

RICCI SOLITONS ON RICCI PSEUDOSYMMETRIC $(LCS)_n$ -MANIFOLDS

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Abstract. The object of the present paper is to study some types of Ricci pseudosymmetric $(LCS)_n$ -manifolds whose metric is Ricci soliton. We found the conditions when Ricci soliton on concircular Ricci pseudosymmetric, projective Ricci pseudosymmetric, W_3 -Ricci pseudosymmetric, conharmonic Ricci pseudosymmetric, conformal Ricci pseudosymmetric $(LCS)_n$ -manifolds to be shrinking, steady and expanding. We also construct an example of concircular Ricci pseudosymmetric $(LCS)_3$ -manifold whose metric is Ricci soliton.

1. Introduction

In 2003, Shaikh [43] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$ -manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [34] and also by Mihai and Rosca [35]. Then Shaikh and Baishya ([46], [47]) investigated the applications of $(LCS)_n$ -manifolds to the general theory of relativity and cosmology. The $(LCS)_n$ -manifolds are also studied by Atceken et. al. ([3], [4], [17]), Hui [16], Hui and Chakraborty [18], Narain and Yadav [37], Prakasha [42], Shaikh and his co-authors ([44], [45], [48]–[50], [52], [53]) and many others.

In 1982, Hamilton [14] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([39], [40]) used Ricci flow

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and its surgery to prove the Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphisms and scaling. To be precise, a Ricci soliton on a Riemannian manifold (M, g) is a triple (g, V, λ) satisfying [15].

$$(1.1) \quad \mathcal{L}_V g + 2S + 2\lambda g = 0,$$

where S is the Ricci tensor, \mathcal{L}_V is the Lie derivative along the vector field V on M and $\lambda \in \mathbb{R}$. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [54] Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi et. al ([1], [2], [5], [30]), Bejan and Crasmareanu [6], Blaga [7], Chen and Deshmukh [9], Deshmukh et. al [11], Hui et. al ([8], [18]-[23], [25]-[29]), Nagaraja and Premalatha [36], Tripathi [55] and many others.

The notion of Ricci pseudosymmetric manifold was introduced by Deszcz ([12],[13]). A geometrical interpretation of Ricci pseudosymmetric manifolds in the Riemannian case is given in [32]. A $(LCS)_n$ -manifold (M^n, g) is called Ricci pseudosymmetric ([12], [13]) if the tensor $R \cdot S$ and the Tachibana tensor $Q(g, S)$ are linearly dependent, where

$$(1.2) \quad (R(X, Y) \cdot S)(Z, U) = -S(R(X, Y)Z, U) - S(Z, R(X, Y)U),$$

$$(1.3) \quad Q(g, S)(Z, U; X, Y) = -S((X \wedge_g Y)Z, U) - S(Z, (X \wedge_g Y)U),$$

and

$$(1.4) \quad (X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y$$

for all vector fields X, Y, Z, U of M , R denotes the curvature tensor of M . Then (M^n, g) is Ricci pseudosymmetric if and only if

$$(1.5) \quad (R(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y)$$

holds on $U_S = \{x \in M : S - \frac{r}{n}g \neq 0 \text{ at } x\}$, for some function L_S on U_S . If $R \cdot S = 0$, then M^n is called Ricci semisymmetric. Every Ricci semisymmetric manifold is Ricci pseudosymmetric but the converse is not true [13]. In this connection it is mentioned that Hui et. al ([24], [51]) studied Ricci pseudosymmetric generalized quasi-Einstein manifolds.

Motivated by the above studies, the object of the present paper is to study Ricci pseudosymmetric $(LCS)_n$ -manifolds whose metric is a Ricci soliton. In this connection it is mentioned that Hui and Chakraborty [23] studied Ricci almost solitons on concircular Ricci pseudosymmetric β -Kenmotsu manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries. In section 3, we investigate, Ricci solitons on concircular Ricci pseudosymmetric $(LCS)_n$ -manifolds, projective Ricci pseudosymmetric $(LCS)_n$ -manifolds, W_3 -Ricci pseudosymmetric $(LCS)_n$ -manifolds, conharmonic Ricci pseudosymmetric $(LCS)_n$ -manifolds, conformal Ricci pseudosymmetric $(LCS)_n$ -manifolds respectively. Here each curvature tensor has geometrical significance and hence each type of Ricci pseudosymmetries has different geometrical interpretation. In each of the cases, we found the value of L_S and hence it turns out that the condition that a Ricci soliton is shrinking, steady, or expanding depends on L_S being less than, equal, or greater than certain value. We call it the *Critical Value* for L_S . In each type of Ricci pseudosymmetry, the critical value for L_S is obtained. Finally we construct an example of concircular Ricci pseudosymmetric $(LCS)_3$ -manifold whose metric is Ricci soliton through which Theorem 3.1 is verified.

2. Preliminaries

An n -dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g , that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \dots, +)$, where $T_p M$ denotes the tangent vector space of M at p . A non-zero vector $v \in T_p M$ is said to be timelike (resp. null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp. $= 0, > 0$) [38].

Definition 2.1 ([43],[46]). *In a Lorentzian manifold (M, g) a vector field P defined by*

$$g(X, P) = A(X),$$

for any $X \in \Gamma(TM)$, the section of all smooth tangent vector fields on M , is said to be a **concircular vector field** if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}$$

where α is a non-zero scalar function and ω is a closed 1-form and ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g .

Let M be an n -dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ , called the characteristic vector field of the manifold. Then we have

$$(2.1) \quad g(\xi, \xi) = -1.$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$(2.2) \quad g(X, \xi) = \eta(X),$$

the equation of the following form holds

$$(2.3) \quad (\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0,$$

and consequently, we get

$$(2.4) \quad \nabla_X \xi = \alpha[X + \eta(X)\xi]$$

for all vector fields X, Y , where α satisfying

$$(2.5) \quad \nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X),$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$(2.6) \quad \phi X = \frac{1}{\alpha}\nabla_X \xi,$$

then from (2.4) and (2.6) we have

$$(2.7) \quad \phi X = X + \eta(X)\xi,$$

from which it follows that ϕ is a symmetric $(1, 1)$ tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ , its associated 1-form η and an $(1, 1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$ -manifold), [44]. In particular, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [34]. In a $(LCS)_n$ -manifold ($n > 2$), the following relations hold ([44], [46], [47], [48]):

$$(2.8) \quad \eta(\xi) = -1, \phi\xi = 0, \eta(\phi X) = 0, g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$(2.9) \quad \phi^2 X = X + \eta(X)\xi,$$

$$(2.10) \quad S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X),$$

$$(2.11) \quad R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y],$$

$$(2.12) \quad R(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y],$$

$$(2.13) \quad (\nabla_X \phi)Y = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\},$$

$$(2.14) \quad (X\rho) = d\rho(X) = \beta\eta(X),$$

$$(2.15) \quad R(X, Y)Z = \phi R(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi,$$

$$(2.16) \quad S(\phi X, \phi Y) = S(X, Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y)$$

for any vector fields X, Y, Z on M and $\beta = -(\xi\rho)$ is a scalar function, where R is the curvature tensor and S is the Ricci tensor of the manifold.

Let (g, ξ, λ) be a Ricci soliton on a $(LCS)_n$ -manifold M . From (2.4), we get

$$\begin{aligned} (\mathcal{L}_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= \alpha[g(X + \eta(X)\xi, Y) + g(X, Y + \eta(Y)\xi)] \\ &= 2\alpha[g(X, Y) + \eta(X)\eta(Y)], \end{aligned}$$

i.e.

$$(2.17) \quad \frac{1}{2}(\mathcal{L}_\xi g)(X, Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}.$$

From (1.1) and (2.17) we have

$$(2.18) \quad S(X, Y) = -(\alpha + \lambda)g(X, Y) - \alpha\eta(X)\eta(Y),$$

which yields

$$(2.19) \quad QX = -(\alpha + \lambda)X - \alpha\eta(X)\xi,$$

$$(2.20) \quad S(X, \xi) = -\lambda\eta(X),$$

$$(2.21) \quad r = -\lambda n - (n-1)\alpha,$$

where Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all X, Y and r is the scalar curvature of M .

3. Ricci solitons on Ricci pseudosymmetric $(LCS)_n$ -manifolds

This section deals with the study of Ricci solitons on concircular (resp., projective, W_3 , conharmonic, conformal) Ricci pseudosymmetric $(LCS)_n$ -manifolds. A concircular curvature tensor is an interesting invariant of a concircular transformation. A transformation of a $(LCS)_n$ -manifold M , which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [56]. A concircular transformation is always a conformal transformation [33]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. The interesting invariant of a concircular transformation is the concircular curvature tensor \tilde{C} , which is defined by [56]

$$(3.1) \quad \tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y],$$

where R is the curvature tensor and r is the scalar curvature of the manifold. Also $(LCS)_n$ -manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a $(LCS)_n$ -manifold to be of constant curvature.

Using (2.2), (2.11) and (2.15), we get

$$(3.2) \quad \tilde{C}(X, Y)\xi = [(\alpha^2 - \rho) - \frac{r}{n(n-1)}][\eta(Y)X - \eta(X)Y],$$

$$(3.3) \quad \eta(\tilde{C}(X, Y)U) = [\frac{r}{n(n-1)} - (\alpha^2 - \rho)][\eta(Y)g(X, U) - \eta(X)g(Y, U)].$$

A $(LCS)_n$ -manifold (M^n, g) is said to be concircular Ricci pseudosymmetric if its concircular curvature tensor \tilde{C} satisfies

$$(3.4) \quad (\tilde{C}(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y)$$

on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, where L_S is some function on U_S .

Let us take a concircular Ricci pseudosymmetric $(LCS)_n$ -manifold whose metric is Ricci soliton. Then by virtue of (3.4) that

$$(3.5) \quad S(\tilde{C}(X, Y)Z, U) + S(Z, \tilde{C}(X, Y)U) = L_S[g(Y, Z)S(X, U) - g(X, Z)S(Y, U) + g(Y, U)S(X, Z) - g(X, U)S(Y, Z)].$$

By virtue of (2.18) it follows from (3.5) that

$$(3.6) \quad \begin{aligned} & \eta(\tilde{C}(X, Y)Z)\eta(U) + \eta(Z)\eta(\tilde{C}(X, Y)U) \\ &= L_S[g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U) \\ & \quad + g(Y, U)\eta(X)\eta(Z) - g(X, U)\eta(Y)\eta(Z)]. \end{aligned}$$

Setting $Z = \xi$ in (3.6) and using (3.2) and (3.3), we get

$$(3.7) \quad [L_S - \{(\alpha^2 - \rho) - \frac{r}{n(n-1)}\}][\eta(Y)g(X, U) - \eta(X)g(Y, U)] = 0.$$

Putting $Y = \xi$ in (3.7) and using (2.8) and (2.21), we get

$$(3.8) \quad [L_S - \{(\alpha^2 - \rho) + \frac{\lambda}{n-1} + \frac{\alpha}{n}\}][g(X, U) + \eta(X)\eta(U)] = 0$$

for all vector fields X and U , which follows that

$$(3.9) \quad L_S = (\alpha^2 - \rho) + \frac{\lambda}{n-1} + \frac{\alpha}{n}.$$

This leads to the following:

Theorem 3.1. *If (g, ξ, λ) is a Ricci soliton on a concircular Ricci pseudosymmetric $(LCS)_n$ -manifold, then $L_S = (\alpha^2 - \rho) + \frac{\lambda}{n-1} + \frac{\alpha}{n}$.*

Again from (3.9), we get

$$(3.10) \quad \lambda = (n-1)[L_S - \{\frac{\alpha}{n} + (\alpha^2 - \rho)\}].$$

Since $n > 1$, we have from (3.10) that $\lambda < 0$, $= 0$ and > 0 according as $L_S < \frac{\alpha}{n} + (\alpha^2 - \rho)$, $L_S = \frac{\alpha}{n} + (\alpha^2 - \rho)$ and $L_S > \frac{\alpha}{n} + (\alpha^2 - \rho)$ respectively. This leads to the following:

Corollary 3.2. *In a concircular Ricci pseudosymmetric $(LCS)_n$ -manifold, the Critical Value for L_S is $\frac{\alpha}{n} + (\alpha^2 - \rho)$.*

Example 3.3. We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in \mathbb{R}^3 . Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame on M given by

$$E_1 = z^2 \frac{\partial}{\partial x}, E_2 = z^2 \frac{\partial}{\partial y}, E_3 = \frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by $g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0$, $g(E_1, E_1) = g(E_2, E_2) = 1$, $g(E_3, E_3) = -1$. Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$ and $\phi E_3 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_3) = -1, \phi U = U + \eta(U)E_3$$

and

$$g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Let ∇ be the Levi-Civita connection for the Lorentzian metric g and R be the curvature tensor of g . Then we have

$$[E_1, E_2] = 0, [E_1, E_3] = -\frac{2}{z}E_1, [E_2, E_3] = -\frac{2}{z}E_2.$$

Using Koszul formula for the Lorentzian metric g , we can easily calculate

$$\nabla_{E_1}E_1 = -\frac{2}{z}E_3, \nabla_{E_1}E_2 = 0, \nabla_{E_1}E_3 = -\frac{2}{z}E_1,$$

$$\nabla_{E_2}E_1 = 0, \nabla_{E_2}E_2 = -\frac{2}{z}E_3, \nabla_{E_2}E_3 = -\frac{2}{z}E_2,$$

$$\nabla_{E_3}E_1 =, \nabla_{E_3}E_2 =, \nabla_{E_3}E_3 = 0.$$

From the above, it can be easily seen that for $E_3 = \xi, (\phi, \xi, \eta, g)$ is a $(LCS)_3$ structure on M . Consequently $M^3(\phi, \xi, \eta, g)$ is a $(LCS)_3$ -manifold with $\alpha = -\frac{2}{z} \neq 0$, such that $X(\alpha) = \rho\eta(X)$, where $\rho = -\frac{2}{z^2}$ [52]. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_1, E_2)E_1 = -\frac{4}{z^2}E_2, R(E_1, E_2)E_2 = \frac{4}{z^2}E_1,$$

$$R(E_1, E_3)E_1 = -\frac{6}{z^2}E_3, R(E_1, E_3)E_3 = -\frac{6}{z^2}E_1,$$

$$R(E_2, E_3)E_2 = -\frac{6}{z^2}E_3, R(E_2, E_3)E_3 = -\frac{6}{z^2}E_2$$

and the components which can be obtained from these by the symmetry properties from which, we can easily calculate the non-vanishing components of the Ricci tensor as follows:

$$S(E_1, E_1) = S(E_2, E_2) = -\frac{2}{z^2}, S(E_3, E_3) = -\frac{12}{z^2}.$$

Also, the scalar curvature r is given by:

$$\begin{aligned} r &= \sum_{i=1}^3 g(E_i, E_i)S(E_i, E_i) \\ &= S(E_1, E_1) + S(E_2, E_2) - S(E_3, E_3) \\ &= \frac{8}{z^2}. \end{aligned}$$

Since $\{E_1, E_2, E_3\}$ forms a basis of the $(LCS)_3$ -manifold, any vector field $X, Y, Z, U \in \chi(M)$ can be written as

$$X = a_1 E_1 + b_1 E_2 + c_1 E_3,$$

$$Y = a_2 E_1 + b_2 E_2 + c_2 E_3,$$

$$Z = a_3 E_1 + b_3 E_2 + c_3 E_3,$$

$$U = a_4 E_1 + b_4 E_2 + c_4 E_3,$$

where $a_i, b_i, c_i \in \mathbb{R}^+$ for all $i = 1, 2, 3$ such that a_i, b_i, c_i are not proportional. Then

$$\begin{aligned} (3.11) \quad R(X, Y)Z &= \frac{2}{z^2} \{2b_3(a_1b_2 - a_2b_1) - 3c_3(a_1c_2 - a_2c_1)\} E_1 \\ &- \frac{2}{z^2} \{2a_3(a_1b_2 - a_2b_1) + 3c_3(b_1c_2 - b_2c_1)\} E_2 \\ &- \frac{6}{z^2} \{b_3(b_1c_2 - b_2c_1) + a_3(a_1c_2 - a_2c_1)\} E_3, \end{aligned}$$

$$\begin{aligned} (3.12) \quad R(X, Y)U &= \frac{2}{z^2} \{2b_4(a_1b_2 - a_2b_1) - 3c_4(a_1c_2 - a_2c_1)\} E_1 \\ &- \frac{2}{z^2} \{2a_4(a_1b_2 - a_2b_1) + 3c_4(b_1c_2 - b_2c_1)\} E_2 \\ &- \frac{6}{z^2} \{b_4(b_1c_2 - b_2c_1) + a_4(a_1c_2 - a_2c_1)\} E_3. \end{aligned}$$

In view of (3.11) we have from (3.1) that

$$\begin{aligned} \tilde{C}(X, Y)Z &= R(X, Y)Z - \frac{r}{6} [g(Y, Z)X - g(X, Z)Y] \\ &= \frac{2}{z^2} [2b_3(a_1b_2 - a_2b_1) - 3c_3(a_1c_2 - a_2c_1) \\ &- \frac{2}{3} \{a_1(b_2b_3 - c_2c_3) - a_2(b_1b_3 - c_3c_1)\}] E_1 \\ &- \frac{2}{z^2} [2a_3(a_1b_2 - a_2b_1) + 3c_3(b_1c_2 - b_2c_1) \\ &+ \frac{2}{3} \{b_1(a_2a_3 - c_2c_3) - b_2(a_1a_3 - c_3c_1)\}] E_2 \\ &- \frac{2}{z^2} [3\{b_3(b_1c_2 - b_2c_1) + a_3(a_1c_2 - a_2c_1)\} \\ &+ \frac{2}{3} \{c_1(a_2a_3 + b_2b_3) - c_2(a_1a_3 + b_1b_3)\}] E_3. \end{aligned}$$

and hence

$$\begin{aligned}
 S(\tilde{C}(X, Y)Z, U) &= \frac{4a_4}{z^4} [2b_3(a_1b_2 - a_2b_1) - 3c_3(a_1c_2 - a_2c_1) \\
 &\quad - \frac{2}{3}\{a_1(b_2b_3 - c_2c_3) - a_2(b_1b_3 - c_3c_1)\}] \\
 &\quad + \frac{4b_4}{z^4} [2a_3(a_1b_2 - a_2b_1) + 3c_3(b_1c_2 - b_2c_1) \\
 (3.13) \quad &\quad + \frac{2}{3}\{b_1(a_2a_3 - c_2c_3) - b_2(a_1a_3 - c_3c_1)\}] \\
 &\quad + \frac{24c_4}{z^4} [3\{b_3(b_1c_2 - b_2c_1) + a_3(a_1c_2 - a_2c_1)\} \\
 &\quad + \frac{2}{3}\{c_1(a_2a_3 + b_2b_3) - c_2(a_1a_3 + b_1b_3)\}].
 \end{aligned}$$

Similarly we obtain,

$$\begin{aligned}
 S(Z, \tilde{C}(X, Y)U) &= -\frac{4a_3}{z^4} [2b_4(a_1b_2 - a_2b_1) - 3c_4(a_1c_2 - a_2c_1) \\
 &\quad - \frac{2}{3}\{a_1(b_2b_4 - c_2c_4) - a_2(b_1b_4 - c_1c_4)\}] \\
 &\quad + \frac{4b_3}{z^4} [2a_4(a_1b_2 - a_2b_1) + 3c_4(b_1c_2 - b_2c_1) \\
 (3.14) \quad &\quad + \frac{2}{3}b_1(a_2a_4 - c_2c_4) - b_2(a_1a_4 - c_1c_4) \\
 &\quad + \frac{24c_3}{z^4} [3\{b_4(b_1c_2 - b_2c_1) + a_4(a_1c_2 - a_2c_1)\} \\
 &\quad + \frac{2}{3}\{c_1(a_2a_4 + b_2b_4) - c_2(a_1a_4 + b_1b_4)\}].
 \end{aligned}$$

Now we have

$$(3.15) \quad \begin{cases} g(Y, Z) = a_2a_3 + b_2b_3 - c_2c_3, \\ g(X, Z) = a_1a_3 + b_1b_3 - c_1c_3, \\ g(Y, U) = a_2a_4 + b_2b_4 - c_2c_4, \\ g(X, U) = a_1a_4 + b_1b_4 - c_1c_4. \end{cases}$$

Also we have

$$(3.16) \quad \begin{cases} S(Y, Z) = -\frac{2}{z^2}(a_2a_3 + b_2b_3 + 6c_2c_3), \\ S(X, Z) = -\frac{2}{z^2}(a_1a_3 + b_1b_3 + 6c_1c_3), \\ S(Y, U) = -\frac{2}{z^2}(a_2a_4 + b_2b_4 + 6c_2c_4), \\ S(X, U) = -\frac{2}{z^2}(a_1a_4 + b_1b_4 + 6c_1c_4). \end{cases}$$

Therefore, from (3.15) and (3.16), we have

$$(3.17) \quad \begin{aligned} & g(Y, Z)S(X, U) - g(X, Z)S(Y, U) + g(Y, U)S(X, Z) - g(X, U)S(Y, Z) \\ &= \frac{14}{z^2}[(a_1c_2 - a_2c_1)(a_3c_4 + a_4c_3) + (b_1c_2 - b_2c_1)(b_3c_4 + b_4c_3)] \neq 0, \end{aligned}$$

since a_i, b_i, c_i are not proportional and assume that

$$(a_1c_2 - a_2c_1)(a_3c_4 + a_4c_3) + (b_1c_2 - b_2c_1)(b_3c_4 + b_4c_3) \neq 0.$$

Also from (3.13) and (3.14) we get

$$(3.18) \quad \begin{aligned} & S(\tilde{C}(X, Y)Z, U) + S(Z, \tilde{C}(X, Y)U) \\ &= \frac{196}{3z^4}[(a_1c_2 - a_2c_1)(a_3c_4 + a_4c_3) + (b_1c_2 - b_2c_1)(b_3c_4 + b_4c_3)] \\ &\neq 0. \end{aligned}$$

Let us consider the function

$$(3.19) \quad L_S = \frac{14}{3z^2}.$$

By virtue of (3.19) we have from (3.17) and (3.18) that

$$\begin{aligned} & S(\tilde{C}(X, Y)Z, U) + S(Z, \tilde{C}(X, Y)U) = L_S[g(Y, Z)S(X, U) \\ & - g(X, Z)S(Y, U) + g(Y, U)S(X, Z) - g(X, U)S(Y, Z)]. \end{aligned}$$

Hence the $(LCS)_3$ -manifold under consideration is concircular Ricci pseudosymmetric. If (g, ξ, λ) is a Ricci soliton on this $(LCS)_3$ -manifold, then from (2.21) we get

$$r = -3\lambda - 2\alpha,$$

i.e.,

$$\frac{8}{z^2} = -3\lambda + \frac{4}{z},$$

i.e.,

$$\lambda = \frac{4}{3}\left(\frac{1}{z} - \frac{2}{z^2}\right)$$

and hence from (3.9) we get

$$L_S = (\alpha^2 - \rho) + \frac{\lambda}{2} + \frac{\alpha}{3} = \frac{14}{3z^2}, \quad \text{as } \alpha = -\frac{2}{z}, \rho = -\frac{2}{z^2},$$

which satisfies (3.19). Thus Theorem 3.1 is verified.

Now we study of Ricci solitons on projective Ricci pseudosymmetric $(LCS)_n$ -manifolds. The projective curvature tensor is an important concept of Riemannian geometry, which one uses to calculate the basic geometric measurements on a manifold, namely, angle, distance and

various invariants on it. The projective transformation on a $(LCS)_n$ -manifold ($n > 1$) is a transformation under which geodesic transforms into geodesic. The Weyl projective curvature tensor is given by [10]

$$(3.20) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} [S(Y, Z)X - S(X, Z)Y],$$

where R and S are the curvature tensor and Ricci tensor of the manifold respectively. Using (2.2), (2.11), (2.15) and (2.18), we get

$$(3.21) \quad P(X, Y)\xi = [(\alpha^2 - \rho) + \frac{\lambda}{n-1}] [\eta(Y)X - \eta(X)Y],$$

$$(3.22) \quad \eta(P(X, Y)U) = [(\alpha^2 - \rho) - \frac{\alpha + \lambda}{n-1}] [\eta(Y)g(X, U) - \eta(X)g(Y, U)].$$

A $(LCS)_n$ -manifold (M^n, g) is said to be projective Ricci pseudosymmetric if its projective curvature tensor P satisfies

$$(3.23) \quad (P(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y).$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, where L_S is some function on U_S .

Let us take a projective Ricci pseudosymmetric $(LCS)_n$ -manifold whose metric is Ricci soliton. Then we get from (3.23) that

$$(3.24) \quad S(P(X, Y)Z, U) + S(Z, P(X, Y)U) = L_S [g(Y, Z)S(X, U) - g(X, Z)S(Y, U) + g(Y, U)S(X, Z) - g(X, U)S(Y, Z)].$$

Using (2.18) in (3.24), we get

$$(3.25) \quad \begin{aligned} & (\alpha + \lambda)[g(P(X, Y)Z, U) + g(Z, P(X, Y)U)] \\ & + \alpha[\eta(P(X, Y)Z)\eta(U) + \eta(Z)\eta(P(X, Y)U)] \\ & = \alpha L_S [g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U) \\ & + g(Y, U)\eta(X)\eta(Z) - g(X, U)\eta(Y)\eta(Z)]. \end{aligned}$$

Setting $Z = \xi$ in (3.25), we get

$$(3.26) \quad [\alpha L_S - (\alpha + 2\lambda)(\alpha^2 - \rho)][\eta(Y)g(X, U) - \eta(X)g(Y, U)] = 0.$$

Putting $Y = \xi$ in (3.26) and using (2.8), we get

$$(3.27) \quad [\alpha L_S - (\alpha + 2\lambda)(\alpha^2 - \rho)][g(X, U) + \eta(X)\eta(U)] = 0$$

for all vector fields X and U , from which it follows that

$$(3.28) \quad L_S = \left(1 + \frac{2\lambda}{\alpha}\right)(\alpha^2 - \rho).$$

This leads to the following:

Theorem 3.4. *If (g, ξ, λ) is a Ricci soliton on a projective Ricci pseudosymmetric $(LCS)_n$ -manifold, then $L_S = (1 + \frac{2\lambda}{\alpha})(\alpha^2 - \rho)$.*

Again from (3.28), we get

$$(3.29) \quad \lambda = \frac{\alpha[L_S - (\alpha^2 - \rho)]}{2(\alpha^2 - \rho)}.$$

This leads to the following:

Corollary 3.5. *In a projective Ricci pseudosymmetric $(LCS)_n$ -manifold, the Critical Value for L_S is $(\alpha^2 - \rho)$, provided $\frac{\alpha}{(\alpha^2 - \rho)} > 0$.*

Remark: In [2] Ashoka, Bagewadi and Ingalahalli studied Ricci solitons in $(LCS)_n$ -manifolds satisfying $R(\xi, X) \cdot \tilde{P} = 0$, where \tilde{P} is the pseudo projective curvature tensor. Thus the present result in our paper are not just special cases of results in [2].

Now we study of Ricci solitons on W_3 -Ricci pseudosymmetric $(LCS)_n$ -manifolds. In 1973 Pokhariyal [41] introduced the notion of a new curvature tensor, denoted by W_3 and studied its relativistic significance. The W_3 -curvature tensor of type (1,3) on a $(LCS)_n$ -manifold is defined by

$$(3.30) \quad W_3(X, Y)Z = R(X, Y)Z + \frac{1}{n-1} [g(Y, Z)QX - S(X, Z)Y],$$

where R is the curvature tensor and Q is the Ricci-operator, i.e.,

$$g(QX, Y) = S(X, Y)$$

for all X, Y . Using (2.11), (2.15), (2.18) and (2.19), we get

$$(3.31) \quad \begin{aligned} W_3(X, Y)\xi &= [(\alpha^2 - \rho) - \frac{\lambda}{n-1}][\eta(Y)X - \eta(X)Y] \\ &\quad - \frac{\alpha}{n-1}\eta(Y)[X + \eta(X)\xi], \end{aligned}$$

(3.32)

$$\begin{aligned} \eta(W_3(X, Y)U) &= [(\alpha^2 - \rho) + \frac{\lambda}{n-1}][\eta(Y)g(X, U) - \eta(X)g(Y, U)] \\ &\quad + \frac{\alpha}{n-1}\eta(Y)\{g(X, U) + \eta(X)\eta(U)\}. \end{aligned}$$

A $(LCS)_n$ -manifold (M^n, g) is said to be W_3 -Ricci pseudosymmetric if it satisfies

$$(3.33) \quad (W_3(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y)$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, where L_S is some function on U_S .

Let us take a W_3 -Ricci pseudosymmetric $(LCS)_n$ -manifold whose metric is Ricci soliton. Then we have from (3.33) that

$$(3.34) \quad \begin{aligned} S(W_3(X, Y)Z, U) + S(Z, W_3(X, Y)U) &= L_S[g(Y, Z)S(X, U) \\ &- g(X, Z)S(Y, U) + g(Y, U)S(X, Z) - g(X, U)S(Y, Z)]. \end{aligned}$$

Using (2.18) in (3.34), we get

$$(3.35) \quad \begin{aligned} &(\alpha + \lambda)[g(W_3(X, Y)Z, U) + g(Z, W_3(X, Y)U)] \\ &+ \alpha[\eta(W_3(X, Y)Z)\eta(U) + \eta(Z)\eta(W_3(X, Y)U)] \\ &= \alpha L_S[g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U) \\ &+ g(Y, U)\eta(X)\eta(Z) - g(X, U)\eta(Y)\eta(Z)]. \end{aligned}$$

Setting $Z = \xi$ in (3.35) and using (3.31) and (3.32), we get

$$(3.36) \quad \begin{aligned} &[(\alpha + 2\lambda)(\alpha^2 - \rho) - \frac{\alpha\lambda}{n-1} - \alpha L_S][\eta(Y)g(X, U) - \eta(X)g(Y, U)] \\ &- \frac{\alpha^2}{n-1}\eta(Y)[g(X, U) + \eta(X)\eta(U)] = 0. \end{aligned}$$

Putting $Y = \xi$ in (3.36) and using (2.8), we get

$$(3.37) \quad [L_S - (1 + \frac{2\lambda}{\alpha})(\alpha^2 - \rho) + \frac{\alpha + \lambda}{n-1}][g(X, U) + \eta(X)\eta(U)] = 0$$

for all vector fields X and U , which follows that

$$(3.38) \quad L_S = (1 + \frac{2\lambda}{\alpha})(\alpha^2 - \rho) - \frac{\alpha + \lambda}{n-1}.$$

This leads to the following:

Theorem 3.6. *If (g, ξ, λ) is a Ricci soliton on a W_3 -Ricci pseudosymmetric $(LCS)_n$ -manifold, then L_S is given by (3.38).*

Again from (3.38), we get

$$(3.39) \quad \lambda = \frac{(n-1)\alpha}{2(n-1)(\alpha^2 - \rho) - \alpha} [L_S - \frac{(n-1)(\alpha^2 - \rho) - \alpha}{n-1}].$$

This leads to the following:

Corollary 3.7. *In a W_3 -Ricci pseudosymmetric $(LCS)_n$ -manifold, the critical value for L_S is $(\alpha^2 - \rho - \frac{\alpha}{n-1})$, provided $\frac{(n-1)\alpha}{2(n-1)(\alpha^2 - \rho) - \alpha} > 0$.*

We now study of Ricci solitons on conharmonic Ricci pseudosymmetric $(LCS)_n$ -manifolds. Of considerable interest is a special type of conformal transformations, conharmonic transformations, which are preserving the harmonicity property of smooth functions. This type of

transformation was introduced by Ishii [31] in 1957 and is now studied from various points of view. It is well known that such transformations have a tensor invariant, the so-called conharmonic curvature tensor. It is easy to verify that this tensor is an algebraic curvature tensor; that is, it possesses the classical symmetry properties of the Riemannian curvature tensor. It is known that a harmonic function is defined as a function whose Laplacian vanishes. A harmonic function is not invariant, in general. The conditions under which a harmonic function remains invariant have been studied by Ishii [31] who introduced the conharmonic transformation as a subgroup of the conformal transformation. A manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from conharmonic flatness. As a special subgroup of the conformal transformation group, Ishii [31] introduced the notion of conharmonic transformation under which a harmonic function transform into a harmonic function. The conharmonic curvature tensor of type (1,3) on a Riemannian manifold (M^n, g) , $n > 3$, is given by [31].

$$(3.40) \quad \begin{aligned} \overline{C}(X, Y)Z = & R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X \\ & - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY], \end{aligned}$$

which is invariant under conharmonic transformation, where S is the Ricci tensor of the manifold of type (0,2).

Using (2.2), (2.11), (2.15) (2.18) and (2.19), we get

$$(3.41) \quad \overline{C}(X, Y)\xi = [(\alpha^2 - \rho) + \frac{\alpha + 2\lambda}{n-2}][\eta(Y)X - \eta(X)Y],$$

$$(3.42) \quad \eta(\overline{C}(X, Y)U) = [(\alpha^2 - \rho) - \frac{\alpha + 2\lambda}{n-2}][\eta(Y)g(X, U) - \eta(X)g(Y, U)].$$

A $(LCS)_n$ -manifold (M^n, g) is said to be conharmonic Ricci pseudosymmetric if its conharmonic curvature tensor \overline{C} satisfies

$$(3.43) \quad (\overline{C}(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y).$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, where L_S is some function on U_S .

Let us take a conharmonic Ricci pseudosymmetric $(LCS)_n$ -manifold whose metric is Ricci soliton. Then we get by virtue of (3.43) that

$$(3.44) \quad \begin{aligned} S(\overline{C}(X, Y)Z, U) + S(Z, \overline{C}(X, Y)U) = & L_S[g(Y, Z)S(X, U) \\ & - g(X, Z)S(Y, U) + g(Y, U)S(X, Z) - g(X, U)S(Y, Z)]. \end{aligned}$$

By virtue of (2.18) it follows from (3.44) that

$$\begin{aligned}
 (3.45) \quad & \eta(\overline{C}(X, Y)Z)\eta(U) + \eta(Z)\eta(\overline{C}(X, Y)U) \\
 & = L_S[g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U) \\
 & \quad + g(Y, U)\eta(X)\eta(Z) - g(X, U)\eta(Y)\eta(Z)].
 \end{aligned}$$

Setting $Z = \xi$ in (3.45) and using (3.41) and (3.42), we get

$$(3.46) \quad [L_S + (\alpha^2 - \rho) - \frac{\alpha + 2\lambda}{n-2}][\eta(Y)g(X, U) - \eta(X)g(Y, U)] = 0.$$

Putting $Y = \xi$ in (3.46) and using (2.8), we get

$$(3.47) \quad [L_S + (\alpha^2 - \rho) - \frac{\alpha + 2\lambda}{n-2}][g(X, U) + \eta(X)\eta(U)] = 0$$

for all vector fields X and U , which follows that

$$(3.48) \quad L_S = \frac{\alpha + 2\lambda}{n-2} - (\alpha^2 - \rho).$$

This leads to the following:

Theorem 3.8. *If (g, ξ, λ) is a Ricci soliton on a conharmonic Ricci pseudosymmetric $(LCS)_n$ -manifold, then $L_S = \frac{\alpha + 2\lambda}{n-2} - (\alpha^2 - \rho)$.*

Again from (3.48), we get

$$(3.49) \quad \lambda = \frac{1}{2}[(n-2)\{L_S + (\alpha^2 - \rho)\} - \alpha].$$

Since $n > 2$, we have from (3.49) that $\lambda < 0$, $= 0$ and > 0 according as $L_S < , = \text{ and } > \frac{\alpha}{n-2} - (\alpha^2 - \rho)$, respectively. This leads to the following:

Corollary 3.9. *In a conharmonic Ricci pseudosymmetric $(LCS)_n$ -manifold, Critical Value for L_S is $\frac{\alpha}{n-2} - (\alpha^2 - \rho)$.*

Now we study of Ricci solitons on conformal Ricci pseudosymmetric $(LCS)_n$ manifolds. In differential geometry, the Weyl curvature tensor, named after Hermann Weyl, is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic. The Weyl tensor is the traceless component of the Riemann curvature tensor [33]. Since the trace component of the Riemann curvature tensor, i.e. the Ricci curvature, contains precisely the information about how volumes change in the presence of tidal forces, the Weyl tensor does not convey information on how the volume of the manifold changes, but rather only how the shape of the body is distorted by the tidal force.

In general relativity, the Weyl curvature is the only part of the curvature that exists in free space—a solution of the vacuum Einstein equation—and it governs the propagation of gravitational radiation through regions of space devoid of matter. More generally, the Weyl curvature is the only component of curvature for Ricci-flat manifolds and always governs the characteristics of the field equations of an Einstein manifold. In dimensions 2 and 3 the Weyl curvature tensor vanishes identically. In dimensions ≥ 4 , the Weyl curvature is generally nonzero. If the Weyl tensor vanishes in dimension ≥ 4 , then the metric is locally conformally flat: there exists a local coordinate system in which the metric tensor is proportional to a constant tensor. This fact was a key component of Nordström's theory of gravitation, which was a precursor of general relativity.

The Weyl tensor has the special property that it is invariant under conformal changes to the metric. For this reason the Weyl tensor is also called the conformal tensor. It follows that the necessary condition for a Riemannian manifold to be conformally flat is that the Weyl tensor vanish. In dimensions ≥ 4 this condition is sufficient as well. In dimension 3 the vanishing of the Cotton tensor is the necessary and sufficient condition for the Riemannian manifold being conformally flat. Any 2-dimensional (smooth) Riemannian manifold is conformally flat, a consequence of the existence of isothermal coordinates. Conformal transformations of a Riemannian structures are an important object of study in differential geometry.

The conformal transformation on a $(LCS)_n$ -manifold is a transformation under which the angle between two curves remains invariant. The Weyl conformal curvature tensor C of type (1,3) of an n -dimensional Riemannian manifold $(LCS)_n$ ($n > 3$) is defined by [10]

$$\begin{aligned}
 (3.50) \quad C(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y \\
 &\quad + g(Y, Z)QX - g(X, Z)QY] \\
 &\quad + \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\},
 \end{aligned}$$

where R , S , Q and r are the Curvature tensor, Ricci tensor, Ricci-operator and scalar curvature of the manifold respectively. Using (2.2), (2.11), (2.15) (2.18) and (2.19), we get

$$(3.51) \quad C(X, Y)\xi = [(\alpha^2 - \rho) + \frac{\lambda}{n-1}][\eta(Y)X - \eta(X)Y],$$

$$(3.52) \quad \eta(C(X, Y)U) = [(\alpha^2 - \rho) - \frac{\lambda}{n-1}][\eta(Y)g(X, U) - \eta(X)g(Y, U)].$$

A $(LCS)_n$ -manifold (M^n, g) is said to be conformal Ricci pseudosymmetric if its conformal curvature tensor C satisfies

$$(3.53) \quad (C(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y).$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, where L_S is some function on U_S .

Let us take a conformal Ricci pseudosymmetric $(LCS)_n$ -manifold whose metric is Ricci soliton. Then we have from (3.53) that

$$(3.54) \quad \begin{aligned} S(C(X, Y)Z, U) + S(Z, C(X, Y)U) &= L_S[g(Y, Z)S(X, U) \\ &\quad - g(X, Z)S(Y, U) + g(Y, U)S(X, Z) - g(X, U)S(Y, Z)]. \end{aligned}$$

By virtue of (2.18) it follows from (3.54) that

$$(3.55) \quad \begin{aligned} \eta(C(X, Y)Z)\eta(U) + \eta(Z)\eta(C(X, Y)U) \\ = L_S[g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U) \\ + g(Y, U)\eta(X)\eta(Z) - g(X, U)\eta(Y)\eta(Z)]. \end{aligned}$$

Setting $Z = \xi$ in (3.55) and using (3.51) and (3.52), we get

$$(3.56) \quad [L_S + (\alpha^2 - \rho) - \frac{\lambda}{n-1}][\eta(Y)g(X, U) - \eta(X)g(Y, U)] = 0.$$

Putting $Y = \xi$ in (3.56) and using (2.8), we get

$$(3.57) \quad [L_S + (\alpha^2 - \rho) - \frac{\lambda}{n-1}][g(X, U) + \eta(X)\eta(U)] = 0$$

for all vector fields X and U , which follows that

$$(3.58) \quad L_S = \frac{\lambda}{n-1} - (\alpha^2 - \rho).$$

This leads to the following:

Theorem 3.10. *If (g, ξ, λ) is a Ricci soliton on a conformal Ricci pseudosymmetric $(LCS)_n$ -manifold, then L_S is given by (3.58).*

Again from (3.58), we get

$$(3.59) \quad \lambda = (n-1)[L_S + (\alpha^2 - \rho)].$$

Since $n > 1$, we have from (3.59) that $\lambda < 0$, $= 0$ and > 0 according as $L_S < -(\alpha^2 - \rho) < 0$, $L_S = -(\alpha^2 - \rho) = 0$ and $L_S > -(\alpha^2 - \rho) > 0$ respectively. This leads to the following:

Corollary 3.11. *In a conformal Ricci pseudosymmetric $(LCS)_n$ -manifold, the Critical Value for L_S is $-(\alpha^2 - \rho)$.*

4. Summary

From Theorem 3.1, Theorem 3.4, Theorem 3.6, Theorem 3.8 and Theorem 3.10, we have the following:

Let (g, ξ, λ) be a Ricci soliton on a $(LCS)_n$ manifold M . Then the following holds:

M	L_S
Concircular Ricci Pseudosymmetric	$(\alpha^2 - \rho) + \frac{\lambda}{n-1} + \frac{\alpha}{n}$
Projective Ricci Pseudosymmetric	$(1 + \frac{2\lambda}{\alpha})(\alpha^2 - \rho)$
W_3 -Ricci Pseudosymmetric	$(1 + \frac{2\lambda}{\alpha})(\alpha^2 - \rho) - \frac{\alpha + \lambda}{n-1}$
Conharmonic Ricci Pseudosymmetric	$\frac{\alpha + 2\lambda}{n-2} - (\alpha^2 - \rho)$
Conformal Ricci Pseudosymmetric	$\frac{\lambda}{n-1} - (\alpha^2 - \rho)$

Again, from Corollary 3.2, Corollary 3.5, Corollary 3.7, Corollary 3.9 and Corollary 3.11, we have the following:

In a $(LCS)_n$ -manifold M , the following holds:

M	Critical value for L_S
Concircular Ricci Pseudosymmetric	$\frac{\alpha}{n} + (\alpha^2 - \rho)$
Projective Ricci Pseudosymmetric	$(\alpha^2 - \rho)$, provided $\alpha > \alpha^2 - \rho$
W_3 -Ricci Pseudosymmetric	$(\alpha^2 - \rho) - \frac{\alpha}{n-1}$, provided $\frac{(n-1)\alpha}{2(n-1)(\alpha^2) - \alpha} > 0$
Conharmonic Ricci Pseudosymmetric	$\frac{\alpha}{n-2} - (\alpha^2 - \rho)$
Conformal Ricci Pseudosymmetric	$-(\alpha^2 - \rho)$

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