT	CFT_{rc}	CFT_{cr}	CFT_{rr}

Throughout this paper, the first and second derivatives of a function f(x) will be expressed as f'(x) and f''(x), respectively.

3. Analysis of Model T (R, C)

Suppose that $B_i = \beta$ for all i where β is a constant with $0 < \beta \le 1$, and that A_i's are i.i.d with $f_A(a)$. Since E[A] $= \frac{\alpha_u}{2}$, and β can be regarded as the average of type II error, ρ can be obtained as $\frac{2\beta}{\alpha_u}$, and it turns out that POT(q) of this case, i.e., $POT_{rc}(q)$ is a function of two variables ρ and q as proved in the following proposition, and as shown in <Figure 2(a)>. In order to compare different results derived from two different distributions, we attach the previous proposition based on the uniform distribution at the end of the following proposition after revising the previous proposition with our notations.

Proposition 1. Under the assumptions that $B_i = \beta$ for all i, where β is a constant with $0 < \beta \le 1$, and A_i's are i.i.d with $f_A(a)$ for $0 < a \le \alpha_u$ and $0 < \alpha_u \le 1$, we have,

(1) $POT_{rc}(q)$

$$\begin{cases} POT_{rc1}(q) = 1 - \frac{\rho^2}{2}\left(\frac{q}{1-q}\right)^2 & \text{for } 0 < q \le \frac{1}{q+\rho}, \\ POT_{rc2}(q) = 2\left\{1 - \frac{\rho}{2}\left(\frac{q}{1-q}\right)\right\}^2 & \text{for } \frac{1}{1+\rho} < q \le \frac{2}{2+\rho} \\ 0 & \text{for } \frac{2}{2+\rho} \le q \le 1, \end{cases}$$

(2) $POT_{rc1}(q)$ is a strictly decreasing concave function of q, and $POT_{rc2}(q)$ is a strictly decreasing convex function of q,

(3) $POT_{rc}(q)$ is continuous and differentiable at q $= \frac{1}{1+\rho}$,

(4) $CFT_{rc} = \frac{1}{1+\rho}$, and

(5) the inspector with ρ overestimates q with PO > 0.5 for $0 \le q < CFT_{rc}$, estimates q with PO = 0.5 for q = CFT_{rc}, and underestimates q with PO < 0.5 for $CFT_{rc} < q \le 1$.

Proof : (1) From (1) and (2), we have,

$$Q = Q_{rc} = (1-q)A + (1-\beta)q$$

$$\begin{aligned} PO(q) &= POT_{rc}(q) \\ &= Pr\ \{Q_{rc} > q\} = Pr\{(1-q)A > \beta q\} \end{aligned} \tag{3}$$

If q=1, $POT_{rc}(1)$ becomes zero since $Q_{rc} = (1-\beta)$ is always smaller than q=1. If q=1 and $\beta=0$, then since $Q_{rc} = q = 1$, every inspector with $\beta = 0$ always estimates correctly regardless of any distribution, and by the definition of PO, $POT_{rc}(1)$ = 0.5. Hence, $POT_{rc}(1)$ has two values, either 0.5 (when $\beta=0$) or zero (when $0 < \beta \le 1$) depending on β. Since we assume $0 < \beta \le 1$, we have, $POT_{rc}(1) = 0$.

If $q \ne 1$, (3) can be further reduced to

$$POT_{rc}(q) = Pr\left\{A > \frac{\beta q}{1-q}\right\}$$

$$= \begin{cases} 1 & \text{for } \frac{\beta q}{1-q} = 0 \\ 1 - F_A\left(\frac{\beta q}{1-q}\right) & \text{for } 0 < \frac{\beta q}{1-q} \le \frac{\alpha_u}{2} \\ 1 - F_A\left(\frac{\beta q}{1-q}\right) & \text{for } \frac{\alpha_u}{2} < \frac{\beta q}{1-q} \le \alpha_u \\ 0 & \text{for } \alpha_u \le \frac{\beta q}{1-q} \end{cases} \tag{4}$$

where $F_A(q) = \int_0^q f_A(a)da$. Note that the value of $F_A\left(\frac{\beta q}{1-q}\right)$ depends upon the value of $\frac{\beta q}{1-q}$.

For $0 < \frac{\beta q}{1-q} \le \frac{\alpha_u}{2}$ (or $0 < q \le \frac{\alpha_u}{\alpha_u + 2\beta} = \frac{1}{1+\rho}$), we have,

$$\begin{aligned} 1 - F_A\left(\frac{\beta q}{1-q}\right) &= POT_{rc1}(q) \\ &= \int_{\frac{\beta q}{1-q}}^{\frac{\alpha_u}{2}} \frac{4a}{\alpha_u^2} da + \int_{\frac{\alpha_u}{2}}^{\alpha_u} \frac{4(\alpha_u - a)}{\alpha_u^2} da \\ &= 1 - 2\left(\frac{\beta}{\alpha_u}\right)^2 \left(\frac{q}{1-q}\right)^2 = 1 - \frac{\rho^2}{2}\left(\frac{q}{1-q}\right)^2 \end{aligned}$$

For $\frac{\alpha_u}{2} \le \frac{\beta q}{1-q} \le \alpha_u$(or $\frac{\alpha_u}{\alpha_u + 2\beta} =, \ \frac{1}{1+\rho} < q \le \frac{\alpha_u}{\alpha_{u+\beta}} = \frac{2}{2+\rho}$) we have,

$$\begin{aligned} 1 - F_A\left(\frac{\beta q}{1-q}\right) &= POT_{rc2}(q) \\ &= \int_{\frac{\beta q}{1-q}}^{\alpha_u} \frac{4(\alpha_u - a)}{\alpha_u^2} da = 2\left\{1 - \frac{\beta q}{\alpha_u(1-q)}\right\}^2 \\ &= 2\left\{1 - \frac{\rho}{2}\left(\frac{q}{1-q}\right)\right\}^2 \end{aligned}$$

(2) Since $POT'_{rc1}(q) = -\frac{\rho^2 q}{(1-q)^3} < 0$ and $POT''_{rc1}(q) = -\frac{\rho^2(1+2q)}{(1-q)^4} < 0$, $POT_{rc1}(q)$ is a strictly decreasing con-

cave function of q for $0 \le q \le \frac{1}{1+\rho}$. Since $POT'_{rc2}(q) = -\frac{\rho\left\{2-\frac{\rho q}{1-q}\right\}}{(1-q)^2} < 0 (\because \rho q \le 2(1-q))$ and $POT''_{rc2}(q) = \frac{\rho\left[\rho+2\left\{2-\frac{\rho q}{1-q}\right\}\right]}{(1-q)^4} > 0$, $POT_{rc2}(q)$ is a strictly decreasing convex function of q for $\frac{1}{1+\rho} < q \le \frac{2}{2+\rho}$.

(3) and (4) Since $POT_{rc1}\left(\frac{1}{1+\rho}\right) = POT_{rc2}\left(\frac{1}{1+\rho}\right) = 0.5$ and $POT'_{rc1}\left(\frac{1}{1+\rho}\right) = POT'_{rc2}(q) = -\frac{(1+\rho)^2}{\rho}$, (3) and (4) hold true.

(5) From (1), (2), (3) and (4), it follows that (5) holds true. □

Proposition 2. (Yang and Chang [5]) Under the assumptions that $B_i = \beta$ for all i, where β is a constant with $0 < \beta \le 1$, and A_i's are i.i.d with $g_A(a) = \left(\frac{1}{\alpha_u}\right) I_{(0, \alpha_u]}(a)$ for $0 < a \le \alpha_u$ and $0 < \alpha_u \le 1$, they have,

(1) $POU_{rc}(q) = \begin{cases} 1 - \frac{\rho}{2}\left(\frac{q}{1-q}\right) & \text{for } 0 < q \le \frac{2}{2+\rho}, \\ 0 & \text{for } \frac{2}{2+\rho} \le q \le 1, \end{cases}$

(2) $POU_{rc}(q)$ is a strictly decreasing concave function of q for $0 \le q \le \frac{2}{2+\rho}$,

(3) $CFU_{rc} = \frac{1}{1+\rho}$, and

(4) the inspector with ρ overestimates q with PO > 0.5 for $0 \le q < CFU_{rc}$, estimates q correctly with PO = 0.5 for $q = CFU_{rc}$, and underestimates q with PO < 0.5 for $CFU_{rc} < q \le 1$.

It can be observed from two propositions above that

(1) $POT_{rc}(q) \ne POU_{rc}(q)$.

(2) $CFT_{rc} = CFU_{rc}$.

(3) In both cases, PO is zero if $q \ge \frac{2}{2+\rho}$. That is, the inspector with ρ "always" underestimates q if and only if $q \ge \frac{2}{2+\rho}$.

(4) $POU_{rc}(q)$ is concave for $0 \le q \le \frac{2}{2+\rho}$ while $POT_{rc}(q)$ is concave for $0 \le q \le \frac{1}{1+\rho}$, and convex for $\frac{1}{1+\rho} \le q \le \frac{2}{2+\rho}$.

As shown in <Figure 2(c)>, it can be observed that $POT_{rc}(q) \ge POU_{rc}(q)$ for $0 \le q \le 16.1\%$ and $POT_{rc}(q) \le POU_{rc}(q)$ for $16.1\% \le q \le 27.7\%$. This property generally holds as proved in the following proposition. That is, PO by an inspector with ($f_A(a)$, β) is greater than or equal to that by an inspector with ($g_A(a)$, β) for $0 \le q \le \frac{1}{1+\rho}$, and vice versa for $0 \le q \le \frac{1}{1+\rho}$.

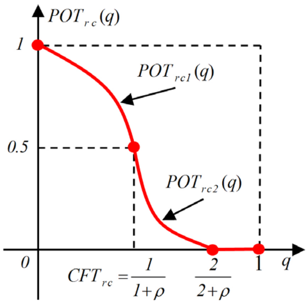
Proposition 3.

(1) $POT_{rc1}(q) \ge POU_{rc}(q)$ for $0 \le q \le \frac{1}{1+\rho}$ and

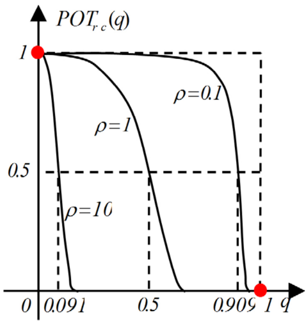
(2) $POT_{rc2}(q) \le POU_{rc}(q)$ for $\frac{1}{1+\rho} \le q \le \frac{2}{2+\rho}$.

Proof : Since $0 \le q \le \frac{1}{1+\rho}$ equivalent to $0 \le \frac{\rho q}{1-q} \le 1$, we have, $POT_{rc1}(q) - POU_{rc}(q) = \frac{\rho q\left\{1-\frac{\rho q}{1-q}\right\}}{2(1-q)} \ge 0$. Thus, (1) holds true. Similarly, $\frac{1}{1+\rho} \le q \le \frac{2}{2+\rho}$ is equivalent to $1 \le \frac{\rho q}{1-q} \le 2$, we have, $POU_{rc}(q) - POT_{rc2}(q) = 0.5\left\{\frac{\rho q}{1-q} - 1\right\}2 - \left\{\frac{\rho q}{1-q}\right\} \ge 0$. Thus, (2) holds true. □

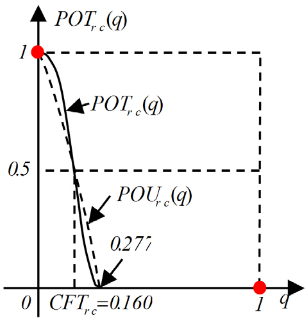
A representative graph of $POT_{rc}(q)$ is shown in <Figure 2(a)>. <Figure 2(b)> shows the changing shape of $POT_{rc}(q)$ as ρ increases from 0.1 to 10. The CFT_{rc}'s for $\rho = 0.1$, 1, and 10, are computed as 9.1%, 50%, and 90.9% respectively. For example, even if (α_u, β) = (2 PPM, 1 PPM), which are extremely small values but correspond to ρ = 1, q is overestimated with more than 50% as long as q is smaller than 50%. <Figure 2(c)> shows the real case of a BLU (back-light unit) company in Korea, where $POU_{rc}(q)$ and $POT_{rc}(q)$ for ρ = 5.23 can be drawn as the dotted curve and the solid curve respectively.



(a) The graph of $POT_{rc}(q)$



(b) Graphs of $POT_{rc}(q)$ when $\rho = 0.1$ 1, and 10



(c) Graphs of $POT_{rc}(q)$ (solid curve) and $POU_{rc}(q)$(dotted curve) when $\rho = 5.23$

<Figure 2> Graphs of $POT_{rc}(q)$

Suppose that both q and ρ are input variables. Since $POT_{rc}(q) > 0.5$ for $0 \le q < \frac{1}{1+\rho}$, it follows that $POT_{rc}(q) > 0.5$ if and only if $0 < \rho < \frac{1-q}{q}$. Hence, both Proposition 1 and 2 implies that every point (q, ρ) in two-dimensional region R_{rc} gives overestimation with PO > 0.5, where R_{rc} can be represented by $\{(q, \rho) | 0 < \rho < \frac{1-q}{q},\ 0 \le q < 1\}$. That is, an inspector with ρ overestimates q if (q, ρ) $\in R_{rc}$, estimates q with PO = 0.5 if $\rho = \frac{1-q}{q}$, and underestimates q otherwise. Note that if $\beta = 0$, then every point (q, 0) for $0 < q < 1$ on the line $\rho = 0$ gives overestimation with PO > 0.5 since $Q_{rc} = (1-q)A + q$ is always greater than q.

Suppose that both α_u and β are input variables and q is given as a constant. In the same manner above, both Proposition 1 and 2 implies that every point (α_u, β) in two-dimensional region $R_{rc}^{'}$ gives overestimation with PO > 0.5 where $R_{rc}^{'} = \{(\alpha_u, \beta) | 0 < \beta < \frac{\alpha_u(1-q)}{2q},\ 0 < \alpha_u \le 1, 0 < \beta \le 1\}$. That is, an inspector with (α_u, β) overestimates with PO > 0.5 if $(\alpha_u, \beta) \in R_{rc}^{'}$, estimates q with PO = 0.5 if $\beta = \frac{\alpha_u(1-q)}{2q}$, and underestimates q with PO < 0.5 otherwise.

4. Analysis of Model T(C, R)

Suppose that $A_i = \alpha$ for all i where α is a constant with $0 < \alpha \le 1$, and that B_i's are i.i.d with $f_B(b)$. Since $E[B] = \frac{\beta_u}{2}$, and α can be regarded as the average of type I error, ρ can be obtained as $\frac{\beta_u}{2\alpha}$, and it turns out that $POT_{cr}(q)$ is a function of two variables ρ as proved in the following proposition. In order to compare different results derived from two different distributions, we attach the previous proposition based on the uniform distribution.

Proposition 4. Under the assumption that $A_i = \alpha$ for all i where α is a constant with $0 < \alpha \le 1$ and B_i's are i.i.d with $f_B(b)$ for $0 < b \le \beta_u$, and $0 < \beta_u \le 1$, we have,

(1) $POT_{cr}(q)$

$$= \begin{cases} 1 & \text{for } 0 \le q \le \frac{1}{1+2\rho}, \\ POT_{cr1}(q) = 1 - 2\left\{1 - \frac{1}{2\rho}\left(\frac{1-q}{q}\right)\right\}^2 & \text{for } \frac{1}{1+2\rho} \le q \le \frac{1}{1+\rho}, \\ POT_{cr2}(q) = \frac{1}{2\rho^2}\left(\frac{1-q}{q}\right)^2 & \text{for } \frac{1}{1+\rho} \le q \le 1, \end{cases}$$

(2) $POT_{cr1}(q)$ is a strictly decreasing concave function of q for $\frac{1}{1+2\rho} \le q \le \frac{1}{1+\rho}$, and $POT_{cr2}(q)$ is a strictly decreasing convex function of q for $\frac{1}{1+\rho} \le q < 1$,

(3) $POT_{cr}(q)$ is continuous and differentiable at $\frac{1}{1+\rho}$,

(4) $CFT_{cr} = \frac{1}{1+\rho}$, and

(5) the inspector with ρ overestimates q with PO > 0.5 for $0 \le q < CFT_{cr}$, estimates q with PO = 0.5 for q = CFT_{cr}, and underestimates q with PO < 0.5 for $CFT_{cr} < q \le 1$.

Proof : (1) From (1) and (2), we have,

$$Q = Q_{cr} = \alpha(1-q) + (1-B)q$$

$$PO(q) = POT_{cr}(q) = \Pr\{Q_{cr} > q\} = \Pr\{qB < \alpha(1-q)\} \quad (5)$$

If q = 0, $POT_{cr}(0)$ becomes one since $Q_{cr} = \alpha$ is always greater than q = 0. If $q = \alpha = 0$, then since Q_{cr}(= 0) is always equal to q(= 0), every inspector with $\alpha=0$ always estimates correctly regardless of any distribution, and by the definition of PO, $POT_{cr}(0)$ is not one but 50% in this case. This result implies that $POT_{cr}(0)$ can be either one or 50% depending on the value of α. Since we assume that $0 < \alpha \leq 1$, we have, $POT_{cr}(0) = 1$.

If $q \neq 0$, (5) can be further reduced to

$$POT_{cR}(q) = \begin{cases} 0 & \text{for } \frac{\alpha(1-q)}{q} \leq 0 \\ F_B\left(\frac{\alpha(1-q)}{q}\right) & \text{for } 0 < \frac{\alpha(1-q)}{q} \leq \frac{\beta_u}{2} \\ F_B\left(\frac{\alpha(1-q)}{q}\right) & \text{for } \frac{\beta_u}{2} < \frac{\alpha(1-q)}{q} \leq \beta_u \\ 1 & \text{for } \beta_u \leq \frac{\alpha(1-q)}{q} \end{cases} \qquad (6)$$

where $F_B(q) = \int_0^q f_B(b)db$. Note that the value of $F_B\left(\frac{\alpha(1-q)}{q}\right)$ depends upon the value of $\frac{\alpha(1-q)}{q}$.

For $\frac{\beta_u}{2} \leq \frac{\alpha(1-q)}{q} \leq \beta_u$(or $\frac{\alpha}{\alpha+\beta_u} = \frac{1}{1+2\rho} \leq q < \frac{2\alpha}{2\alpha+\beta_u} = \frac{1}{1+\rho}$), we have,

$$F_B\left(\frac{\alpha(1-q)}{q}\right) = POT_{cr1}(q) = \int_0^{\frac{\beta_u}{2}} \frac{4b}{\beta_u^2}db + \int_{\frac{\beta}{2}}^{\frac{\alpha(1-q)}{q}} \frac{4(\beta_u-b)}{\beta_u^2}db = 1-2\left\{1-\frac{1}{2\rho}\left(\frac{1-q}{q}\right)\right\}^2$$

For $0 < \frac{\alpha(1-q)}{q} \leq \frac{\beta_u}{2}$ (or $\frac{2\alpha}{2\alpha+\beta_u} = \frac{1}{1+\rho} \leq q < 1$), we have,

$$F_B\left(\frac{\alpha(1-q)}{q}\right) = POT_{cr2}(q) = \int_0^{\frac{\alpha(1-q)}{q}} \frac{4a}{\beta_u^2}da\, \frac{1}{2\rho^2}\left(\frac{1-q}{q}\right)^2$$

(2) Since $POT_{cr1}'(q) = -\frac{\left\{2-\frac{1-q}{\rho q}\right\}}{\rho q^2} < 0$ and $POT_{cr1}''(q) = \frac{h(q)}{\rho^2 q^4} < 0$ where $h(q) = 2q(2\rho q-1+q)+1 < 0$ for $\frac{1}{1+2\rho} \leq q < \frac{1}{1+\rho}$. $POT_{cr1}(q)$ is a strictly decreasing concave function of q. Since $POT_{cr2}'(q) = -\frac{1-q}{\rho^2 q^3} < 0$ and $POT_{cr2}''(q) = \frac{3-2q}{\rho^2 q^4} > 0$, $POT_{cr2}(q)$ is a strictly decreasing convex function of q for $\frac{1}{1+\rho} \leq q \leq 1$.

(3) and (4) Since $POT_{cr1}\left(\frac{1}{1+\rho}\right) = POT_{cr2}\left(\frac{1}{1+\rho}\right) = 0.5$ and $POT'_{cr1}\left(\frac{1}{1+\rho}\right) = POT'_{cr2}\left(\frac{1}{1+\rho}\right) = -\frac{(1+\rho)^2}{\rho}$, (3) and (4) hold true.

(5) From (1), (2), (3) and (4), it follows that (5) holds true. □

Proposition 5. (Yang and Chang [5]) Under the assumption that $A_i = \alpha$ for all i where α is a constant with $0 < \alpha \leq 1$ and B_i's are i.i.d with $g_B(b) = \left(\frac{1}{\beta_u}\right)I_{(0,\beta_u]}(b)$ for $0 < b \leq \beta_u$, and $0 < \beta_u \leq 1$, they have,

(1) $POU_{cr}(q) = \begin{cases} 1 & \text{for } 0 \leq q \leq \frac{1}{1+2\rho} \\ \frac{1}{2\rho}\left(\frac{1-q}{q}\right) & \text{for } \frac{1}{1+2\rho} \leq q \leq 1, \end{cases}$

(2) $POU_{cr}(q)$ is a strictly decreasing convex function of q for $\frac{1}{1+2\rho} \leq q \leq 1$,

(3) CFU_{cr} $\frac{1}{1+\rho}$, and

(4) the inspector with ρ overestimates q with PO > 0.5 for $0 \leq q < CFU_{cr}$, estimates q with PO = 0.5 for $q = CFU_{cr}$, and underestimates q with PO < 0.5 for $CFU_{cr} < q \leq 1$.

It can be observed from two propositions above that

(1) $POT_{cr}(q) \neq POU_{cr}(q)$

(2) In both cases, PO is one if $q \leq \frac{1}{1+2\rho}$.

(3) $CFT_{cr} = CFU_{cr}$

(4) $POU_{cr}(q)$ is convex for $\frac{1}{1+2\rho} \leq q \leq 1$ while $POT_{cr}(q)$ 5 is concave for $\frac{1}{1+2\rho} \leq q \leq \frac{1}{1+\rho}$, and convex for $\frac{1}{1+\rho} \leq q \leq 1$.

As shown in <Figure 3(c)>, it can be observed that $POT_{cr1}(q) \geq POU_{cr}(q)$ for $8.7\% \leq q \leq 16\%$ and $POT_{cr2}(q) \leq POU_{cr}(q)$ for $16\% \leq q \leq 100\%$. This property generally holds as proved in the following proposition. That is, PO by an inspector with (α, $f_B(b)$) is greater than or equal to that by an inspector with (α, $g_B(b)$) for $\frac{1}{1+2\rho} \leq q \leq \frac{1}{1+\rho}$, and vice versa for

$\frac{1}{1+\rho} \leq q \leq \frac{2}{2+\rho}$. Note that equality holds when $q = \frac{1}{1+\rho}$.

Proposition 6.

(1) $POT_{cr1}(q) \geq POU_{cr}(q)$ for $\frac{1}{1+2\rho} \leq q \leq \frac{1}{1+\rho}$, and

(2) $POT_{cr2}(q) \leq POU_{cr}(q)$ for $\frac{1}{1+\rho} \leq q \leq 1$.

Proof : Since $\frac{1}{1+2\rho} \leq q \leq \frac{1}{1+\rho}$ is equivalent to $1 \leq \frac{1-q}{\rho q} \leq 2$, we have,

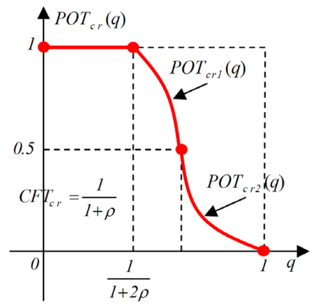
$$POT_{cr1}(q) - POU_{cr}(q) = 0.5\left\{\frac{1-q}{\rho q} - 1\right\}\left\{2 - \frac{1-q}{\rho q}\right\} \geq 0.$$

Thus, (1) holds true. Similarly, $\frac{1}{1+\rho} \leq q \leq 1$ is equivalent to $0 \leq \frac{1-q}{\rho q} \leq 1$, we have, $POU_{cr}(q) - POT_{cr2}(q) = \frac{(1-q)\left\{1 - \frac{1-q}{\rho q}\right\}}{2\rho q} \geq 0$. Thus, (2) holds true. □

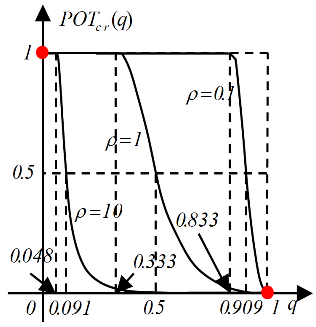
A representative graph of $POT_{cr}(q)$ is shown in <Figure 3(a)>. <Figure 3(b)> shows the changing shape of $POT_{cr}(q)$ as ρ increases from 0.1 to 10. The values of CFT_{cr}'s for ρ = 0.1, 1, and 10, are computed as 9.1%, 50%, and 90.9% respectively. Note that they are exactly the same as those values of the CFT_{rc}'s as proved in Proposition 1. <Figure 3(c)> shows the case of the BLU company in Korea, where $POU_{cr}(q)$ and $POT_{cr}(q)$ for $\rho = 5.23$ can be drawn as the dotted curve and the solid curve respectively.

Suppose that both q and ρ are input variables. Since $POT_{cr}(q) > 0.5$ for $0 \leq q < \frac{1}{1+\rho}$ (or equivalently, $0 < \rho < \frac{1-q}{q}$), both Proposition 4 and 5 implies that every point (q, ρ) in two-dimensional region R_{cr} gives overestimation with PO > 0.5 where $R_{cr} = \{(q, \rho) \mid 0 < \rho < \frac{1-q}{q}, 0 \leq q < 1\}$. That is, an inspector with ρ overestimates q if $(q, \rho) \in R_{cr}$, estimates q with PO = 0.5 if $\rho = \frac{1-q}{q}$, and underestimates q with PO < 0.5 otherwise. Note that if $\alpha = 0$, then every point $(q, 0)$ for $0 < q < 1$ gives underestimation since $\Pr\{Q_{cr} > q\} = \Pr\{B < 0\} = 0$.

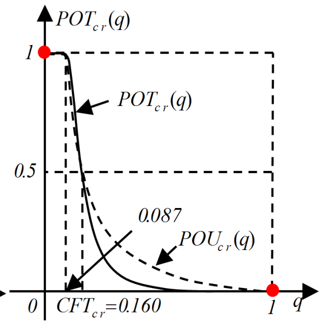
Suppose that both α and β_u are input variables and q is given as a constant. Similarly, since the condition above $0 < \rho < \frac{1-q}{q}$ is equivalent to $0 < \beta_u < \frac{2\alpha(1-q)}{q}$, both Proposition 4 and 5 implies that every point (α, β_u) in two-dimensional region R'_{cr} gives overestimation with PO > 0.5 where $R'_{cr} = \left\{(\alpha, \beta_u) \mid 0 < \beta_u < \frac{2\alpha(1-q)}{q}, 0 < \alpha < 1, 0 < \beta_u \leq 1\right\}$. That is, an inspector with (α, β_u) overestimates with PO > 0.5 if $(\alpha, \beta_u) \in R'_{cr}$, estimates q with PO = 0.5 if $\beta = \frac{2\alpha_u(1-q)}{q}$, and underestimates q with PO < 0.5 otherwise.



(a) The graph of $POT_{cr}(q)$



(b) Graphs of $POT_{cr}(q)$ when ρ = 0.1 1, and 10



(c) Graphs of $POT_{cr}(q)$ (solid curve) and $POU_{cr}(q)$(dotted curve) when ρ = 5.23

<Figure 3> Graphs of $POT_{cr}(q)$

5. Analysis of Model T (R, R)

Suppose that A_i's and B_i's are i.i.d with $f_A(a)$ and $f_B(b)$ respectively. Since $E[A] = \frac{\alpha_u}{2}$ and $E[B] = \frac{\beta_u}{2}$, ρ can be obtained as $\frac{\beta_u}{\alpha_u}$, and it turns out that $POT_{rr}(q)$ is a function of two variables ρ and q as shown in <Figure 4(a)>. In order to compare different results derived from two different distributions, we attached the previous result based on the uniform distributions at the end of Proposition 7.

Proposition 7. Under the assumption that for all i, A_i's and B_i's are i.i.d with $f_A(a)$ and $f_B(b)$ respectively for $0 < a \le \alpha_u$, $0 < b \le \beta_u$ and $0 < \alpha_u, \beta_u \le 1$, we have,

(1) $POT_{rr}(q)$ =

$$\begin{cases} POT_{rr1}(q) = 1-\frac{7\rho^2}{12}\left(\frac{q}{1-q}\right)^2 & \text{for } 0 \le q \le \frac{1}{1+2\rho} \\ POT_{rr2}(q) = \frac{3\rho^2}{4}\left(\frac{q}{1-q}\right)^2-\frac{8\rho}{3}\left(\frac{q}{1-q}\right)+3-\frac{2}{3\rho}\left(\frac{1-q}{q}\right) + \frac{1}{12\rho^2}\left(\frac{1-q}{q}\right)^2 & \text{for } \frac{1}{1+2\rho} \le q \le \frac{1}{1+\rho} \\ POT_{rr3}(q) = -\frac{3}{4\rho^2}\left(\frac{1-q}{q}\right)^2+\frac{8}{3\rho}\left(\frac{1-q}{q}\right)-2+\frac{2\rho}{3}\left(\frac{q}{1-q}\right) - \frac{\rho^2}{12}\left(\frac{q}{1-q}\right)^2 & \text{for } \frac{1}{1+\rho} \le q \le \frac{2}{2+\rho} \\ POT_{rr4}(q) = \frac{7}{12\rho^2}\left(\frac{1-q}{q}\right)^2 & \text{for } \frac{2}{2+\rho} \le q \le 1 \end{cases}$$

(2) $POT_{rr}(q)$ is a strictly decreasing function of q, and $POT_{rr1}(q)$ is concave while $POT_{rr4}(q)$ is convex.

(3) $POT_{rr}(q)$ is continuous and differentiable in the open interval (0, 1), i.e., we have

$$POT_{rr1}\left(\frac{1}{1+2\rho}\right)=POT_{rr2}\left(\frac{1}{1+2\rho}\right)=\frac{41}{48},$$

$$POT_{rr2}\left(\frac{1}{1+\rho}\right)=POT_{rr3}\left(\frac{1}{1+\rho}\right)=\frac{1}{2},$$

$$POT_{rr3}\left(\frac{2}{2+\rho}\right)=POT_{rr4}\left(\frac{2}{2+\rho}\right)=\frac{7}{48},$$

$$\frac{41}{48}\le POT_{rr1}\left(\frac{1}{1+2\rho}\right)\le 1,\ \frac{1}{2}\le POT_{rr2}(q)\le\frac{41}{48}$$

$$\frac{7}{48}\le POT_{rr3}(q)\le\frac{1}{2},\ 0\le POT_{rr4}(q)\le\frac{7}{48}$$

$$POT'_{rr1}\left(\frac{1}{1+2\rho}\right)=POT'_{rr2}\left(\frac{1}{1+2\rho}\right)=-\frac{7(1+2\rho)^2}{48\rho}$$

$$POT'_{rr2}\left(\frac{1}{1+\rho}\right)=POT'_{rr3}\left(\frac{1}{1+\rho}\right)=-\frac{2(1+\rho)^2}{3\rho}$$

$$POT'_{rr3}\left(\frac{2}{2+\rho}\right)=POT'_{rr4}\left(\frac{2}{2+\rho}\right)=-\frac{7(2+\rho)^2}{48\rho}.$$

(4) $CFT_{rr} = \frac{1}{1+\rho}$, and

(5) the inspector with ρ overestimates q with PO > 0.5 for $0 \le q < CFT_{rr}$, estimates q with PO = 0.5 for $q = CFT_{rr}$, and underestimates q with PO < 0.5 for $CFT_{rr} < q \le 1$.

Proof : Since the proof is too long, it is attached in Appendix. □

Proposition 8. (Yang and Chang [5]) Under the assumption that for all i, A_i's and B_i's are i.i.d with $g_A(a)=\frac{1}{\alpha_u}I_{(0,\alpha_u]}(a)$ and $g_{B(b)}=\frac{1}{\beta_u}I_{(0,\beta_u]}(b)$ respectively for $0 < a \le \alpha_u$, $0 < b \le \beta_u$ and $0 < \alpha_u, \beta_u \le 1$, they have,

(1) $POU_{rr}(q)$

$$=\begin{cases} POU_{rr1}(q)=1-\frac{\rho}{2}\left(\frac{q}{1-q}\right) & \text{for } 0\le q\le\frac{1}{1+\rho} \\ POU_{rr2}(q)=\frac{1}{2\rho}\left(\frac{1-q}{q}\right) & \text{for } \frac{1}{1+\rho}\le q\le 1 \end{cases}$$

(2) $POU_{rr1}(q)$ is a strictly decreasing concave function of q with $POU_{rr1}(0) = 1$, and $POU_{rr2}(q)$ is a strictly decreasing convex function of q with $POU_{rr2}(1) = 0$,

(3) $CFUrr = \frac{1}{1+\rho}$, and

(4) the inspector with ρ overestimates q with PO > 0.5 for $0 \le 0 < CFU_{rr}$, estimates q with PO = 0.5 for $q = CFU_{rr}$, and underestimates q with PO < 0.5 for $CFU_{rr} < q \le 1$.

It can be observed from two propositions above that

(1) $POT_{rr}(q) \ne POU_{rr}(q)$.
(2) CFTrr = CFUrr.
(3) $POU_{rr1}(q)$ is concave for $0 \le q < \frac{1}{1+\rho}$ and $POU_{rr1}(q)$ is convex for $\frac{1}{1+\rho} \le q < 1$ while $POT_{rr1}(q)$ is concave for $0 \le q < \frac{1}{1+2\rho}$ and $POT_{rr4}(q)$ is convex for $\frac{1}{2+\rho} \le q \le 1$..

As shown in <Figure 4(c)>, it can be observed that $POT_{rr}(q) \ge POU_{rr}(q)$ for $0 \le q \le 16\%$ and $POT_{rr}(q) \le POU_{rr}(q)$ for $16\% \le q \le 100\%$. This property generally holds as proved in the following proposition. In other words, PO by an inspector with $(f_A(a), f_B(b))$ is greater than or equal to that by an inspector with $g_A(a), g_B(b)$ for $0 \le q \le \frac{1}{1+\rho}$, and vice versa for $\frac{1}{1+\rho} \le q \le 1$. Note that equality holds when $q = \frac{1}{1+\rho}$.

Proposition 9.

(1) $POT_{rr1}(q) \ge POU_{rr1}(q)$ for $0 \le q \le \frac{1}{1+2\rho}$,

(2) $POT_{rr2}(q) \ge POU_{rr1}(q)$ for $\frac{1}{1+2\rho} \le q \le \frac{1}{1+\rho}$,

(3) $POT_{rr3}(q) \le POU_{rr2}(q)$ for $\frac{1}{1+\rho} \le q \le \frac{2}{2+\rho}$, and

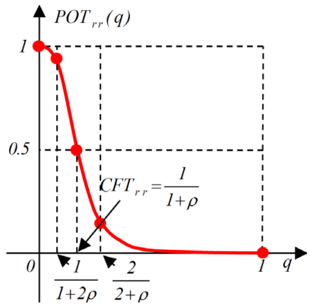
(4) $POT_{rr3}(q) \le POU_{rr2}(q)$ for $\frac{2}{2+\rho} \le q \le 1$.

Proof : See Appendix.

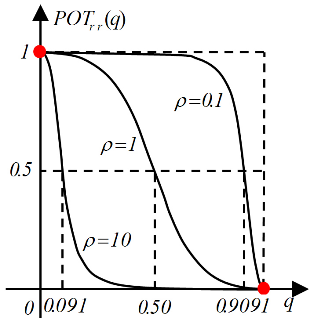
A representative graph of $POT_{rr}(q)$ can be drawn as shown in <Figure 4(a)>. <Figure 4(b)> shows the changing shape of $POT_{rr}(q)$ as ρ increases from 0.1 to 10. The CFT_{rr}'s for ρ = 0.1, 1, and 10, are computed as 9.1%, 50%, and 90.9% respectively. These results are exactly the same as those of two previous models. Even though it may be not easy to detect from the graphs only with the eye, the convexity and concavity of $POT_{rr2}(q)$ and $POT_{rr3}(q)$, it turns out that (1) for ρ = 0.1, $POT_{rr2}(q)$ is concave while $POT_{rr3}(q)$ is concave in the first part of its own interval and convex in the last part, (2) for ρ = 1, $POT_{rr2}(q)$ is concave while $POT_{rr3}(q)$ is convex, and (3) for ρ = 10, $POT_{rr2}(q)$ is concave in the first part of its own interval and convex in the last part while $POT_{rr3}(q)$ is convex. In other words, $POT_{rr2}(q)$ and $POT_{rr3}(q)$ can be either concave or convex or both depending upon the value of ρ. <Figure 4(c)> shows the case of the BLU company in Korea, where $POU_{rr}(q)$ and $POT_{rr}(q)$ for ρ = 5.23 can be drawn as the dotted curve and the solid curve respectively.

Suppose that both q and ρ are input variables. Since $POT_{rr}(q) > 0.5$ for $0 \le q < \frac{1}{1+\rho}$ (or equivalently, $0 < \rho < \frac{1-q}{q}$), both Proposition 7 and 8 implies that every point (q, ρ) in two-dimensional region R_{rr} gives overestimation with PO > 0.5 where $R_{rr} = \{(q, \rho) \mid 0 < \rho < q_s, 0 \le q < 1\}$ and $q_s = \frac{1-q}{q}$. That is, an inspector with ρ overestimates q if (q, ρ) $\in R_{rr}$, estimates q with PO = 0.5 if $\rho = q_s$, and underestimates q with PO < 0.5 otherwise. Note that if $\alpha = 0$, then every point (q, 0) for 0 < q < 1 gives underestimation since $\Pr\{Q_{rr} > q\} = \Pr\{B < 0\} = 0$.

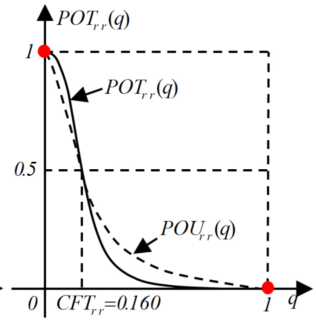
Suppose that both α_u and β_u are input variables and q is given as a constant. In the similar manner above, both Proposition 7 and 8 implies that every point (α_u, β_u) in two-dimensional region R'_{rr} gives overestimation with PO > 0.5 where $R'_{rr} = \{(\alpha_u, \beta_u) \mid 0 < \beta_u < 2\alpha_u q_s, 0 < \alpha_u \le 1, 0 < \beta_u \le 1\}$. That is, an inspector with (α_u, β_u) overestimates with



(a) The graph of $POT_{rr}(q)$



(b) Graphs of $POT_{rr}(q)$ when ρ = 0.1, 1, and 10



(c) Graphs of $POT_{rr}(q)$ (solid curve) and $POU_{rr}(q)$(dotted curve) when ρ = 5.23

<Figure 4> Graphs of $POT_{rr}(q)$

PO > 0.5 if $(\alpha_u, \beta_u) \in R'_{cr}$, estimates q with PO = 0.5 if $\beta_u = \frac{2\alpha_u(1-q)}{q}$, and underestimates q with PO < 0.5 otherwise.

6. Summary

From the propositions proved or attached in sections through 3 to 6, the following theorem holds true.

Theorem 7. Assuming an infinite sequence of items with a known FD q, and nonzero inspection errors with either the uniform or the triangular distribution, we have,

(1) an imperfect inspector with ρ has his/her/its own PO curve and CF,

(2) PO is a function of two variables q and ρ,

(3) PO is a decreasing function of q, PO(q), with PO(0) = 1 and PO(1) = 0,

(4) there always exists a unique $CF = \frac{1}{1+\rho}$, which depends only on inspection errors and not q, and

(5) the inspector overestimates q with PO > 0.5 for 0 ≤ q < CF, estimates q with PO = 0.5 for q = CF, and underestimates q with PO < 0.5 for CF < q ≤ 1,

(6) PO by an inspector with $(f_A(a), \beta)$ or $(\alpha, f_B(b))$ or $(fA(a), fB(b))$ is greater than or equal to that by an inspector with $(g_A(a), \beta)$ or $(\alpha, g_B(b))$ or $(g_A(a), g_B(b))$ for $0 \leq q \leq CF$, and the former is smaller than or equal to the latter for $CF \leq q \leq 1$. Note that equality holds when q = CF.

As conjectured by Yang and Chang [4], we have proved that their conjecture still holds true for the case of the symmetrical triangular distribution. However, statement (2) may be changed to a statement that PO is a function of one combined variable, for example, $x = \frac{\rho q}{1-q}$, since all formulas above can be expressed as the function of the combined variable. Note that this conversion gives $x \geq 0$ and that $\frac{1}{1+2\rho}$, $\frac{1}{1+\rho}$, and $\frac{2}{2+\rho}$ can be converted to 0.5, 1, and 2 respectively. For example, $POT_{rr2}(q)$ can be expressed as

$$\begin{aligned} POT_{rr2}(q) &= \frac{3\rho^2}{4}\left(\frac{q}{1-q}\right)^2 - \frac{8\rho}{3}\left(\frac{q}{1-q}\right) + 3 - \frac{2}{3\rho}\left(\frac{1-q}{q}\right) \\ &\quad + \frac{1}{12\rho^2}\left(\frac{1-q}{q}\right)^2 \text{ for } \frac{1}{1+2\rho} \leq q \leq \frac{1}{1+\rho} \\ &= \frac{3}{4}x^2 - \frac{8}{3}x + 3 - \frac{2}{3}\left(\frac{1}{x}\right) + \frac{1}{12}\left(\frac{1}{x}\right)^2 \\ &= POT_{rr2}(x) \text{ for } \frac{1}{2} \leq x \leq 1. \end{aligned}$$

This implies that PO depends ultimately upon $\frac{\rho q}{1-q}$, and that when the values of (ρ, q) and (ρ', q') are given respectively, the value of PO is same if and only if $\frac{\rho q}{1-q} = \frac{\rho' q'}{1-q'}$.

7. Discussion and Concluding Remarks

We started this research expecting both at least significantly different results from Yang and Chang [4]. However, regardless of the uniform or the triangular error distribution, the critical fraction defectives were same except the value of PO and the concavity/convexity of a PO curve. Our strong conjecture is that this phenomenon might result from two assumptions : the symmetry of the probability density functions and the error intervals with a zero lower bound. In other words, our strong conjecture is that the critical fraction defective could be derived differently if the symmetry is not satisfied and/or the interval with a nonzero lower bound would be assumed. Further research may be concentrated on a skewed triangular distribution defined in an interval with a nonzero lower bound or a zero lower bound, at the cost of mathematical complexity. If PO is to be greater than 50% even under the new assumptions, we cannot help but agree with their conclusion that all commercial inspection plans should be revised with the concept of PO in the near future, for the fairness of commercial trades. Furthermore, as mentioned in Yang and Chang [4], since our mathematical models do not consider any related costs, a cost-based optimization model with the PO concept could be constructed to determine a trade-off point between buyers and sellers. We hope that the concept of PO should become one of the major criteria in the future, and that our PO functions would be widely used since they are more accurate than the previous PO functions.

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<Appendix>

Proof of Proposition 7.

(1) From (1) and (2), Q_{rr} and PO can be expressed respectively as

$$Q = Q_{rr} = A(1-q) + (1-B)q.$$
$$PO(q) = POT_{rr}(q) = \Pr\{Q_{rr} > q\} = \Pr\{(1-q)A > qB\} \tag{7}$$

If q = 1, then $POT_{rr}(1) = 0$ since $POT_{rr}(q) = \Pr\{B<0\} = 0$. On the other hand, if q = 0, then $POT_{rr}(0) = 1$ since $POT_{rr}(0) = \Pr\{A > 0\} = 1$.

For 0 < q < 1, (7) can be reduced to

$$POT_{rr}(q) = \int_0^{\beta_u} \Pr\{(1-q)A > qB \mid B = b\} f_B(b)db = \int_0^{\beta_u} \Pr\left\{\frac{qb}{1-q} < A \le \alpha_u\right\} f_B(b)db$$
$$= \int_0^{\beta_u} \left\{ \int_{\frac{qb}{1-q}}^{\alpha_u} < f_A(a)da \right\} f_B(b)db = \iint_{(a,b)\in S_{rr}} f_A(a) f_B(b) da\, db \tag{8}$$

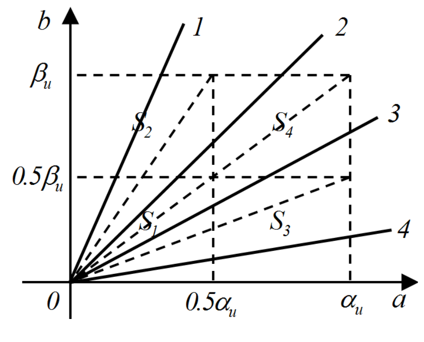
where $S_{rr} = \{(a, b) \mid b < q_s a,\ 0 \le a \le \alpha_u,\ 0 \le b \le \beta_u, \text{ and } 0 < \alpha_u, \beta_u \le 1\}$ and $q_s = \frac{1-q}{q}$. As shown in <Figure 5>, since $POT_{rr}(q)$ depends upon the shape of S_{rr}, which changes as q_s changes, PO can be computed by considering four cases :

Case 1 for $q_s \ge \frac{2\beta_u}{\alpha_u}$, where a straight line b = q_sa passes through two regions S_1 and S_2,

Case 2 for $\frac{\beta_u}{\alpha_u} \le q_s \le \frac{2\beta_u}{\alpha_u}$, where the straight line passes through three regions S_1, S_2 and S_4,

Case 3 for $\frac{B_u}{\alpha_u} \le q_s \le \frac{\beta_u}{\alpha_u}$, where the straight line passes through two regions S_1, S_3 and S_4, and

Case 4 for $0 < q_s \le \frac{\beta_u}{2\alpha_u}$, where the straight line passes through two regions S_1 and S_3.



<Figure 5> Four Split Feasible Regions (S_1, S_2, S_3, S_4) and Possible Straight Lines Depending on q_s

(a) Case 1 for $q_s > \frac{2\beta_u}{\alpha_u}$ (or $0 \le q \le \frac{1}{1+2\rho}$) : Since the joint probability density function of A and B changes depending upon regions, and since S_{rr} includes S_1, S_2, S_3, and S_4, (8) can be derived as

$$POT_{rr1}(q) = \int\int_{(a,b)\in S_{rr}} f_A(a) f_B(b) da\, db = M_{11} + M_{12} + M_{13} + M_{14} = 1 - \frac{7\rho^2}{12}\left(\frac{q}{1-q}\right)^2$$

where

$$M_{11} = \iint_{(a,b)\in S_1} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_0^{\frac{\beta_u q}{2(1-q)}}\int_0^{\frac{(1-q)a}{q}} ab\,db\,da + \int_{\frac{\beta_u q}{2(1-q)}}^{\frac{\alpha_u}{2}}\int_0^{\frac{\beta_u}{2}} ab\,db\,da\right]$$

$$= \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{1}{64}\alpha_u^2\beta_u^2 - \frac{\beta_u^4}{128}\left(\frac{q}{1-q^2}\right)^2\right] = \frac{1}{4} - \frac{\rho^2}{8}\left(\frac{q}{1-q}\right)^2$$

$$M_{12} = \iint_{(a,b)\in S_2} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_{\frac{\beta_u q}{2(1-q)}}^{\frac{\beta_u q}{(1-q)}}\int_{\frac{\beta_u}{2}}^{\frac{(1-q)a}{q}} a(\beta_u - b)\,db\,da + \int_{\frac{\beta_u q}{(1-q)}}^{\frac{\alpha_u}{2}}\int_{\frac{\beta_u}{2}}^{\beta_u} a(\beta_u - b)\,db\,da\right]$$

$$= \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{\alpha_u^2\beta_u^2}{64} - \frac{11\beta_u^4}{384}\left(\frac{q}{1-q^2}\right)^2\right] = \frac{1}{4} - \frac{11\rho^2}{24}\left(\frac{q}{1-q}\right)^2$$

$$M_{13} = \iint_{(a,b)\in S_3} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_{\frac{\alpha_u}{2}}^{\alpha_u}\int_0^{\frac{\beta_u}{2}} (\alpha_u - a) b\ db\,da\right] = \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{\alpha_u^2\beta_u^2}{64}\right] = \frac{1}{4}$$

$$M_{14} = \iint_{(a,b)\in S_4} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_{\frac{\alpha_u}{2}}^{\alpha_u}\int_{\frac{\beta_u}{2}}^{\frac{\beta_u}{2}} (\alpha_u - a)(\beta_u - b) db\ da\right] = \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{\alpha_u^2\beta_u^2}{64}\right] = \frac{1}{4}$$

(b) Case 2 for $\frac{\beta_u}{\alpha} \le q_s \le \frac{2\beta_u}{\alpha_u}$ (or $\frac{1}{1+2\rho} \le q \le \frac{1}{1+\rho}$) : Since S_{rr} includes S_1, S_2, S_3, and S_4, (8) can be derived as

$$POT_{rr}(q) = \iint_{(a,b)\in S_{rr}} f_A(a) f_B(b) da\ db = \int_0^{\beta_u} \Pr\left\{\frac{qb}{1-q} < A \le \alpha_u\right\} f_B(b) db$$

$$= M_{21} + M_{22} + M_{23} + M_{24} = \frac{3\rho^2}{4}\left(\frac{q}{1-q}\right)^2 - \frac{8\rho}{3}\left(\frac{q}{1-q}\right) + 3 - \frac{2}{3\rho}\left(\frac{1-q}{q}\right) + \frac{1}{12\rho^2}\left(\frac{1-q}{q}\right)^2$$

where

$$M_{21} = \iint_{(a,b)\in S_1} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_0^{\frac{\beta_u q}{2(1-q)}}\int_0^{\frac{(1-q)a}{q}} ab\,db\,da + \int_{\frac{\beta_u q}{2(1-q)}}^{\frac{\alpha_u}{2}}\int_0^{\frac{\beta_u}{2}} ab\,db\,da\right]$$

$$M_{11} = \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{1}{64}\alpha_u^2\beta_u^2 - \frac{\beta_u^4}{128}\left(\frac{q}{1-q}\right)^2\right] = \frac{1}{4} - \frac{\rho^2}{8}\left(\frac{q}{1-q}\right)^2$$

$$M_{22} = \iint_{(a,b)\in S_2} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_{\frac{\beta_u q}{2(1-q)}}^{\frac{\alpha_u}{2(1-q)}}\int_{\frac{\beta_u}{2}}^{\frac{(1-q)a}{q}} a(\beta_u - b) db\ da\right]$$

$$= \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{5\beta_u^4}{384}\left(\frac{q}{1-q}\right)^2 - \frac{3\alpha_u^2\beta_u^2}{64} + \frac{\alpha_u^3\beta_u}{24}\left(\frac{1-q}{q}\right) - \frac{\alpha_u^4}{128}\left(\frac{1-q}{q^2}\right)^2\right] = \frac{5\rho^2}{24}\left(\frac{q}{1-q}\right)^2 - \frac{3}{4} + \frac{2}{3\rho}\left(\frac{1-q}{q}\right) - \frac{1}{8\rho^2}\left(\frac{1-q}{q}\right)^2$$

$$M_{24} = \iint_{(a,b)\in S_4} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_{\frac{\alpha_u}{2}}^{\frac{\beta_u q}{1-q}}\int_{\frac{\beta_u}{2}}^{\frac{(1-q)a}{q}} (\alpha_u - a)(\beta_u - b) db\ da + \int_{\frac{\beta_{uq}}{1-q}}^{\alpha_u}\int_{\frac{\beta_u}{2}}^{\beta_u} (\alpha_u - a)(\beta_u - b) db\ da\right]$$

$$= \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{5\alpha_u^4}{384}\left(\frac{1-q}{q}\right)^2 - \frac{\alpha_u^3\beta_u}{12}\left(\frac{1-q}{q}\right) - \frac{\alpha_u\beta_u^3}{6}\left(\frac{q}{1-q}\right) + \frac{\beta_u^4}{24}\left(\frac{q}{1-q}\right)^2 + \frac{13\alpha_u^2\beta_u^2}{64}\right]$$

$$= \frac{5}{24\rho^2}\left(\frac{1-q}{q^2}\right) - \frac{4}{3\rho}\left(\frac{1-q}{q}\right) - \frac{8\rho}{3}\left(\frac{q}{1-q}\right) + \frac{2\rho^2}{3}\left(\frac{q}{1-q}\right)^2 + \frac{13}{4}$$

(c) Case 3 for $\frac{\beta_u}{2\alpha_u} \le q_s \le \frac{\beta_u}{\alpha_u}$: Since S_{rr} includes S_1, S_3, and S_4, (8) can be derived as

$$POT_{rr3}(q) = \iint_{(a,b)\in S_{rr}} f_A(a) f_B(b) da\ db = M_{31} + M_{32} + M_{32} + M_{34}$$
$$= -\frac{3}{4\rho^2}\left(\frac{1-q}{q}\right)^2 + \frac{8}{3\rho}\left(\frac{1-q}{q}\right) - 2 + \frac{2\rho}{3}\left(\frac{q}{1-q}\right) - \frac{\rho^2}{12}\left(\frac{q}{1-q}\right)^2$$

where

$$M_{31} = \iint_{(a,b)\in S_1} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_0^{\frac{\alpha_u}{1-q}} \int_0^{\frac{(1-q)a}{q}} ab\,db\,da\right] = \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{\alpha_u^4}{128}\left(\frac{1-q}{q}\right)^2\right] = \frac{\alpha_u^2}{8\beta_u^2}\left(\frac{1-q}{q}\right)^2 = \frac{1}{8\rho^2}\left(\frac{1-q}{q}\right)^2,$$

$$M_{32} = \iint_{(a,b)\in S_2} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}[0],$$

$$M_{33} = \iint_{(a,b)\in S_3} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_{\frac{\alpha_u}{2}}^{\frac{\beta_u q}{2(1-q)}} \int_0^{\frac{(1-q)a}{q}} (\alpha_u - a) b\ db\ da + \int_{\frac{\beta_u q}{2(1-q)}}^{\alpha_u} \int_0^{\frac{\beta_u}{2}} (\alpha_u - a) b\ db\ da\right]$$
$$= \frac{16}{\alpha_u^2\beta_u^2}\left[-\frac{5\alpha_u^4}{384}\left(\frac{1-q}{q}\right)^2 - \frac{\alpha_u^2\beta_u^2}{16} - \frac{\alpha_u\beta_u^3}{24}\left(\frac{q}{1-q}\right) + \frac{\beta_u^4}{128}\left(\frac{q}{1-q}\right)^2\right] = -\frac{5}{24\rho^2}\left(\frac{1-q}{q}\right)^2 + 1 - \frac{2\rho}{3}\left(\frac{1-q}{q}\right) + \frac{\rho^2}{8}\left(\frac{q}{1-q}\right)^2$$

and

$$M_{34} = \iint_{(a,b)\in S_4} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_{\frac{\beta_u q}{2(1-q)}}^{\alpha_u} \int_{\frac{\beta_u}{2}}^{\frac{(1-q)a}{q}} (\alpha_u - a)(\beta_u - b) db\ da\right]$$
$$= \frac{16}{\alpha_u^2\beta_u^2}\left[-\frac{\alpha_u^4}{24}\left(\frac{1-q}{q}\right)^2 + \frac{\alpha_u^3\beta_u}{6}\left(\frac{1-q}{q}\right) - \frac{3\alpha_u^2\beta_u^2}{16} + \frac{\alpha_u\beta_u^3}{12}\left(\frac{q}{1-q}\right) - \frac{5\beta_u^4}{384}\left(\frac{q}{1-q}\right)^2\right]$$
$$= -\frac{2}{3\rho^2}\left(\frac{1-q}{q}\right)^2 + \frac{8}{3\rho}\left(\frac{1-q}{q}\right) - 3 + \frac{4\rho}{3}\left(\frac{q}{1-q}\right) - \frac{5\rho^2}{24}\left(\frac{q}{1-q}\right)^2.$$

(d) Case 4 for $0 \le q_s \le \frac{\beta_u}{2\alpha_u}$: Since S_{rr} includes S_1 and S_3, (8) can be derived as

$$POT_{rr4}(q) = \iint_{(a,b)\in S_{rr}} f_A(a) f_B(b) da\ db = M_{41} + M_{42} + M_{42} + M_{44} = \frac{7}{12\rho^2}\left(\frac{1-q}{q}\right)^2$$

where

$$M_{41} = \iint_{(a,b)\in S_1} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_0^{\frac{\alpha_u}{2}} \int_0^{\frac{(1-q)a}{q}} ab\ db\ da\right] = \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{\alpha_u^4}{128}\left(\frac{1-q}{q^2}\right)^2\right] = \frac{\alpha_u^2}{8\beta_u^2}\left(\frac{1-q}{q}\right)^2 = \frac{1}{8\rho^2}\left(\frac{1-q}{q}\right)^2$$

$$M_{42} = \iint_{(a,b)\in S_2} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}[0] = 0$$

$$M_{43} = \iint_{(a,b)\in S_3} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}\left[\int_{\frac{\alpha_u}{2}}^{\alpha_u} \int_0^{\frac{(1-q)a}{q}} (\alpha_u - a) b\,db\,da\right]$$
$$= \frac{16}{\alpha_u^2\beta_u^2}\left[\frac{11\alpha_u^4}{384}\left(\frac{1-q}{q^2}\right)^2\right] = \frac{11\alpha_u^2}{24\beta_u^2}\left(\frac{1-q}{q}\right)^2 = \frac{11}{24\rho_{rr}^2}\left(\frac{1-q}{q}\right)^2$$

$$M_{44} = \iint_{(a,b)\in S_4} f_A(a) f_B(b) da\ db = \frac{16}{\alpha_u^2\beta_u^2}[0] = 0$$

(2) Since $POT'_{rr1}(q) = \frac{7\rho^2 q}{6(1-q)^3} < 0$ and $POT''_{rr1}(q) = \frac{7\rho^2(1+2q)}{6(1-q)^4} < 0$, $POT_{rr1}(q)$ is a strictly decreasing concave function of q for $0 \le q \le \frac{1}{1+2\rho}$. Since $POT''_{rr4}(q) = \frac{7(1-q)}{6\rho^2 q^3} < 0$, and $POT''_{rr4}(q) = \frac{7(3-2q)}{6\rho^2 q^4} > 0$, $POT_{rr4}(q)$ is a strictly decreasing convex function of q for $\frac{2}{2+2\rho} \le q \le 1$. $POT'_{rr2}(q)$ and $POT_{rr3}(q)$ can be derived respectively as

$$POT'_{rr2}(q) = \frac{3\rho^2}{2}\frac{q}{(1-q)^3} - \frac{8\rho_{rr}}{3}\frac{1}{(1-q)^2} + \frac{2}{3\rho}\frac{1}{q^2} - \frac{1}{6\rho^2}\frac{(1-q)}{q^3}$$
$$= \frac{1}{6\rho^2 q^3(1-q)^3}[9\rho^4 q^4 - 16\rho^3 q^3(1-q) + 4\rho q(1-q)^3 - (1-q)^4]$$

$$\text{for } \frac{1}{1+2\rho} \le q \le \frac{1}{1+\rho}$$

$$POT'_{rr3}(q) = \frac{1}{q^3(1-q)^3}[\frac{3}{2\rho^2}(1-q)^4 - \frac{8}{3\rho}(1-q)^3 q + \frac{2\rho}{3}(1-q)q^3 - \frac{\rho^2}{6}q^4]$$
$$= \frac{1}{6\rho^2 q^3(1-q)^3}[9(1-q)^4 - 16\rho(1-q)^3 q + 4\rho^3(1-q)q^3 - \rho^4 q^4]$$

$$\text{for } \frac{1}{1+\rho} \le q \le \frac{2}{2+\rho}$$

However, it seems to be mathematically hard to prove directly from the above results that $POT'_{rr2}(q) < 0$ and $POT'_{rr3}(q) < 0$. Since $q \ne 0$ in the interval $\frac{1}{1+2\rho} \le q \le \frac{2}{2+2\rho}$, by replacing the ratio of B to A with a random variable Z, (7) can be reduced to

$$POT_{rr}(q) = \Pr\{QI < q\} = \Pr\{\frac{B}{A} < \frac{1-q}{q}\} = \Pr\{Z < \frac{1-q}{q}\} = F_Z\left(\frac{1-q}{q}\right)$$

Since $POT'_{rr}(q)$ can be derived as $POT'_{rr}(q) = -\frac{f_Z\left(\frac{1-q}{q}\right)}{q^2} < 0$, it follows that $POT'_{rr2}(q)$ and $POT'_{rr3}(q)$ are strictly decreasing functions of q. Note that this method can be applied $POT_{rr1}(q)$ and $POT_{rr4}(q)$.

(3) Since $POT_{rr1}\left(\frac{1}{1+2\rho}\right) = POT_{rr2}\left(\frac{1}{1+2\rho}\right) = \frac{41}{48}$, $POT_{rr2}\left(\frac{1}{1+\rho}\right) = POT_{rr3}\left(\frac{1}{1+\rho}\right) = 0.5$, and $POT_{rr3}\left(\frac{2}{2+2\rho}\right) = POT_{rr4}\left(\frac{2}{2+2\rho}\right) = \frac{7}{48}$, $POT_{rr}(q)$ is continuous in the open interval (0, 1) and it follows that $\frac{41}{48} \le POT_{rr1}\left(\frac{1}{1+2\rho}\right) \le 1$, $0.5 \le POT_{rr2}(q) \le \frac{41}{49}$, $\frac{7}{48} \le POT_{rr3}(q) \le 0.5$, and $0 \le POT_{rr4}(q) \le \frac{7}{48}$. $POT_{rr}(q)$ is differentiable in the open interval (0, 1) since we have, $POT'_{rr1}\left(\frac{1}{1+2\rho}\right) = POT'_{rr2}\left(\frac{1}{1+2\rho}\right) = -7\frac{(1+2\rho)^2}{48\rho}$, $POT'_{rr2}\left(\frac{1}{1+\rho}\right) = POT'_{rr3}\left(\frac{1}{1+\rho}\right) = -\frac{2(1+\rho)^2}{3}$, and $POT'_{rr3}\left(\frac{1}{2+2\rho}\right) = POT'_{rr4}\left(\frac{2}{2+2\rho}\right) = -\frac{7(2+\rho)^2}{48\rho}$.

(4) Since $POT_{rr2}\left(\frac{1}{1+\rho}\right) = POT_{rr3}\left(\frac{1}{1+\rho}\right) = 0.5$, CFT_{rr} is $\frac{1}{1+\rho}$.

(5) From (1), (2), (3) and (4), it follows that (5) holds true. □

Proof of Proposition 9.

(1) Since $0 \leq q \leq \frac{1}{1+2\rho}$ is equivalent to $0 \leq \frac{\rho q}{1-\rho} \leq 0.5$, we have, $POT_{rr1}(q) - POU_{rr1}(q) = \frac{7\rho q\left\{\frac{6}{7} - \frac{\rho q}{1-q}\right\}}{12} \geq 0$. Hence, (1) holds true.

(2) The inequality $\frac{1}{1+2\rho} \leq q \leq \frac{1}{1+\rho}$ is equivalent to $1 \leq \frac{1-q}{\rho q} \leq 2$, and we have,

$$POT_{rr2}(q) - POU_{rr1}(q) = -\frac{3}{4}\left(\frac{\rho q}{1-q}\right)^2 - \frac{13}{6}\left(\frac{\rho q}{1-q}\right) + 2 - \frac{2}{3}\left(\frac{1-q}{\rho q}\right) + \frac{1}{12}\left(\frac{1-q}{\rho q}\right)^2.$$

Replacing $\frac{1-q}{\rho q}$ with x, and letting $h(x) = POT_{rr2}(q) - POU_{rr1}(q)$, we have,

$$h(x) = \frac{k(x)}{12x^2} \text{ for } 1 \leq x \leq 2,$$

where $k(x) = x^4 - 8x^3 + 24x^2 - 26x + 9$.

It is enough to prove that $k(x) \geq 0$ for $1 \leq x \leq 2$. Since $k''(x) = 12(x-2)^2 \geq 0$, $k'(x)$ is an increasing function. Since $k'(1) = 2$, we have, $k'(x) \geq 0$ for $1 \leq x \leq 2$, and it follows that $k(x)$ is an increasing function for $1 \leq x \leq 2$. Since $k(1) = 0$, we have,

$$k(x) \geq 0 \text{ for } 1 \leq x \leq 2. \tag{9}$$

It follows that (2) holds true.

(3) The inequlity $\frac{1}{1+\rho} \leq q \leq \frac{2}{2+\rho}$ is equivalent to $1 \leq \frac{\rho q}{1-q} \leq 2$, and we have,

$$POT_{rr2}(q) - POU_{rr3}(q) = -\frac{3}{4}\left(\frac{1-q}{\rho q}\right)^2 - \frac{13}{6}\left(\frac{1-q}{\rho q}\right) + 2 - \frac{2}{3}\left(\frac{\rho q}{1-q}\right) + \frac{1}{12}\left(\frac{\rho q}{1-q}\right)^2.$$

Replacing $\frac{\rho q}{1-q}$ with y, and letting $s(x) = POU_{rr2}(q) - POT_{rr3}(q)$, we have,

$$s(x) = \frac{1}{12x^2}(x^4 - 8x^3 + 24x^2 - 26x + 9) \text{ for } 1 \leq x \leq 2$$

From (9), we have, $s(x) \geq 0$ and it follows that (3) holds true.

(4) Since $\frac{2}{2+\rho} \leq q \leq 1$ is equivalent to $0 \leq \frac{1-q}{\rho q} \leq 0.5$, we have, $POU_{rr2}(q) - POT_{rr4}(q) = \frac{7(1-q)\left\{\frac{6}{7} - \frac{1-q}{\rho q}\right\}}{12\rho q} \geq 0$. Hence, (4) holds true.