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SUBORDINATION RESULTS FOR CERTAIN SUBCLASSES BY USING INTEGRAL OPERATOR DEFINED IN THE SPACE OF ANALYTIC FUNCTIONS

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Abstract. In this study, firstly we introduce generalized differential and integral operator, also using integral operator two classes are presented. Furthermore, some subordination results involving the Hadamard product (Convolution) for these subclasses of analytic function are proved. A number of consequences of some of these subordination results are also discussed.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad |z| < 1 \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. In [6], Darus and Faisal introduced the following differential operator. For $f \in \mathcal{A}$,

$$\begin{aligned}
D_{\lambda}^{0}(\alpha,\beta,\mu)f(z) &= f(z)\\
D_{\lambda}^{1}(\alpha,\beta,\mu)f(z) &= \left(\frac{\alpha-\mu+\beta-\lambda}{\alpha+\beta}\right) f(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right) zf'(z)\\
D_{\lambda}^{2}(\alpha,\beta,\mu)f(z) &= D\left(D_{\lambda}^{1}(\alpha,\beta,\mu)f(z)\right)\\
&\vdots
\end{aligned}$$

$$D_{\lambda}^{n}(\alpha,\beta,\mu)f(z) = D\left(D_{\lambda}^{n-1}(\alpha,\beta,\mu)f(z)\right). \tag{2}$$

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If f is given by (1) then from (2), it can obtained

$$D_\lambda^n(\alpha,\beta,\mu)f(z) = z + \sum_{k=2}^{\infty}\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n a_k z^k \tag{3}$$

where $f \in \mathcal{A}; \alpha,\beta,\mu,\lambda \geq 0; \alpha+\beta \neq 0;\ n \in \mathbb{N}_0$.

For special cases of the parameters of $D_\lambda^n(\alpha,\beta,\mu)$, it can obtained the well-known differential operators in [1],[2],[3],[4],[5],[11],[13].

Also, in [7], Faisal et al. defined the following new integral operator. For, $f \in \mathcal{A}$,

$$C^0(\alpha,\beta,\mu,\lambda)f(z) = f(z)$$

$$C^1(\alpha,\beta,\mu,\lambda)f(z) = \left(\frac{\alpha+\beta}{\mu+\lambda}\right) z^{1-(\frac{\alpha+\beta}{\mu+\lambda})} \int_0^z t^{(\frac{\alpha+\beta}{\mu+\lambda})-2} f(t)dt$$

$$C^2(\alpha,\beta,\mu,\lambda)f(z) = \left(\frac{\alpha+\beta}{\mu+\lambda}\right) z^{1-(\frac{\alpha+\beta}{\mu+\lambda})} \int_0^z t^{(\frac{\alpha+\beta}{\mu+\lambda})-2} C^1(\alpha,\beta,\mu,\lambda)f(t)dt$$

$$\vdots$$

(4)

$$C^m(\alpha,\beta,\mu,\lambda)f(z) = \left(\frac{\alpha+\beta}{\mu+\lambda}\right) z^{1-(\frac{\alpha+\beta}{\mu+\lambda})} \int_0^z t^{(\frac{\alpha+\beta}{\mu+\lambda})-2} C^{m-1}(\alpha,\beta,\mu,\lambda)f(t)dt.$$

If f is given by (1) then from (4) it can be written as

$$C^m(\alpha,\beta,\mu,\lambda)f(z) = z + \sum_{k=2}^{\infty}\left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k-1)+\beta}\right)^m a_k z^k \tag{5}$$

where $f \in \mathcal{A}; \alpha,\beta,\mu,\lambda \geq 0; \alpha+\beta \neq 0; \mu+\lambda \neq 0; m \in \mathbb{N}_0$.

For special cases of the parameters of $C^m(\alpha,\beta,\mu,\lambda)$, it can obtained the well-known integral operators in [8],[9],[10].

Definition 1.1 ([7])**.** *The function $f \in \mathcal{A}$ is said to **belong to the class** $M(\alpha,\beta,\mu,\lambda,\delta)$ if it satisfies the following analytic criterion*

$$\Re\left\{\frac{z(C^m(\alpha,\beta,\mu,\lambda)f(z))'}{C^m(\alpha,\beta,\mu,\lambda)f(z)}\right\} > \delta; \qquad 0 \leq \delta < 1. \tag{6}$$

Definition 1.2 ([7]). *The function $f \in \mathcal{A}$ is said to* ***belong to the class*** $N(\alpha, \beta, \mu, \lambda, \delta)$ *if it satisfies the following analytic criterion*

$$\Re\left\{\frac{(z(C^m(\alpha,\beta,\mu,\lambda)f(z))')'}{(C^m(\alpha,\beta,\mu,\lambda)f(z))'}\right\} > \delta; \qquad 0 \le \delta < 1. \tag{7}$$

Theorem 1.3 ([7]). *If an analytic function $f \in \mathcal{A}$ satisfies the following inequality*

$$\sum_{k=2}^{\infty}(k-\delta)\left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k-1)+\beta}\right)^m |a_k| \le 1-\delta \tag{8}$$

then $f \in M(\alpha, \beta, \mu, \lambda, \delta)$.

Theorem 1.4 ([7]). *If an analytic function $f \in \mathcal{A}$ satisfies the following inequality*

$$\sum_{k=2}^{\infty}k(k-\delta)\left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k-1)+\beta}\right)^m |a_k| \le 1-\delta \tag{9}$$

then $f \in N(\alpha, \beta, \mu, \lambda, \delta)$.

In view of Theorem 1.3 and Theorem 1.4, we now introduce the subclasses $M^*(\alpha, \beta, \mu, \lambda, \delta) \subset M(\alpha, \beta, \mu, \lambda, \delta)$ and $N^*(\alpha, \beta, \mu, \lambda, \delta) \subset N(\alpha, \beta, \mu, \lambda, \delta)$ which consist of functions $f \in \mathcal{A}$ whose Taylor-Maclaurin coefficients a_k satisfy the inequalities (8) and (9), respectively. In our proposed investigation of functions in the classes $M^*(\alpha, \beta, \mu, \lambda, \delta)$ and $N^*(\alpha, \beta, \mu, \lambda, \delta)$, we shall also make use of the following definitions and theorem.

Definition 1.5. *(Hadamard Product or Convolution) Given two functions $f, g \in \mathcal{A}$ where f is given by (1) and g is given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the* ***Hadamard product (or convolution)*** *$f * g$ is defined (as usual) by*

$$(f*g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g*f)(z). \tag{10}$$

Definition 1.6. *(Subordination Principle) For two functions f and g, analytic in $\mathbb{U}$, we say that the* ***function f is subordinate to*** *g, written $f \prec g$, if there exists an analytic Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{U}$, then the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.*

Definition 1.7. *(Subordinating factor sequences) A sequence* $\{c_k\}_{k=1}^{\infty}$ *of complex numbers is called a* ***subordinating factor sequence*** *if, whenever* f *is analytic, univalent and convex in* $\mathbb{U}$, *we have the subordination is given by*

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) \qquad (z \in \mathbb{U},\ a_1 = 1). \tag{11}$$

Theorem 1.8 ([14])**.** *Let the sequence* $\{c_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$\Re\{1 + 2\sum_{k=1}^{\infty} c_k z^k\} > 0 \qquad (z \in \mathbb{U}). \tag{12}$$

In this paper, we prove an interesting subordination results for the classes $M^*(\alpha,\beta,\mu,\lambda,\delta)$ and $N^*(\alpha,\beta,\mu,\lambda,\delta)$.

2. SUBORDINATION RESULTS FOR THE CLASSES $M^*(\alpha,\beta,\mu,\lambda,\delta)$ AND $M(\alpha,\beta,\mu,\lambda,\delta)$

Theorem 2.1. *Let the function* f *defined by* (1) *be in the class* $M^*(\alpha,\beta,\mu,\lambda,\delta)$. *Also, let* K *denotes familiar class of functions* $f \in \mathcal{A}$ *which are univalent and convex in* $\mathbb{U}$. *Then for* $z \in \mathbb{U}, m \in \mathbb{N}_0$ *and every function* h *in* K

$$\frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]}(f * h)(z) \prec h(z), \tag{13}$$

and

$$\Re\{f(z)\} > -\left\{1 + \frac{1-\delta}{2-\delta}\left(1 + \frac{\mu+\lambda}{\alpha+\beta}\right)^m\right\}. \tag{14}$$

The following constant factor

$$\frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]} \tag{15}$$

in the subordination result (13) *can not be replaced by a larger one.*

Proof. Let $f \in M^*(\alpha,\beta,\mu,\lambda,\delta)$ and assume that $h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in K$. Then we have

$$\frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]}(f*h)(z)$$
$$= \frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]}\left(z + \sum_{k=2}^{\infty} a_k c_k z^k\right).$$

Thus, by Definition 1.7, the subordination result (13) will hold true if the sequence

$$\left\{\frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Theorem 1.8, this equivalent to the following inequality,

$$\Re\left\{1 + \sum_{k=1}^{\infty} \frac{(2-\delta)(\alpha+\beta)^m}{(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m} a_k z^k\right\} > 0, (z \in \mathbb{U}). \tag{16}$$

Now, since $(k-\delta)(\alpha+\beta)^m$ $(k \geq 2)$ is an increasing function of k, we have

$$\Re\left\{1 + \sum_{k=1}^{\infty} \frac{(2-\delta)(\alpha+\beta)^m}{(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m} a_k z^k\right\}$$
$$= \Re\left\{1 + \frac{(2-\delta)(\alpha+\beta)^m}{(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m} z\right.$$
$$\left. + \frac{(\alpha+\mu+\lambda+\beta)^m}{(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m} \sum_{k=2}^{\infty} \frac{(2-\delta)(\alpha+\beta)^m}{(\alpha+\mu+\lambda+\beta)^m} a_k z^k\right\}$$
$$\geq 1 - \frac{(2-\delta)(\alpha+\beta)^m}{(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m} r$$
$$- \frac{(\alpha+\mu+\lambda+\beta)^m}{(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m}$$
$$\times \sum_{k=2}^{\infty} (k-\delta)\left(\frac{\alpha+\beta}{\alpha+(\mu+\lambda)(k-1)+\beta}\right)^m |a_k| r^k$$
$$> 1 - \frac{(2-\delta)(\alpha+\beta)^m}{(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m} r$$
$$- \frac{(1-\delta)(\alpha+\mu+\lambda+\beta)^m}{(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m} r$$

$= 1-r > 0 \qquad (|z| = r < 1),$

where we have also made use of assertion (8) of Theorem 1.3. Thus, the inequality (16) holds true in $\mathbb{U}$, this evidently proves the inequality (13). The inequality (14) follows from (13) upon setting $h(z) = \frac{z}{1-z} = \sum_{k=1}^{\infty} z^k \in K$. To prove the sharpness of the constant

$$\frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]},$$

we consider the function

$$f_0(z) = z - \frac{(1-\delta)}{(2-\delta)} \left(\frac{\alpha+\mu+\lambda+\beta}{\alpha+\beta} \right)^m z^2 \tag{17}$$

which is a member of the class $M^*(\alpha, \beta, \mu, \lambda, \delta)$. Then by using (13), we have

$$\frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]} f_0(z) \prec \frac{z}{1-z}, (z \in \mathbb{U}).$$

Moreover, it can be easily be verified for the function $f_0(z)$ given by (17) that

$$\min_{|z|\le r} \left\{ \Re \frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]} f_0(z) \right\} = -\frac{1}{2},$$

which shows that the constant $\frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]}$ is the best estimate. Thus proof is complete. □

Corollary 2.2. *Let the function f defined by* (1) *be in the class $M(\alpha, \beta, \mu, \lambda, \delta)$. Then the assertions* (13) *and* (14) *of Theorem 2.1 hold true. Furthermore, the following constant factor*

$$\frac{(2-\delta)(\alpha+\beta)^m}{2[(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m]}$$

can not be replaced by a larger one.
If we take $m = 0$ and $\delta = 0$ in Theorem 2.1, we have the following corollaries, respectively.

Corollary 2.3. *Let the function f is in the class $S^*(\delta)$ which is the class of starlike functions of order δ. Then we have $\frac{(2-\delta)}{2(3-2\delta)}(f * h)(z) \prec h(z)$, $h \in K$. In particular, $\Re\{f(z)\} > -\frac{3-2\delta}{2-\delta}$. The constant $\frac{(2-\delta)}{2(3-2\delta)}$ is the best estimate.*

Corollary 2.4 ([12]). *Let the function f is in the class S^* which is well-known the class of starlike functions, then we have $\frac{1}{3}(f * h)(z) \prec h(z)$, $h \in K$. In particular, $\Re\{f(z)\} > -\frac{3}{2}$. The constant $\frac{1}{3}$ is the best estimate.*

3. SUBORDINATION RESULTS FOR THE CLASSES $N^*(\alpha, \beta, \mu, \lambda, \delta)$ AND $N(\alpha, \beta, \mu, \lambda, \delta)$

Theorem 3.1. *Let the function f defined by (1) be in the class $N^*(\alpha, \beta, \mu, \lambda, \delta)$. Also, let K denotes familiar class of functions $f \in \mathcal{A}$ which are univalent and convex in $\mathbb{U}$. Then for $z \in \mathbb{U}, m \in \mathbb{N}_0$ and every function h in K*

$$\frac{(2-\delta)(\alpha+\beta)^m}{2(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m}(f * h)(z) \prec h(z), \tag{18}$$

and

$$\Re\{f(z)\} > -\left\{1 + \frac{1-\delta}{2(2-\delta)}\left(1 + \frac{\mu+\lambda}{\alpha+\beta}\right)^m\right\}. \tag{19}$$

The following constant factor

$$\frac{(2-\delta)(\alpha+\beta)^m}{2(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m} \tag{20}$$

in the subordination result (18) can not be replaced by a larger one.

Since the proof of the Theorem 3.1 is similar to the proof of Theorem 2.1, we will avoid doing the proof of Theorem 3.1.

Corollary 3.2. *Let the function f defined by (1) be in the class $N(\alpha, \beta, \mu, \lambda, \delta)$. Then the assertions (18) and (19) of Theorem 3.1 hold true. Furthermore, the following constant factor,*

$$\frac{(2-\delta)(\alpha+\beta)^m}{2(2-\delta)(\alpha+\beta)^m + (1-\delta)(\alpha+\mu+\lambda+\beta)^m}$$

can not be replaced by a larger one.

If we take $m = 0$ and $\delta = 0$ in Theorem 3.1, we have the following corollaries, respectively.

Corollary 3.3. *Let the function f is in the class $K(\delta)$ which is the class of convex functions of order δ, then we have $\frac{2-\delta}{5-3\delta}(f * h)(z) \prec h(z)$, $h \in K$. In particular, $\Re\{f(z)\} > -\frac{5-3\delta}{2(2-\delta)}$. The constant $\frac{2-\delta}{5-3\delta}$ is the best estimate.*

Corollary 3.4. *Let the function f is in the class K which is well-known the class of convex functions, then we have $\frac{2}{5}(f * h)(z) \prec h(z)$, $h \in K$. In particular, $\Re\{f(z)\} > -\frac{5}{4}$. The constant $\frac{2}{5}$ is the best estimate.*

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