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SRB MEASURES IN CHAOTIC DYNAMICAL SYSTEMS

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Abstract. In this paper, we present the construction of natural invariant measures so called SRB(Sinai-Ruelle-Bowen) measures by the properties of geometric t-potential and Bowen's equation for the hyperbolic attractors.

1. Introduction

In this paper, we consider SRB(Sinai-Ruelle-Bowen) measures which are one of the most important classes of natural invariant measures with chaotic (hyperbolic) behavior in dynamical systems. In the 1960s and 70s, Sinai, Ruelle, and Bowen [2, 12, 14] introduced a remarkable way of investigating chaotic topological attractors by statistical mechanics and smooth ergodic theory and used Markov partitions due to Sinai to construct invariant Borel probability measures, which we now call SRB measures, for uniformly hyperbolic attractors. At the end of the 1990s, Young [15] introduced an extension of Markov partitions, known as Young towers, which have been used to prove the existence of SRB measures for some specific classes of hyperbolic dynamical systems. Recently, Climenhaga, Dolgopyat, and Pesin [4] prove that Young towers exist and can be used to construct SRB measures for surface diffeomorphisms satisfying non-uniformly hyperbolicity assumption. They deal

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with the most general situation of chaotic attractors with non-zero Lyapunov exponents and construct SRB measures for this attractors. The theory of hyperbolic dynamical systems investigates the mechanism that drives the time evolution from predictability to randomness. The theory of uniformly hyperbolic dynamical system is presently well-understood, but this uniformly hyperbolic condition is so restrictive that it rarely applies to physically relevant examples. So several different approaches have been used to study more realistic non-uniformly hyperbolic dynamics. The goal of this paper is to present geometric approach due to Climenhaga, Dolgopyat, and Pesin of the existence of SRB measures and their ergodic and statistical properties for general hyperbolic dynamical systems. This paper is organized as follows. In section II, We will define SRB measures and survey some basic ergodic properties of it as a natural invariant probability measure. In section III, we introduce some ergodic properties of topological hyperbolic attractors. In section IV, we discuss statistical properties of SRB measures for non-uniformly hyperbolic attractors. Finally, In section V, we will prove our main result by applying some unifying geometrical methods to non-uniformly hyperbolic dynamical systems.

2. SRB measures

Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and $U \subset M$ an open subset with $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n>=0} f^n(U)$ a topological attractor for f. Then Λ is compact, maximal f-invariant set. We will study the statistical properties of the dynamics in U. Let m be normalized Lebesgue measure on M and let μ be an arbitrary Borel probability measure on Λ. Then the set

$$B_\mu = \{x \in U : \frac{1}{n}\sum_{k=0}^{n-1} h(f^k(x)) \to \int_\Lambda h d\mu, \forall h \in C^1(M)\}$$

is called basin of attraction of μ. We say that μ is a physical measure if $m(B_\mu) > 0$.

Definition 2.1. [9,10] (1) Given $x \in \Lambda$ and $v \in T_x M$, the Lyapunov exponent of v at x is defined by

$$\lambda(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|df^n v\|.$$

(2) A Borel invariant measure μ on Λ is hyperbolic if $\lambda_i(x) \neq 0$ and $\lambda_1(x) < 0 < \lambda_m(x)$, where $m = dimM$.

(3) Given l, define regular set of level l by

$$\Lambda_l = \{x \in \Lambda : C(x) \leq l, K(x) \geq \frac{1}{l}\},$$

for some positive Borel functions $C(x)$ and $K(x)$ on Λ.

(4) A measure μ on Λ is called an SRB measure if μ is hyperbolic and for every l with $\mu(\Lambda_l) > 0$, almost every $x \in \Lambda_l$ and almost every $y \in B(x, \delta_l) \cap \Lambda_l$, the conditional measure $\mu^u(y)$ on local unstable manifold $V^u(y)$ at y is absolutely continuous with respect to the measure $m_{V^u}(y)$.

Theorem 2.2. *[11] Let μ be an SRB measure on Λ, then the density $d^u(x, \cdot)$ of the conditional measure $\mu^u(x)$ with respect to the leaf-volume $m_{V^u}(x)$ is given by $d^u(x, y) = \rho^u(x)^{-1}\rho^u(x, y)$, where $\rho^u(x) = \int_{V^u(x)} \rho^u(x, y)dm^u(x)(y)$ is the normalizing factor.*

Pesin and Climenhaga introduced more general concept of non-uniform hyperbolicity. This theory includes many chaotic phenomena [8].

Definition 2.3. [10] A diffeomorphism f of a manifold M is called non-uniformly hyperbolic with respect to an invariant ergodic probability measure μ if μ-almost every $x \in M$ and for every unit vector $v \in T_xM$, the Lyapunov exponet at x is non-zero.

Using results of non-uniform hyperbolicity theory, we can obtain some ergodic properties of SRB measures

Theorem 2.4. [7, 10] *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M with the topological attractor Λ and let μ be an SRB measure on Λ. Then there exist $\Lambda_0, \Lambda_1, \cdots \subset \Lambda$ such that*

1. *$\Lambda = \bigcup_{i \geq 0} \Lambda_i$, $\Lambda_i \bigcap \Lambda_j = \emptyset$;*
2. *$\mu(\Lambda_0) = 0$ and $\mu(\Lambda_i) > 0$ for $i > 0$;*
3. *$f|\Lambda_i$ is ergodic for $i > 0$;*
4. *for each $i > 0$, there exists $n_i > 0$ such that $\Lambda_i = \bigcup_{j=1}^{n_i} \Lambda_{i,j}$, where the union is disjoint (modulo μ-null sets) , $f(\Lambda_{i,j}) = \Lambda_{i,j+1}$, $f(\Lambda_{n_i,1}) = \Lambda_{i,1}$ and $f^{n_i}|\Lambda_{i,1}$ is Bernoulli ;*
5. *if μ is ergodic, then the basin of attraction B_μ has positive Lebesgue measure in U.*

The geometric approach for constructing SRB measures is to follow the Bogolyubov-Krylov theorem for the existence of invariant measures by pushing forward a given reference measure.

Theorem 2.5. **[Bogolyubov-Krylov Theorem]** [4] *If X is a compact metric space and $f : X \to X$ is a continuous map, then there exists at least one Borel invariant measure μ.*

To study the relation between the metric entropy $h_\mu(f)$ and the Lyapunov exponent $\lambda_i(x)$. We define Lyapunov exponents $\lambda(x, v)$.

Definition 2.6. [7, 10] We define the Lyapunov exponent of v at x by

$$\lambda(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log |d_x f^n v|$$

$\forall x \in M, v \in T_x M$.

For each $x \in \Lambda$, the function $\lambda(x, \cdot)$ takes on finitely many values $\lambda_1(x) \leq \cdots \leq \lambda_m(x)$, where m is the dimension of M. and the functions $\lambda_i(x)$ are f-invariant Borel measurable.

Theorem 2.7. **[Pesin's Entropy Formula]** [8] *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism with an f-invariant Borel probability measure μ, then, the entropy formula*

$$h_\mu(f) = \int_M \sum_{\lambda_i > 0} \lambda_i(x) m_i(x) d\mu(x)$$

holds if and only if μ is an SRB measure.

3. Topological hyperbolic attractors

Let M be a m-dimensional smooth Riemannian manifold, $U \subset M$ an open set that $\overline{U}$ is compact, and $f : U \to M$ a $C^{1+\alpha}$ diffeomorphism onto its image such that $\overline{f(U)} \subset U$.

Definition 3.1. [4]

1. The topological attractor for f is defined by $\Lambda = \bigcap_{n\geq 0} f^n(U)$.
2. Let μ be an invariant probability measure on Λ and let the basin of attraction of μ be the set

$$B_\mu = \{x \in U : \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) \to \int_\Lambda \phi d\mu\}$$

then μ is a physical measure if the Lebesgue measure m of B_μ is positve, that is, $m(B_\mu)$> 0.

3. An attractor with physical measure is said to be a Milnor attractor.
4. μ is a hyperbolic measure on Λ if μ has non-zero Lyapunov exponents.

The topological attractor Λ contains all the global unstable manifolds $W^u(x)$ of its points x since the local unstable manifold $V^u(x)$ at each $x \in \Lambda$ is contained in Λ. On the other hand, the intersection of Λ with the global stable manifolds $V^s(x)$ of its points x is usually a Cantor set. Results of non-uniformly hyperbolic theory allow us to construct for almost every x $\in \Lambda$ stable $E^s(x)$ and unstable $E^u(x)$ subspaces which integrate locally into local stable $V^s(x)$ and local unstable $V^u(x)$ manifolds. These give rise to global stable $W^s(x)$ and global unstable $W^u(x)$ manifolds :

$$W^s(x) = \bigcup_{n\in\mathbb{Z}} f^{-n}(V^s(f^n((x)))$$

and

$$W^u(x) = \bigcup_{n\in\mathbb{Z}} f^{n}(V^u(f^{-n}((x)))$$

4. Non-uniformly hyperbolic maps

A general theory of thermodynamic formalism for non-uniformly hyperbolic maps is far from being complete, although some examples are well-understood [11].

Definition 4.1. [7] Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M. then, f is called a non-uniformly hyperbolic map on an invariant Borel subset $S \subset M$ if there exists a measurable df-invariant decomposition of the tangent space $T_xM = E^s(x) \bigoplus E^u(x)$ for each $x \in S$ and a measurable f-invariant function $\epsilon(x) > 0$ and $0 < \lambda(x) < 1$ such that for every $0 < \epsilon < \epsilon(x)$. We can find measurable functions $c(x) > 0$ and $k(x) > 0$ satisfying for every $x \in U \subset M$,

1. $\|df^n v\| \leq c(x)\lambda(x)^n \|v\|$ for $v \in E^s(x)$, $n \geq 0$;
2. $\|df^{-n} v\| \leq c(x)\lambda(x)^n \|v\|$ for $v \in E^u(x), n \geq 0$;
3. $\angle(E^s(x), E^u(x)) \geq k(x)$;
4. $c(f^m x) \leq e^{\epsilon\ |m|} c(x)$, $k(f^m x) \geq\ e^{-\epsilon\ |m|} k(x)$, $n \in \mathbb{Z}$.

Assume that μ is an invariant measure for f with $\mu(S) = 1$ and if μ is a hyperbolic measure, then, by the Oseledec's Mulitiplicative Ergodic

Theorem, f is a non-uniformly hyperbolic map on S. Recently Climenhaga and Pesin introduce the notion of Effective Hyperbolicity that can be used to prove Hadamard-Perron Theorem [4] and establish existence of SRB measure for non-uniformly hyperbolic maps, even without the method of countable Markov partitions and a dominated splitting.

THEOREM 4.2. **[Climenhaga, Dolgopyat, and Pesin, 2016]** [4, 5, 9] *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and $U \subset \Lambda$, $\overline{f(U)} \subset U$, and $D \subset U$ be a forward invariant set with invariant measurable cone families $K^{s(u)}(x)$, respectively. If $S \subset D$ is the set of $x \in D$ for which forward trajectory of x is effectively hyperbolic and $K^s(x)$ has negative Lyapunov exponents and if the Lebesgue measure of S is positive, then, f admits an SRB measure supported on attractor Λ.*

5. SRB Measures for Non-uniformly Hyperbolic Dynamics

The construction of SRB measures for non-uniformly hyperbolic dynamical systems has two important approaches. The first method is to construct a subshift of countable type due to Sarig or a tower due to Young. This approach allow us to obtain the exponential decay of correlations and the Central Limit Theorem. On the other hand, the second method is based on choosing an appropriate natural invariant measure and then pushing it forward under the non-uniformly hyperbolic dynamics. So, the limit measure in the topology of Λ resulting from this geometric evolution procedure is an SRB measure. This approach allow us to construct SRB measures under more natural settings that is physically relevant for non-uniformly hyperbolic dynamics. Our main result is the construction of SRB measures using geometric t-potential functions of the thermodynamic formalism.

DEFINITION 5.1. [2, 12, 14] Let M be a compact smooth Riemannian manifold with the Borel σ-algebra and let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism and let $\phi : M \to \mathbb{R}$ be a continuous potential, then, An f-invariant probability measure μ_ϕ is called an equilibrium measure for ϕ if the pressure of ϕ

$$P(\phi) = h_{\mu_\phi} + \int \phi d\mu_\phi = \sup\{h_\mu + \int \phi d\mu\}$$

where the supremum is taken over all f-invariant Borel probability measure μ.

THEOREM 5.2. [1,16] *Let $f : \Lambda \to \Lambda$ be a hyperbolic map restricted to a topological attractor Λ and let $\phi : \Lambda \to \mathbb{R}$ be Hölder continuous potential, then there exists a unique equilibrium measure μ_ϕ for ϕ*

THEOREM 5.3. [10,15] *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and $\Lambda \subset M$ be a topological attractor, then, there exists a unique SRB measure μ supported on Λ and it is the equilibrium measure of geometric Hölder continuous potential $\phi(x) = -\log \|Df|_{E^u(x)}\|$ for local unstable manifold $E^u(x)$ of x.*

DEFINITION 5.4. [2,12] Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and let $f : M \to \mathbb{R}$ be a Hölder continuous potential and let μ be an f-invariant Borel probability measure on M, then μ satisfies the Gibbs property for some $P \in \mathbb{R}$ if for each $\epsilon > 0$, there exists $K > 0$ such that

$$\frac{1}{K} \leq \frac{\mu(B_n(x,\epsilon))}{e^{S_n\phi(x)+nP}} \leq K$$

where $B_n(x,\epsilon) = \{y \in M : d(f^k x, f^k y) < \epsilon, \forall 0 \leq k < n\}$ is the Bowen ball around x of order n and radius ϵ, and $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(f^k x)$ is the nth ergodic sum along the orbit of x.

Since the geometric potential $\phi(x) = -\log \|Df|_{E^u(x)}\|$ is Hölder continuous, its equilibrium measure is absolutely continuous invariant measure on unstable manifolds. So it is an SRB measure satisfying the Gibbs property for $C^{1+\alpha}$ diffeomorphism f.

The most important potential function is the geometric t-potential : a family of potential functions $\phi_t(x) = -t \log |df|E^u(x))|$ for all $t \in \mathbb{R}$.

THEOREM 5.5. [6,11] *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism and let the pressure function $P(t) = P(\phi_t)$ for all $t \in \mathbb{R}$. then, the following statements hold :*

1. *$P(t)$ is well defined for all t, and decreasing, convex, and real analytic in t,*
2. *If $t = 0$, $P(0)$ is the topological entropy $h_{top}(f)$ and the corresponding equilibrium measure μ_0 is the unique measure of maximal entropy,*
3. *Using Bowen's equation, there exists $t_0 \in (0,1]$ such that $P(t_0) = 0$,*

4. *When Λ is a topological attractor for f, we have that $t = 1$ and μ_1 is an SRB measure for f.*

Proof. (1) For simplicity, let $P(t) = P(\phi_t) = P(-t\log|Df_x|) = h_{top}(f) - t\lambda$, where λ is Lyapunov exponent, then, P is well defined for all $t \in \mathbb{R}$ and convex, decreasing, and real analytic in t. (2) When $t = 0$, then $P(0) = h_{top}(f)$, and by the Variational Principle, the corresponding equilibrium measure μ_0 is the unique measure of maximal entropy since $h_\mu(f) < h_{\mu_0}(f)$. (3) Since $P(t)$ is monotone decreasing and $P(0) = h_{top}(f)$ and $P(t) \to -\infty$ as $t \to \infty$, there exists a unique number $t_0 > 0$ such that $P(t_0) = 0$, where the equation $P(t) = 0$ is a Bowen's equation. (4) If Λ is a topological hyperbolic attractor, then, the equilibrium measure μ_1 is a hyperbolic ergodic measure satisfying Pesin's entropy formula :

$$h_\mu(f) = \sum_{i:\lambda_i \geq 0} \lambda_i(\mu)$$

So,by Ledrappier and Young, this implies that μ_1 has absolutely continous conditional measures on unstable manifolds. Hence SRB measures are the equlibrium measures for the geometric t-potential ϕ_t at $t = 1$. □

One way to prove the uniqueness of SRB measures is to show that its ergodic component is open (mod 0) in the topology of Λ and $f_{|\Lambda}$ is topologically transitive, that is, for any two open set U and V, there exists n such that $f^n(U) \cap V \neq \emptyset$.

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