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SOLUTIONS OF VECTOR VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we prove the existence results of the solutions for *vector variational inequality problems* by using the $\|\cdot\|$-sequentially continuous mapping.

1. Introduction

Based on the research works originated by Hartmann and Stampacchia [12] in finite dimensional Euclidean spaces, Giannessi [11] studied the vector version of scalar variational inequalities. Vector variational inequalities have been developed and extended in several areas including vector equilibrium problems and vector optimization problems, *see* [1, 4, 6, 9, 10, 15].
Inspired and motivated by recent works [2, 5, 8, 10, 13, 14, 17, 18], in this paper we prove the existence of solutions for *vector variational inequality problems* by using the $\|\cdot\|$-sequentially continuous mapping.

Suppose that X and Y are two Banach spaces. A nonempty subset P of X is called convex cone, if $\lambda P \subseteq P$ for all $\lambda \geq 0$ and $P + P \subset P$. A cone P is called pointed cone if P is a cone and $P \cap (-P) = \{0\}$, where 0 denotes the zero vector. Also, a cone P is called proper if it is properly contained in X. Let K be a nonempty closed convex subset of X and $\mathcal{C} : K \to 2^Y$ be a multivalued mapping such that for each $x \in K, \mathcal{C}(x)$

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is a closed convex cone with $int\mathcal{C}(x) \neq \emptyset$, where $int\mathcal{C}(x)$ denotes the interior of $\mathcal{C}(x)$. The partial order $\leq_{\mathcal{C}(x)}$ on Y induced by $\mathcal{C}(x)$ is defined by declaring $y \leq_{\mathcal{C}(x)} z$ if and only if $z - y \in \mathcal{C}(x)$ for all $x, y, z \in K$. We will write $y \leq_{\mathcal{C}(x)} z$ if $z - y \in int\mathcal{C}(x)$ in the case $int\mathcal{C}(x) \neq \emptyset$. Let $\mathcal{A} : K \subseteq X \to L(X, Y)$ be a mapping where $L(X, Y)$ be the family of all bounded linear mapping from X to Y and $\zeta : K \to X$ be a given operator. The *vector variational inequality problems* for finding $x \in K$ such that

$$\langle \mathcal{A}(x), \zeta(y) - \zeta(x) \rangle \notin -int\mathcal{C}(x), \ \forall y \in K. \tag{1.1}$$

Special Cases:

(i) We note that $\zeta \equiv id_K, id_K : K \to K, id_K(x) = x$. Then (1.1) reduces to the *vector variational inequality problems* for finding $x \in K$ such that

$$\langle \mathcal{A}(x), y - x \rangle \notin -int\mathcal{C}(x), \ \forall y \in K. \tag{1.2}$$

(ii) If $\mathcal{C}(x) = \mathbb{R}_+$ for all $x \in X$, then (1.1) reduces to *general variational inequality problems* for finding $x \in K$ such that

$$\langle \mathcal{A}(x), \zeta(y) - \zeta(x) \rangle \geq 0, \ \forall y \in K. \tag{1.3}$$

(iii) If $\mathcal{C}(x) = \mathbb{R}_+$ for all $x \in X$, then (1.2) reduces to *variational inequality problems* for finding $x \in K$ such that

$$\langle \mathcal{A}(x), y - x \rangle \geq 0, \ \forall y \in K \tag{1.4}$$

studied by Hartmann and Stampacchia [12].

Definition 1.1. Let $\mathcal{C} : K \to 2^Y$ be a multifunction such that $\mathcal{C}(x)$ is a proper closed convex cone with $int\mathcal{C}(x) \neq \emptyset$, then a mapping $g : K \to X$ is called $\mathcal{C}_x$-convex if for each $x, y \in K$ and $\lambda \in [0, 1]$,

$$(1 - \lambda)g(x) + \lambda g(y) - g((1 - \lambda)x + \lambda y) \in \mathcal{C}(x),$$

and called affine if for each $x, y \in K$ and $\lambda \in R$,

$$g((1 - \lambda)x + \lambda y) = \lambda g(x) + (1 - \lambda)g(y).$$

Remark 1.2. If $g : K \to Y$ is a $\mathcal{C}_x$-convex vector valued function, then

$$\sum_{i=1}^{n} \lambda_i g(y_i) - g(\sum_{i=1}^{n} \lambda_i \ y_i) \in \mathcal{C}(x), \ \forall y_i \in K, \ \lambda_i \in [0, 1], \ i = 1, 2, \cdots, n$$

with $\sum_{i=1}^{n} \lambda_i = 1$.

DEFINITION 1.3. Suppose X and Y are two Banach spaces and $T : D \subseteq X \to L(X,Y)$ is said to be weak to $\|\cdot\|$-sequentially continuous at $x \in D$ if for every sequence $\{x_n\} \subseteq D$ that converges weakly to $x \in D$, the sequence $\{T(x_n)\} \subseteq L(X,Y)$ converges to $T(x) \in L(X,Y)$ in the topology of the norm $L(X,Y)$. We say that T is weak to $\|\cdot\|$-sequentially continuous on $D \subseteq X$ and it has the property at every point $x \in D$. The operator $T : D \subseteq X \to X$ is said to be weak to weak-sequentially continuous at $x \in D$, if for every sequence $\{x_n\} \subseteq D$ that converges weakly to $x \in D$, then the sequence $\{T(x_n)\} \subseteq X$ is converges weakly to $T(x) \subseteq X$. We say that T is weak to weak-sequentially continuous on $D \subseteq X$, then it has property at every point $x \in D$.

PROPOSITION 1.4. [13] *Let $\mathcal{A} : K \subseteq X \to L(X,Y)$ be a given operator. If $\mathcal{A}$ is weak to $\|\cdot\|$-sequentially continuous and K is weakly compact and convex. Then variational inequality admits a solution.*

Let Z and Y be two arbitrary sets. The inverse of a mapping $f : Z \to Y$ is defined as the set valued mapping $f^{-1} : Y \rightrightarrows Z$,

$$f^{-1}(y) = \{z \in Z : f(z) = y\}.$$

A single valued selection of a multivalued mapping $F : Z \rightrightarrows Y$ is the single valued mapping $f : Z \to Y$ satisfying

$$f(z) \in F(z), \ \forall z \in Z.$$

THEOREM 1.5. [7] *Let Y be a topological vector space with a pointed closed and convex cone $\mathcal{C}$ such that $int\mathcal{C} \neq \emptyset$, then for all $x, y, z \in Y$, we have*

(i) $x - y \in -int\mathcal{C}$ *and* $x \notin -int\mathcal{C} \Rightarrow y \notin -int\mathcal{C}$;
(ii) $x + y \in -\mathcal{C}$ *and* $x + z \notin -int\mathcal{C} \Rightarrow z - y \notin -int\mathcal{C}$;
(iii) $x + z - y \notin -int\mathcal{C}$ *and* $-y \in -\mathcal{C} \Rightarrow x + z \notin -int\mathcal{C}$;
(iv) $x + y \notin -int\mathcal{C}$ *and* $y - z \in -\mathcal{C} \Rightarrow x + z \notin -int\mathcal{C}$.

2. Main Results

THEOREM 2.1. *Let K be a nonempty subset of X. Let $\mathcal{A} : K \subseteq X \to L(X,Y)$ and $\zeta : K \to X$ be the given operators. Assume that $\zeta(K)$ is weakly compact and convex. Assume further that for every sequence $\{x_n\} \subseteq K$ the following condition holds:*
if the sequence $\{\zeta(x_n)\} \subseteq \zeta(K)$ converges weakly to $\zeta(x) \subseteq \zeta(K)$ then

the sequence $\{\mathcal{A}(x_n)\} \subseteq L(X,Y)$ *is the norm convergent to* $\mathcal{A}(x) \subseteq L(X,Y)$.
Then (1.1) *admits a solution.*

Proof. Consider $\beta : \zeta(K) \to K$ is a single valued selection of ζ^{-1}. Let $\{u_n\} \subseteq \zeta(K)$ be a weakly convergent sequence to $u \in X$. From the weak compactness of $\zeta(K)$, we have $u \in \zeta(K)$. We show that

$$(\mathcal{A} \circ \beta)(u_n) \to (\mathcal{A} \circ \beta)(u), \text{ as } n \to \infty.$$

Since $\{u_n\} \subseteq \zeta(K)$, there exists a sequence $\{x_n\} \subseteq K$ such that $u_n = \zeta(x_n), n \in \mathbb{N}$.
Analogously, $u = \zeta(x)$ for some $x \in K$, then

$$\zeta(\beta(u_n)) = u_n, n \in \mathbb{N} \text{ and } \zeta(\beta(u)) = u.$$

Hence the sequence $\{\zeta(\beta(u_n))\}$ is converges weakly to $\zeta(\beta(u))$. From the hypothesis of the theorem

$$(\mathcal{A} \circ \beta)(u_n) \to (\mathcal{A} \circ \beta)(u), n \to \infty.$$

Hence the operator $\mathcal{A} \circ \beta : \zeta(K) \to L(X,Y)$ is weak to $\|\cdot\|$-sequentially continuous. From Proposition 1.4, there exists $u \in \zeta(K)$ such that

$$\langle (\mathcal{A} \circ \beta)(u), v - u \rangle \notin -int\mathcal{C}(x), \forall v \in \zeta(K).$$

Since for every $y \in K$, there exists $v \in \zeta(K)$ such that

$$\zeta(y) = v,$$

and

$$\langle (\mathcal{A} \circ \beta)(u), \zeta(y) - u \rangle \notin -int\mathcal{C}(x), \forall y \in K.$$

Since $\zeta(\beta(u)) = u$. Thus

$$\langle \mathcal{A}(\beta(u)), \zeta(y) - \zeta(\beta(u)) \rangle \notin -int\mathcal{C}(x), \forall y \in K,$$

or equivalently $\beta(u) \in K$ is a solution of (1.1) □

Remark 2.2. The condition $\{\zeta(x_n)\} \subseteq \zeta(K)$ is converges weakly to $\zeta(x) \subseteq \zeta(K)$, then the sequence $\{\mathcal{A}(x_n)\} \subseteq L(X,Y)$ is norm convergent to $\mathcal{A}(x) \subseteq L(X,Y)$. From Theorem 2.1 implies that

$$\zeta^{-1}(\zeta(x)) \subseteq \mathcal{A}^{-1}(\mathcal{A}(x)) \text{ for } x \in K.$$

Let $x \in K$ and $\zeta(K)$ be the weakly sequentially closed, there exists a sequence $\{\zeta(x_n)\} \subseteq \zeta(K)$ converges to $\zeta(x)$ in the weak topology of X. But the sequence $\{\mathcal{A}(x_n)\}$ converges strongly to $\mathcal{A}(x)$. Let $y \in \zeta^{-1}(\zeta(x))$, then

$$\zeta(y) = \zeta(x),$$

hence $\{\mathcal{A}(x_n)\}$ is converges strongly to $\mathcal{A}(y)$.
Therefore $\mathcal{A}(y) = \mathcal{A}(x)$, hence $y \in \mathcal{A}^{-1}(\mathcal{A}(x))$.

Corollary 2.3. *Let $K \subseteq X$ be a weakly compact, $\mathcal{A} : K \subseteq X \to L(X,Y)$ and $\zeta : K \to X$ be the given operators. Assume that $\zeta(K)$ is convex, ζ is weak to weak-sequentially continuous. Further assume that for every sequence $\{x_n\} \subseteq K$ the following condition holds:*
if the sequence $\{\zeta(x_n)\} \subseteq \zeta(K)$ is converges weakly to $\zeta(x) \subseteq \zeta(K)$ then the sequence $\{\mathcal{A}(x_n)\} \subseteq L(X,Y)$ is norm convergent to $\mathcal{A}(x) \subseteq L(X,Y)$. Then (1.1) *admits a solution.*

Proof. We prove that $\zeta(K)$ is weakly compact and conclusion follows from Theorem 2.1. From Eberlein Smulian Theorem [3], $\zeta(K)$ is weakly compact if and only if, it is weakly sequentially compact. To prove that $\zeta(K)$ is weakly sequentially compact, let $\{u_n\}$ be an arbitrary sequence in $\zeta(K)$. Then there exists a sequence $\{x_n\} \subseteq K$ such that

$$u_n = \zeta(x_n), n \in \mathbb{N}.$$

We show that $\{\zeta(x_n)\}$ has a weakly convergent subsequence in $\zeta(K)$. Since $\{x_n\}$ is a sequence in the weakly compact set K and $\{x_n\}$ has a weakly convergent subsequence. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$, that is weakly converges to $x \in K$. Since ζ is weak to weak sequentially continuous, then $\{\zeta(x_{n_i})\}$ is converges weakly to $\zeta(x)$ and proof is completed. □

Definition 2.4. [11] An operator $T : D \subseteq X \to L(X,Y)$ is called monotone if for all $x, y \in D$, we have

$$\langle T(x) - T(y), y - x\rangle \geq 0.$$

T is monotone relative to the operator $\gamma : D \to X$ if for all $x, y \in D$, we have

$$\langle T(x) - T(y), \gamma(y) - \gamma(x)\rangle \geq 0.$$

We note that $\gamma = id_D$, then T is called continuous on finite dimensional subspaces if for every finite dimensional subspace $M \subseteq X$, the restriction of T to $D \cap M$ is weak continuous, that is for every sequence $\{x_n\} \subseteq D \cap M$ converges to $x \in M$, the sequence $\{A(x_n)\} \subseteq L(X,Y)$ is converges to $A(x)$ in the weak topology of $L(X,Y)$, see [16].

Theorem 2.5. *Let X and Y be the two reflexive Banach spaces. Let $\mathcal{A} : K \subseteq X \to L(X,Y)$ be the monotone relative to $\zeta : K \to X$, where $\zeta(K)$ is weakly compact and convex. Assume that for every*

finite dimensional subset $L \subseteq \zeta(K)$ *and for every sequence* $\{x_n\} \subseteq X$ *such that* $\zeta(x_n) \subset L$, *and if the sequence* $\{\zeta(x_n)\} \subseteq L$ *is converges to* $\zeta(x) \subseteq \zeta(K)$, *then the sequence* $\{\mathcal{A}(x_n)\} \subseteq L(X,Y)$ *is weakly converges to* $\mathcal{A}(x) \subseteq L(X,Y)$.
Then (1.1) *admits a solution.*

Proof. Suppose $\beta : \zeta(K) \to K$ is a single valued selection of ζ^{-1} and $u, v \in \zeta(K)$. Then

$$\langle (\mathcal{A} \circ \beta)(u) - (\mathcal{A} \circ \beta)(v), u - v \rangle = \langle \mathcal{A}(x) - \mathcal{A}(y), \zeta(x) - \zeta(y) \rangle$$

where $x = \beta(u), y = \beta(v)$. Since $\mathcal{A}$ is monotone relative to ζ, we have

$$\langle \mathcal{A}(x) - \mathcal{A}(y), \zeta(x) - \zeta(y) \rangle \notin -int\mathcal{C}(x).$$

Hence the operator $\mathcal{A} \circ \beta : \zeta(K) \to L(X,Y)$ is monotone. Let M be a finite dimensional subspace of X and $L = M \cap \zeta(K)$. Let $\{u_n\} \subseteq L$ be a sequence converges to $u \in \zeta(K)$. Since M is finite dimensional subspace then it is closed. Hence from weak compactness of $\zeta(K)$, we get that $u \in L$.
Now, we have to show that the sequence $\{(\mathcal{A} \circ \beta)(u_n)\} \subseteq L(X,Y)$ converges to $\{(\mathcal{A} \circ \beta)(u)\} \subseteq L(X,Y)$ in the weak topology of $L(X,Y)$. Since $\{u_n\} \subseteq \zeta(K)$, there exists $\{x_n\} \subseteq K$ such that $u_n = \zeta(x_n)$. Analogously $u = \zeta(x)$ for some $x \in K$, since $\beta : \zeta(K) \to K$ is a single valued selection of ζ^{-1}. Observe that $\zeta(\beta(u_n)) = u_n \in L$ and $\zeta(\beta(u)) = u \in L$. Hence $\{\zeta(\beta(u_n))\}$ is converges to $\zeta(\beta(u))$. From the hypothesis of the theorem, the sequence $\{\mathcal{A}(\beta(u_n))\} \subseteq L(X,Y)$ converges weakly to $\mathcal{A}(\beta(u)) \subseteq L(X,Y)$ as $n \to \infty$, which show that $\mathcal{A} \circ \beta$ is continuous on finite dimensional subspace, there exists $u \in \zeta(K)$ such that

$$\langle (\mathcal{A} \circ \beta)(u), v - u \rangle \notin -int\mathcal{C}(x), \forall v \in \zeta(K).$$

Since for every $y \in K$, there exists $v \in \zeta(K)$ such that

$$\zeta(y) = v,$$

we have

$$\langle (\mathcal{A} \circ \beta)(u), \zeta(y) - u \rangle \notin -int\mathcal{C}(x), \forall y \in K.$$

Observe that $\zeta(\beta(u)) = u$. Thus

$$\langle \mathcal{A}(\beta(u)), \zeta(y) - \zeta(\beta(u)) \rangle \notin -int\mathcal{C}(x), \forall v \in K,$$

or equivalently $\beta(u) \in K$ is a solution of (1.1). □

COROLLARY 2.6. *Let X and Y be the two reflexive Banach spaces. Assume that K is weakly compact, $\zeta(K)$ is convex and weak to weak sequentially continuous. Let $\mathcal{A}$ be a monotone relation to ζ. Further, assume that for every finite dimensional subset $L \subseteq \zeta(K)$ and for every sequence $\{x_n\} \subseteq K$ such that $\zeta(x_n) \subseteq L$, and if the sequence $\{\zeta(x_n)\} \subseteq L$ converges to $\zeta(x) \subseteq \zeta(K)$, then the sequence $\{\mathcal{A}(x_n)\} \subseteq L(X,Y)$ is weakly converges to $\mathcal{A}(x) \subseteq L(X,Y)$.*
Then (1.1) *admits a solution.*

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