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REIDEMEISTER TORSION AND ORIENTABLE PUNCTURED SURFACES

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Abstract. Let $\Sigma_{g,n,b}$ denote the orientable surface obtained from the closed orientable surface Σ_g of genus $g \geq 2$ by deleting the interior of $n \geq 1$ distinct topological disks and $b \geq 1$ points. Using the notion of symplectic chain complex, the present paper establishes a formula for computing Reidemeister torsion of the surface $\Sigma_{g,n,b}$ in terms of Reidemeister torsion of the closed surface Σ_g, Reidemeister torsion of disk, and Reidemeister torsion of punctured disk.

1. Introduction

Reidemeister torsion is not only a topological invariant but also an invariant of the basis of the homology of a manifold [2]. It was first introduced by K. Reidemeister [5], where he classified 3-dimensional lens spaces with the help of this invariant. In 1935, W. Franz was able to classify higher dimensional lens spaces [1] by extending the notion of Reidmeister torsion. For more information about Reidemeister torsion, we refer the reader [4, 11].

In the present article, we consider the homologies of the surfaces with untwisted coefficients. Using the definition of Reidemeister torsion and homological algebraic computations, we prove formulas for computing Reidemeister torsion of unit disk (Proposition 3.4) and once punctured unit disk (Proposition 3.5). Moreover, with the help of these formulas, we establish a new formula to compute Reidemeister torsion of orientable surfaces $\Sigma_{g,n,b}$ (Theorem 3.10). The techniques to prove these results are similar therefore one of them will be proved completely and others will be stated.

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2. Preliminaries

This section provides the required definitions and the basic facts about Reidemeister torsion and symplectic chain complex. Further information and the detailed proofs can be found in [2, 4, 6, 10–12] and the references therein.

Assume that $C_* : 0 \to C_n \overset{\partial_n}{\to} C_{n-1} \to \cdots \to C_1 \overset{\partial_1}{\to} C_0 \to 0$ is a chain complex of finite dimensional vector spaces over the field of real numbers $\mathbb{R}$. For $p = 0, \ldots, n$, let $B_p(C_*)$, $Z_p(C_*)$, and $H_p(C_*) = Z_p(C_*)/B_p(C_*)$ denote $\mathrm{Im}\partial_{p+1}$, $\mathrm{Ker}\partial_p$, and p-th homology of the chain complex, respectively. Using the definition of $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$, we have the following short-exact sequences

$$0 \to Z_p(C_*) \overset{\imath}{\to} C_p \overset{\partial_p}{\to} B_{p-1}(C_*) \to 0, \tag{1}$$

$$0 \to B_p(C_*) \overset{\imath}{\to} Z_p(C_*) \overset{\varphi_p}{\to} H_p(C_*) \to 0. \tag{2}$$

Here, $\imath$ and φ_p are the inclusion and the natural projection, respectively.

Throughout the paper, we denote by $s_p : B_{p-1}(C_*) \to C_p$ and $\ell_p : H_p(C_*) \to Z_p(C_*)$ section of $\partial_p : C_p \to B_{p-1}(C_*)$ and $\varphi_p : Z_p(C_*) \to H_p(C_*)$, respectively. By the Splitting Lemma for the sequences (1) and (2), we get

$$C_p = B_p \oplus \ell_p(H_p(C_*)) \oplus s_p(B_{p-1}(C_*)). \tag{3}$$

If $\mathbf{c_p}$, $\mathbf{b_p}$, and $\mathbf{h_p}$ are bases of C_p, $B_p(C_*)$, and $H_p(C_*)$, respectively, then by equation (3), we obtain the basis $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ for C_p, $p = 0, \ldots, n$.

Reidemeister torsion of the chain complex C_* with respect to bases $\{\mathbf{c}_p\}_0^n$, $\{\mathbf{h}_p\}_0^n$ is defined by

$$\prod_{p=0}^{n} [\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p]^{(-1)^{(p+1)}},$$

where $[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p]$ is the determinant of the change-base-matrix from basis $\mathbf{c}_p$ to $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ of C_p.

In [2], J. Milnor proved that Reidemeister torsion is independent of the bases $\mathbf{b}_p$, and sections s_p, ℓ_p.

Remark 2.1. Let $\mathbf{c}'_p, \mathbf{h}'_p$ be also bases of C_p, $H_p(C_*)$, respectively. Then, the following change-base formula holds [2]

$$\mathbb{T}\left(C_*, \{\mathbf{c}'_p\}_0^n, \{\mathbf{h}'_p\}_0^n\right) = \prod_{p=0}^{n} \left(\frac{[\mathbf{c}'_p, \mathbf{c}_p]}{[\mathbf{h}'_p, \mathbf{h}_p]}\right)^{(-1)^p} \mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right).$$

Consider the short-exact sequence of chain complexes $0 \to A_* \overset{\imath}{\to} B_* \overset{\pi}{\to} D_* \to 0$, and the corresponding long-exact sequence obtained by the Snake Lemma

$$C_* : \cdots \to H_p(A_*) \overset{\imath_p}{\to} H_p(B_*) \overset{\pi_p}{\to} H_p(D_*) \overset{\delta_p}{\to} H_{p-1}(A_*) \to \cdots.$$

Here, $C_{3p} = H_p(D_*)$, $C_{3p+1} = H_p(A_*)$, and $C_{3p+2} = H_p(B_*)$. Clearly, the bases $\mathbf{h}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^B$ serve as bases for C_{3p}, C_{3p+1}, and C_{3p+2}, respectively.

Theorem 2.2 ([2]). *Let $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ be bases of A_p, B_p, D_p, $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. Suppose $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, and $\mathbf{c}_p^D$ are compatible in the sense that $[\mathbf{c}_p^B, \mathbf{c}_p^A \sqcup \widetilde{\mathbf{c}_p^D}] = \pm 1$, where $\pi_p\left(\widetilde{\mathbf{c}_p^D}\right) = \mathbf{c}_p^D$. Then,*

$$\begin{aligned}\mathbb{T}(B_*, \{\mathbf{c}_p^B\}_0^n, \{\mathbf{h}_p^B\}_0^n) = {} & \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\, \mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n) \\ & \times\ \mathbb{T}(C_*, \{\mathbf{c}_{3p}\}_0^{3n+2}).\end{aligned}$$

Clearly, Theorem 2.2 immediately yields the following result:

Lemma 2.3. *Suppose that A_*, D_* are chain complexes, and $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^D$ are bases of A_p, D_p, $H_p(A_*)$, $H_p(D_*)$, respectively. Then, the formula is valid*

$$\begin{aligned}&\mathbb{T}(A_* \oplus D_*, \{\mathbf{c}_p^A \sqcup \mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^A \sqcup \mathbf{h}_p^D\}_0^n) \\ &= \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n).\end{aligned}$$

A detailed proof of Lemma 2.3 can also be found in [8].

A *symplectic chain complex* $(C_*, \partial_*, \{\omega_{*,q-*}\})$ of length q is a chain complex C_* with the following properties: $q \equiv 2 (\mathrm{mod}\ 4)$ and for $p = 0, \dots, q/2$, there is a non-degenerate bilinear form $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{R}$ such that

- ∂-*compatible*: $\omega_{p,q-p}(\partial_{p+1}a, b) = (-1)^{p+1}\omega_{p+1,q-(p+1)}(a, \partial_{n-p}b)$,
- *anti-symmetric*: $\omega_{p,q-p}(a, b) = (-1)^{p(q-p)}\omega_{q-p,p}(b, a)$.

Note that by $q \equiv 2 \pmod 4$, we easily have $\omega_{p,q-p}(a, b) = (-1)^p\omega_{q-p,p}(b, a)$. Clearly, it follows from ∂-compatibility that the non-degenerate anti-symmetric bilinear maps $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{R}$ can be extended to homologies [6].

Let C_* be a symplectic chain complex. Let $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ be bases of C_p and C_{q-p}, respectively. We say these bases are ω-*compatible*, if the matrix of $\omega_{p,q-p}$ in bases $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ equals to $I_{k\times k}$ when $p \neq q/2$ and it equals to $\begin{pmatrix} 0_{\ell\times\ell} & I_{\ell\times\ell} \\ -I_{\ell\times\ell} & 0_{\ell\times\ell}\end{pmatrix}$ when $p = q/2$. Here, $0_{\ell\times\ell}$ and $I_{\ell\times\ell}$ are respectively zero and the identity matrix, and also $k = \dim C_p = \dim C_{q-p}$, $2\ell = \dim C_{q/2}$.

Using the fact that every symplectic chain complex has a ω-compatible basis, the following result was proved in [6].

Theorem 2.4. *Let $(C_*, \partial_*, \{\omega_{*,q-*}\})$ be a symplectic chain complex and $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ be ω-compatible bases of C_p, C_{q-p}, respectively. For $p = 0, \dots, q$, let $\mathbf{h}_p$ be a basis of $H_p(C_*)$. Then,*

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q) = \prod_{p=0}^{(q/2)-1} (\det[\omega_{p,q-p}])^{(-1)^p} \sqrt{\det[\omega_{q/2,q/2}]}^{(-1)^{q/2}}.$$

Here, $\det[\omega_{p,q-p}]$ is the determinant of the matrix of the non-degenerate pairing $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathbb{R}$ in the bases $\mathbf{h}_p$, $\mathbf{h}_{q-p}$.

Suppose M is a smooth m-manifold with a cell decomposition K. Let $\mathbf{c}_i$ be the geometric basis for the i-cells $C_i(K)$, $i = 0, \ldots, m$. Associated to M there is the chain-complex $0 \to C_m(K) \overset{\partial_m}{\to} C_{m-1}(K) \to \cdots \to C_1(K) \overset{\partial_1}{\to} C_0(K) \to 0$, where ∂_i is the usual boundary operator. $\mathbb{T}(C_*(K), \{\mathbf{c}_i\}_0^m, \{\mathbf{h}_i\}_0^m)$ is called *Reidemeister torsion* of M. Here, $\mathbf{h}_i$ is a basis of $H_i(M) = H_i(M; \mathbb{R})$, $i = 0, \ldots, m$.

Following the arguments introduced in [6, Lemma 2.0.5], one can also conclude that Reidemeister torsion of a manifold M is independent of the cell-decomposition K of M. Hence, instead of $\mathbb{T}(C_*(K), \{\mathbf{c}_i\}_0^m, \{\mathbf{h}_i\}_0^m)$, we write $\mathbb{T}(M, \{\mathbf{h}_i\}_0^m)$.

It was proved in [8] that Theorem 2.4 yields the following formula for computing Reidemeister torsion of even dimensional smooth manifolds in terms of the intersection number pairings:

Theorem 2.5. *Assume that M is an orientable closed connected $2m$-manifold ($m \geq 1$). Assume also that $\mathbf{h}_i$ is a basis of $H_i(M)$, $i = 0, \ldots, 2m$. Then, the following formula holds*

$$
\begin{aligned}
|\mathbb{T}(M, \{\mathbf{h}_i\}_0^{2m})| = & \prod_{i=0}^{m-1} \left|\det \triangle_{i,2m-i}^M(\mathbf{h}_i, \mathbf{h}_{2m-i})\right|^{(-1)^i} \\
& \times \sqrt{|\det \triangle_{m,m}^M(\mathbf{h}_m, \mathbf{h}_m)|}^{(-1)^m}.
\end{aligned}
\tag{4}
$$

Here, $\triangle_{i,2m-i}^M(\mathbf{h}_i, \mathbf{h}_{2m-i})$ is the matrix of intersection pairing $(\cdot,\cdot)_{i,2m-i}$: $H_p(M) \times H_{2m-i}(M) \to \mathbb{R}$ in bases $\mathbf{h}_i$, $\mathbf{h}_{2m-i}$.

Remark 2.6. In the case of odd dimensional smooth manifolds it was proved in [8] that $|T(M, \{\mathbf{h}_i\}_0^{2m+1})| = 1$, where $\mathbf{h}_i$ is a basis of $H_i(M)$, $i = 0, \ldots, 2m+1$. Thus, in particular for the circle $\mathbb{S}^1$, we have $|\mathbb{T}(\mathbb{S}^1, \{\mathbf{h}_i\}_0^1)| = 1$, where $\mathbf{h}_i$ is a basis of $H_i(\mathbb{S}^1)$, $i = 0, 1$.

For further applications of Theorem 2.4, we refer the reader to [3,6–10] and the references therein.

3. Main result

In this section, we first improve the formula (4) in Theorem 2.5, where the absolute values are removed. By the notion of symplectic chain complex, we establish novel formulas to compute Reidemeister torsion of disk and punctured disk. Then applying these formulas, we also prove a formula for computing Reidemeister torsion of orientable surface $\Sigma_{g,n,b}$ in terms of Reidemeister torsion of orientable closed surface Σ_g, Reidemeister torsion of disk, and Reidemeister torsion of punctured disk.

Lemma 3.1. *Let M be an orientable closed connected $2m$-manifold. Then, there exists a basis $\mathbf{h}_i^{M,o}$ of $H_i(M)$, $i = 0, \ldots, 2m$ such that*

$$\mathbb{T}(M, \{\mathbf{h}_i^{M,o}\}_0^{2m}) = 1.$$

Proof. Let M be an orientable closed connected $2m$-manifold. Let us first consider the case, where m is odd. By the non-degeneracy of the intersection number pairings, there is a basis $\mathbf{h}_i^{M,o}$ of $H_i(M)$, $i=0,\ldots,2m$ so that for $i\neq m$ the matrix of $\triangle^M_{i,2m-i}(\mathbf{h}_i^{M,o},\mathbf{h}_{2m-i}^{M,o})$ is the identity matrix, for $i=m$ the matrix of $\triangle^M_{m,m}(\mathbf{h}_m^{M,o},\mathbf{h}_m^{M,o})$ equals to $\begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ -I_{d\times d} & 0_{d\times d}\end{pmatrix}$. Here, $d=\dim_{\mathbb{R}} H_m(M)$.

By Theorem 2.5, we have $|\mathbb{T}(M,\{\mathbf{h}_i^{M,o}\}_0^{2m})|=1$. Without loss of generality, we can assume $\mathbb{T}(M,\{\mathbf{h}_i^{M,o}\}_0^{2m})$ is 1. Otherwise, we can rearrange the basis $\mathbf{h}_i^{M,o}$, $i=0,\ldots,2m$ so that Reidemeister torsion of M in the new basis is 1. More precisely, if $\mathbb{T}(M,\{\mathbf{h}_i^{M,o}\}_0^{2m})$ is (-1), we can rearrange the bases $\{\mathbf{h}_i^{M,o}\}_0^{2m}$ as follows: let $i_k\in\{0,\ldots,2m\}$ be the first index such that the space $H_{i_k}(M)$ is non-zero. If dimension of $H_{i_k}(M)$ is 1, then replace $\mathbf{h}_{i_k}^{M,o}$ with $-\mathbf{h}_{i_k}^{M,o}$. Otherwise, switch the first two basis elements in $\mathbf{h}_{i_k}^{M,o}$. We denote the new basis again by $\mathbf{h}_{i_k}^{o}$. Thus, by Remark 2.1, Reidemeister torsion of M in the new bases is 1.

Let us now consider the case, where M is an orientable closed connected $2m$-manifold with m even. There exists a basis $\mathbf{h}_i^{M,o}$ of $H_i(M)$, $i=0,\ldots,2m$ so that for $i\neq m$, the matrix of $\triangle^M_{i,2m-i}(\mathbf{h}_i^{M,o},\mathbf{h}_{2m-i}^{M,o})$ equals to identity matrix, and for $i=m$, the matrix of $\triangle^M_{m,m}(\mathbf{h}_m^{M,o},\mathbf{h}_m^{M,o})$ equals to the diagonal matrix $\mathrm{D}=\mathrm{Diag}(\overbrace{1,\ldots,1}^{\mathbf{p}_D},\overbrace{-1,\ldots,-1}^{\mathbf{n}_D})_{k\times k}$, where k is $\dim H_m(M)$, $\mathbf{p}_D$ is the number of positive elements, and $\mathbf{n}_D$ is the number of negative elements. By using the arguments as stated in the odd case, we obtain the required result. ☐

To alleviate the cumbersome, we introduce the following notation. Let M and $\mathbf{h}_i^{M,o}$ be as in Lemma 3.1. If $\mathbf{h}_i^{M}$ is a basis of $H_i(M)$, $i=0,\ldots,2m$, then we set

$$\delta(\mathbf{h}_i^{M},\mathbf{h}_i^{M,o})=\begin{cases} 1 & ;\text{if } \mathbf{h}_i^{M},\mathbf{h}_i^{M,o} \text{ are in the same orientation class,}\\ -1 & ;\text{if } \mathbf{h}_i^{M},\mathbf{h}_i^{M,o} \text{ are not in the same orientation class.}\end{cases}$$

Let us denote by $n_M=n_M(\{\mathbf{h}_i^{M}\}_0^{2m})$ the cardinality of the set

$$\{i:\mathbf{h}_i^{M},\mathbf{h}_i^{M,o} \text{ are not in the same orientation class}\}.$$

Proposition 3.2. *Let M be an orientable closed connected $2m$-manifold. Assume that $\mathbf{h}_i^{M}$ is a basis of $H_i(M)$, $i=0,\ldots,2m$. Assume also that $\mathbf{h}_i^{M,o}$ is a basis of $H_i(M)$ as in Lemma 3.1. Then, we have the following formulas*:

(i) *for m odd,*

$$\mathbb{T}(M,\{\mathbf{h}_i^{M}\}_0^{2m})=\delta(\mathbf{h}_m^{M,o},\mathbf{h}_m^{M})\prod_{i=0}^{m-1}(\det\triangle^M_{i,2m-i}(\mathbf{h}_i^{M},\mathbf{h}_{2m-i}^{M}))^{(-1)^i}$$
$$\times\sqrt{\det\triangle^M_{m,m}(\mathbf{h}_m^{M},\mathbf{h}_m^{M})}^{(-1)^m},$$

(ii) *for m even,*

$$\mathbb{T}(M,\{\mathbf{h}_i^M\}_0^{2m}) = (-1)^{n_M}\ \delta(\mathbf{h}_m^{M,o},\mathbf{h}_m^M)\ \prod_{i=0}^{m-1}(\det\triangle_{i,2m-i}^M(\mathbf{h}_i^M,\mathbf{h}_{2m-i}^M))^{(-1)^i}$$
$$\times\ \sqrt{(-1)^{\boldsymbol{n}_D}\det\triangle_{m,m}^M(\mathbf{h}_m^M,\mathbf{h}_m^M)}^{(-1)^m}.$$

Here, $\boldsymbol{n}_D$ is the number of negative elements of the matrix of $\triangle_{m,m}^M(\mathbf{h}_m^{M,o},\mathbf{h}_m^{M,o})$ which equals to $k\times k$ diagonal matrix $\mathrm{D}=\mathrm{Diag}(1,\dots,1,-1,\dots,-1)$.

Proof. First, we consider the case, where m is odd. By using Lemma 3.1 and Remark 2.1, we get

$$\mathbb{T}\left(M,\{\mathbf{h}_i^M\}_0^{2m}\right) = \prod_{i=0}^{2m}[\mathbf{h}_i^M,\mathbf{h}_i^{M,o}]^{(-1)^{i+1}}. \tag{5}$$

Moreover, from Theorem 2.5 and the fact that m is odd it follows that

$$|\mathbb{T}(M,\{\mathbf{h}_i^M\}_0^{2m})| = \prod_{i=0}^{m-1}\left|\det\triangle_{i,2m-i}^M(\mathbf{h}_i^M,\mathbf{h}_{2m-i}^M)\right|^{(-1)^i}$$
$$\times\ \sqrt{\det\triangle_{m,m}^M(\mathbf{h}_m^M,\mathbf{h}_m^M)}^{(-1)^m}. \tag{6}$$

From equation (5) it follows that

$$|\mathbb{T}(M,\{\mathbf{h}_i^M\}_0^{2m})| = (-1)^{n_M}\ \mathbb{T}(M,\{\mathbf{h}_i^M\}_0^{2m}). \tag{7}$$

By the property of the basis $\mathbf{h}_i^{M,o}$, $i=0,\dots,2m$, we have for $i\neq m$

$$\det\triangle_{i,2m-i}^M(\mathbf{h}_i^M,\mathbf{h}_{2m-i}^M) = [\mathbf{h}_i^{M,o},\mathbf{h}_i^M]\ [\mathbf{h}_{2m-i}^{M,o},\mathbf{h}_{2m-i}^M] \tag{8}$$

for $i=m$, we get

$$\det\triangle_{m,m}^M(\mathbf{h}_m^M,\mathbf{h}_m^M) = [\mathbf{h}_m^{M,o},\mathbf{h}_m^M]^2. \tag{9}$$

Considering equations (8) and (9), we obtain the following equalities

$$|\det\triangle_{i,2m-i}^M(\mathbf{h}_i^M,\mathbf{h}_{2m-i}^M)| = \delta(\mathbf{h}_i^{M,o},\mathbf{h}_i^M)\ \delta(\mathbf{h}_{2m-i}^{M,o},\mathbf{h}_{2m-i}^M)$$
$$\times\ [\mathbf{h}_i^{M,o},\mathbf{h}_i^M]\ [\mathbf{h}_{2m-i}^{M,o},\mathbf{h}_{2m-i}^M], \tag{10}$$

$$\sqrt{\det\triangle_{m,m}^M(\mathbf{h}_m^M,\mathbf{h}_m^M)} = \delta(\mathbf{h}_m^{M,o},\mathbf{h}_m^M)\ [\mathbf{h}_m^{M,o},\mathbf{h}_m^M]. \tag{11}$$

Furthermore, it follows from equation (11) that

$$[\mathbf{h}_m^{M,o},\mathbf{h}_m^M] = \delta(\mathbf{h}_m^{M,o},\mathbf{h}_m^M)\ \sqrt{\det\triangle_{m,m}^M(\mathbf{h}_m^M,\mathbf{h}_m^M)}. \tag{12}$$

By equations (6), (10), and (11), we obtain

$$|\mathbb{T}(M,\{\mathbf{h}_i^M\}_0^{2m})| = \prod_{i=0}^{2m}\delta(\mathbf{h}_i^M,\mathbf{h}_i^{M,o})^{(-1)^i}\ [\mathbf{h}_m^M,\mathbf{h}_m^M]^{(-1)^m}$$

$$\times \prod_{i=0}^{m-1} (\det \triangle^M_{i,2m-i}(\mathbf{h}^M_i, \mathbf{h}^M_{2m-i}))^{(-1)^i}. \tag{13}$$

Combining equations (12) and (13), we get

$$|\mathbb{T}(M, \{\mathbf{h}^M_i\}_0^{2m})| = (-1)^{n_M}\ \delta(\mathbf{h}^{M,o}_m, \mathbf{h}^M_m) \prod_{p=0}^{m-1} (\det \triangle^M_{i,2m-i}(\mathbf{h}^M_i, \mathbf{h}^M_{2m-i}))^{(-1)^i}$$

$$\times \sqrt{\det \triangle^M_{m,m}(\mathbf{h}^M_m, \mathbf{h}^M_m)}^{(-1)^m}. \tag{14}$$

Hence, from equations (7) and (14) it follows

$$\mathbb{T}(M, \{\mathbf{h}^M_i\}_0^{2m}) = \delta(\mathbf{h}^{M,o}_m, \mathbf{h}^M_m) \prod_{i=0}^{m-1} (\det \triangle^M_{i,2m-i}(\mathbf{h}^M_i, \mathbf{h}^M_{2m-i}))^{(-1)^i}$$

$$\times \sqrt{\det \triangle^M_{m,m}(\mathbf{h}^M_m, \mathbf{h}^M_m)}^{(-1)^m}. \tag{15}$$

Now, we consider the case, where M is an orientable closed connected $2m$-manifold with even m. By using Lemma 3.1, we get $\mathbb{T}(M, \{\mathbf{h}^{M,o}_i\}_0^{2m}) = 1$.

Recall that the number of negative elements of D denoted by $\mathbf{n}_D$. Then, we obtain the following formula

$$\mathbb{T}(M, \{\mathbf{h}^M_i\}_0^{2m}) = (-1)^{n_M}\ \delta(\mathbf{h}^{M,o}_m, \mathbf{h}^M_m) \prod_{i=0}^{m-1} (\det \triangle^M_{i,2m-i}(\mathbf{h}^M_i, \mathbf{h}^M_{2m-i}))^{(-1)^i}$$

$$\times \sqrt{(-1)^{\mathbf{n}_D} \det \triangle^M_{m,m}(\mathbf{h}^M_m, \mathbf{h}^M_m)}^{(-1)^m}. \tag{16}$$

This finishes the proof of Proposition 3.2. $\square$

Applying Proposition 3.2 yields following:

Proposition 3.3. *Let Σ_g be an orientable closed surface with genus $g \geq 0$. For $i = 0, 1, 2$ let $\mathbf{h}^{\Sigma_g}_i$ be a basis of $H_i(\Sigma_g)$ and $\mathbf{h}^{\Sigma_g,o}_i$ be the basis of $H_i(\Sigma_g)$ as in Lemma 3.1. Then, we have*

$$\mathbb{T}(\Sigma_g, \{\mathbf{h}^{\Sigma_g}_i\}_0^2) = \delta(\mathbf{h}^{\Sigma_g,o}_1, \mathbf{h}^{\Sigma_g}_1)\ \det \triangle^{\Sigma_g}_{0,2}(\mathbf{h}^{\Sigma_g}_0, \mathbf{h}^{\Sigma_g}_2) \sqrt{\det \triangle^{\Sigma_g}_{1,1}(\mathbf{h}^{\Sigma_g}_1, \mathbf{h}^{\Sigma_g}_1)}^{(-1)}.$$

Note that by Proposition 3.3, Reidemeister torsion of sphere $\mathbb{S}^2$ satisfies the formula $T(\mathbb{S}^2, \{\mathbf{h}^{\mathbb{S}^2}_0, 0, \mathbf{h}^{\mathbb{S}^2}_2\}) = (\mathbf{h}^{\mathbb{S}^2}_0, \mathbf{h}^{\mathbb{S}^2}_2)_{0,2}$. Here, $\delta(\mathbf{h}^{\mathbb{S}^2}_1, \mathbf{h}^{\mathbb{S}^2}_1) = \delta(0, 0) = 1$ and we use the convention that $1 \cdot 0 = 0$.

3.1. Reidemeister torsion of unit disk

In this subsection, we establish a formula for computing Reidemeister torsion of closed unit disk $\mathbb{D}$.

Proposition 3.4. *Let $\mathbb{D}$ be the closed unit disk in the plane and $d(\mathbb{D})$ be the double of $\mathbb{D}$. Let $\mathbf{h}_0^{\mathbb{D}}$ be a basis of $H_0(\mathbb{D})$ and $\mathbf{h}_1^{\mathbb{S}^1}$ be an arbitrary basis of $H_1(\mathbb{S}^1)$. Assume that $f : H_2(d(\mathbb{D})) \to H_1(\mathbb{S}^1)$ is the isomorphism obtained by Mayer-Vietoris long-exact sequence associated to the short-exact sequence*

$$0 \to C_*(\mathbb{S}^1) \longrightarrow C_*(\mathbb{D}) \oplus C_*(\mathbb{D}) \longrightarrow C_*(d(\mathbb{D})) \to 0,$$

and $\mathbf{h}_2^{d(\mathbb{D})} = f^{-1}(\mathbf{h}_1^{\mathbb{S}^1})$ is the basis of $H_2(d(\mathbb{D}))$. If $\mathbf{h}_0^{\mathbb{D},o}$ is a basis of $H_0(\mathbb{D})$ so that $\mathbb{T}(\mathbb{D}, \{\mathbf{h}_0^{\mathbb{D},o}\})$ is 1, then there is a basis $\mathbf{h}_0^{\mathbb{S}^1}$ of $H_0(\mathbb{S}^1)$ and a basis $\mathbf{h}_0^{d(\mathbb{D})}$ of $H_0(d(\mathbb{D}))$ so that Reidemeister torsion of $\mathbb{D}$ satisfies the following equality

$$\mathbb{T}(\mathbb{D}, \{\mathbf{h}_0^{\mathbb{D}}\}) = \delta(\mathbf{h}_0^{\mathbb{D},o}, \mathbf{h}_0^{\mathbb{D}}) \sqrt{\delta(\mathbf{h}_0^{\mathbb{S}^2}, \mathbf{h}_0^{\mathbb{S}^2,o})\, \delta(\mathbf{h}_2^{\mathbb{S}^2}, \mathbf{h}_2^{\mathbb{S}^2,o})\, (\mathbf{h}_0^{\mathbb{S}^2}, \mathbf{h}_2^{\mathbb{S}^2})_{0,2}}.$$

Here, $\mathbf{h}_i^{\mathbb{S}^2}$ is the basis of $H_i(\mathbb{S}^2)$ associated to $\mathbf{h}_i^{d(\mathbb{D})}$ of $H_i(d(\mathbb{D}))$ by a fixed homeomorphism $\alpha : \mathbb{D} \to \mathbb{S}^2$ and $(\cdot,\cdot)_{0,2} : H_0(\mathbb{S}^2) \times H_2(\mathbb{S}^2) \to \mathbb{R}$ is the intersection number pairing, and $\mathbf{h}_i^{\mathbb{S}^2,o}$ is the basis of $H_i(\mathbb{S}^2)$, $i = 0, 2$ such that

$$\mathbb{T}(\mathbb{S}^2, \{\mathbf{h}_i^{\mathbb{S}^2,o}, 0, \mathbf{h}_i^{\mathbb{S}^2,o}\}) = 1.$$

Proof. Let us consider the natural short-exact sequence of chain complexes

$$0 \to C_*(\mathbb{S}^1) \longrightarrow C_*(\mathbb{D}) \oplus C_*(\mathbb{D}) \longrightarrow C_*(d(\mathbb{D})) \to 0 \tag{17}$$

and corresponding long-exact sequence obtained by the Snake Lemma

$$\begin{aligned} \mathcal{H}_* : 0 \to H_2(d(\mathbb{D})) &\xrightarrow{f} H_1(\mathbb{S}^1) \xrightarrow{g} 0 \xrightarrow{h} H_0(\mathbb{S}^1) \xrightarrow{i} H_0(\mathbb{D}) \oplus H_0(\mathbb{D}) \\ &\xrightarrow{j} H_0(d(\mathbb{D})) \xrightarrow{k} 0. \end{aligned} \tag{18}$$

Note that from exactness of the sequence (18), f and j become isomorphism. By using the bases $\mathbf{h}_0^{\mathbb{D}}$ of $H_0(\mathbb{D})$, $\mathbf{h}_1^{\mathbb{S}^1}$ of $H_1(\mathbb{S}^1)$, $\mathbf{h}_2^{d(\mathbb{D})}$ of $H_2(d(\mathbb{D}))$, and the isomorphisms f and j, we will obtain bases of $H_0(d(\mathbb{D}))$ and $H_0(\mathbb{S}^1)$ so that Reidemeister torsion of $\mathcal{H}_*$ in the corresponding bases is 1.

Let us first denote the vector spaces in the long-exact sequence (18) by $C_p(\mathcal{H}_*)$, $p = 0, \dots, 5$. Clearly, for all p, we have the following short-exact sequence

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \twoheadrightarrow B_{p-1}(\mathcal{H}_*) \to 0. \tag{19}$$

For each p, let us consider the isomorphism $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*))$ obtained by the First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$. Then, we obtain

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{20}$$

Consider the space $C_0(\mathcal{H}_*) = H_0(d(\mathbb{D}))$ in (20). The fact that $\operatorname{Im} k = 0$ yields

$$C_0(\mathcal{H}_*) = \operatorname{Im} j \oplus s_0(\operatorname{Im} k) = \operatorname{Im} j. \tag{21}$$

Note that there is a non-zero vector (α_1, α_2) in the plane such that $\mathbf{h}_0^{d(\mathbb{D})} = \alpha_1 j(\mathbf{h}_0^{\mathbb{D}}, 0) + \alpha_2 j(0, \mathbf{h}_0^{\mathbb{D}})$ is a basis of $C_0(\mathcal{H}_*)$. Let us consider the initial basis $\mathbf{h}_0$ of $C_0(\mathcal{H}_*)$ as $\mathbf{h}_0^{d(\mathbb{D})}$. As $\operatorname{Im} j$ is equal to $H_0(d(\mathbb{D}))$, $\mathbf{h}_0$ can be chosen as the basis of $\operatorname{Im} j$. From equation (21) it follows that $\mathbf{h}_0$ is also the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. Thus, we get

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{22}$$

Using (20) for the space $C_1(\mathcal{H}_*) = H_0(\mathbb{D}) \oplus H_0(\mathbb{D})$, we have

$$C_1(\mathcal{H}_*) = \operatorname{Im} i \oplus s_1(\operatorname{Im} j). \tag{23}$$

As $\operatorname{Im} i$ is a 1-dimensional subspace of $C_1(\mathcal{H}_*)$, there exists a non-zero vector (a_{11}, a_{12}) in the plane so that $\{a_{11}(\mathbf{h}_0^{\mathbb{D}}, 0) + a_{12}(0, \mathbf{h}_0^{\mathbb{D}})\}$ is a basis of $\operatorname{Im} i$. Let us take the basis $\mathbf{h}^{\operatorname{Im} i}$ of $\operatorname{Im} i$ as $\{K_1[a_{11}(\mathbf{h}_0^{\mathbb{D}}, 0) + a_{12}(0, \mathbf{h}_0^{\mathbb{D}})]\}$, where the non-zero constant K_1 will be determined later.

Since 1-dimensional space $s_1(\operatorname{Im} j)$ is also a subspace of $C_1(\mathcal{H}_*)$, there is a non-zero vector (a_{21}, a_{22}) in the plane such that $\{a_{21}(\mathbf{h}_0^{\mathbb{D}}, 0) + a_{22}(0, \mathbf{h}_0^{\mathbb{D}})\}$ is a basis of $s_1(\operatorname{Im} j)$. Recall that $\mathbf{h}'_0 = \alpha_1 j(\mathbf{h}_0^{\mathbb{D}}, 0) + \alpha_2 j(0, \mathbf{h}_0^{\mathbb{D}})$ was chosen as the basis of $\operatorname{Im} j$ in the previous step. Then, $s_1(\mathbf{h}'_0)$ becomes a basis of $s_1(\operatorname{Im} j)$ and hence $s_1(\mathbf{h}'_0) = \mu[a_{21}(\mathbf{h}_0^{\mathbb{D}}, 0) + a_{22}(0, \mathbf{h}_0^{\mathbb{D}})]$ for some real number $\mu \neq 0$.

Note that $\{(\mathbf{h}_0^{\mathbb{D}}, 0), (0, \mathbf{h}_0^{\mathbb{D}})\}$ is the initial basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$. From equation (23), it follows that $\{\mathrm{K}_1[a_{11}(\mathbf{h}_0^{\mathbb{D}}, 0) + a_{12}(0, \mathbf{h}_0^{\mathbb{D}})], \mu[a_{21}(\mathbf{h}_0^{\mathbb{D}}, 0) + a_{22}(0, \mathbf{h}_0^{\mathbb{D}})]\}$ becomes the obtained basis $\mathbf{h}'_1$ of $C_1(\mathcal{H}_*)$ and determinant of the 2×2 matrix $A = [a_{ij}]$ is non-zero. Therefore, the choice of K_1 as $(\mu \det A)^{-1}$ yields

$$[\mathbf{h}'_1, \mathbf{h}_1] = \mathrm{K}_1 \ \det A = 1. \tag{24}$$

We now consider the space $C_2(\mathcal{H}_*) = H_0(\mathbb{S}^1)$ in (20). By the fact that $\operatorname{Im} h = 0$, we get

$$C_2(\mathcal{H}_*) = \operatorname{Im} h \oplus s_2(\operatorname{Im} i) = s_2(\operatorname{Im} i). \tag{25}$$

Taking the initial basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$ as $s_2(\mathbf{h}^{\operatorname{Im} i})$ and using equation (25), the obtained basis $\mathbf{h}'_2$ of $C_2(\mathcal{H}_*)$ becomes $s_2(\mathbf{h}^{\operatorname{Im} i})$. Hence, we have

$$[\mathbf{h}'_2, \mathbf{h}_2] = 1. \tag{26}$$

Next, let us consider the case of $C_3(\mathcal{H}_*) = 0$. Clearly, the initial basis $\mathbf{h}_3$ and the obtained basis $\mathbf{h}'_3$ of $C_3(\mathcal{H}_*)$ are equal to zero. Therefore, the equality holds

$$[\mathbf{h}'_3, \mathbf{h}_3] = 1, \tag{27}$$

where we use the convention that $0 = 1 \cdot 0$.

Consider now (20) for $C_4(\mathcal{H}_*) = H_1(\mathbb{S}^1)$. Since $\operatorname{Im} g$ is zero, we obtain

$$C_4(\mathcal{H}_*) = \operatorname{Im} f \oplus s_4(\operatorname{Im} g) = \operatorname{Im} f. \tag{28}$$

Recall that the initial basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$ is $\mathbf{h}_1^{\mathbb{S}^1}$. If the basis of $\operatorname{Im} f$ is chosen as $\mathbf{h}_1^{\mathbb{S}^1}$, then $\mathbf{h}_1^{S_1}$ becomes the obtained basis $\mathbf{h}'_4$ of $C_4(\mathcal{H}_*)$. Thus,

$$[\mathbf{h}'_4, \mathbf{h}_4] = 1. \tag{29}$$

Finally, let us consider the case of $C_5(\mathcal{H}_*) = H_2(d(\mathbb{D}))$ in (20). For $B_5(\mathcal{H}_*)$ being zero, we get

$$C_5(\mathcal{H}_*) = B_5(\mathcal{H}_*) \oplus s_5(\operatorname{Im} f) = s_5(\operatorname{Im} f). \tag{30}$$

Furthermore, $\mathbf{h}_2^{d(\mathbb{D})} = f^{-1}(\mathbf{h}_1^{\mathbb{S}^1})$ is the initial basis $\mathbf{h}_5$ of $C_5(\mathcal{H}_*)$, and $\mathbf{h}_1^{\mathbb{S}^1}$ was chosen basis of $\operatorname{Im} f$ in the previous step. Hence, from equation (30) and the fact that $s_5 = f^{-1}$, it follows that $s_5(\mathbf{h}_1^{\mathbb{S}^1}) = f^{-1}(\mathbf{h}_1^{\mathbb{S}^1})$ becomes the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. Clearly, the equality holds

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{31}$$

Combining equations (22), (24), (26), (27), (29), (31), we have

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_i\}_0^5, \{0\}_0^5\right) = \prod_{i=0}^{5} [\mathbf{h}'_i, \mathbf{h}_i]^{(-1)^{(i+1)}} = 1. \tag{32}$$

From compatibility of the natural bases in the sequence (17), Theorem 2.2, Lemma 2.3 and equation (32) it follows that

$$\mathbb{T}(\mathbb{D}, \{\mathbf{h}_0^{\mathbb{D}}\})^2 = \mathbb{T}(\mathbb{S}^1, \{\mathbf{h}_0^{\mathbb{S}^1}, \mathbf{h}_1^{\mathbb{S}^1}\})\ \mathbb{T}(d(\mathbb{D}), \{\mathbf{h}_0^{d(\mathbb{D})}, 0, \mathbf{h}_2^{d(\mathbb{D})}\}). \tag{33}$$

Since $d(\mathbb{D})$ is homeomorphic to unit sphere $\mathbb{S}^2$, let us fix a homeomorphism $\alpha : d(\mathbb{D}) \to \mathbb{S}^2$. Considering the isomorphism $H_i[\alpha] : H_i(d(\mathbb{D})) \to H_i(\mathbb{S}^2)$, let $\mathbf{h}_i^{\mathbb{S}^2} = H_i[\alpha](\mathbf{h}_i^{d(\mathbb{D})})$ be the corresponding basis of $H_i(\mathbb{S}^2)$. The fact that Reidemeister torsion is a topological invariant yields

$$\mathbb{T}(d(\mathbb{D}), \{\mathbf{h}_0^{d(\mathbb{D})}, 0, \mathbf{h}_2^{d(\mathbb{D})}\}) = \mathbb{T}(\mathbb{S}^2, \{\mathbf{h}_0^{\mathbb{S}^2}, 0, \mathbf{h}_2^{\mathbb{S}^2}\}). \tag{34}$$

From equations (33) and (34) it follows

$$\mathbb{T}(\mathbb{D}, \{\mathbf{h}_0^{\mathbb{D}}\})^2 = \mathbb{T}(\mathbb{S}^1, \{\mathbf{h}_0^{\mathbb{S}^1}, \mathbf{h}_1^{\mathbb{S}^1}\})\ \mathbb{T}(\mathbb{S}^2, \{\mathbf{h}_0^{\mathbb{S}^2}, 0, \mathbf{h}_2^{\mathbb{S}^2}\}). \tag{35}$$

Considering equation (35) and Remark 2.6, we have

$$|\mathbb{T}(\mathbb{D}, \{\mathbf{h}_0^{\mathbb{D}}\})| = \sqrt{|\mathbb{T}(\mathbb{S}^2, \{\mathbf{h}_0^{\mathbb{S}^2}, 0, \mathbf{h}_2^{\mathbb{S}^2}\})|}. \tag{36}$$

Note that $|\mathbb{T}(\mathbb{D}, \{\mathbf{h}_0^{\mathbb{D}}\})| = \delta(\mathbf{h}_0^{\mathbb{D},o}, \mathbf{h}_0^{\mathbb{D}})\ \mathbb{T}(\mathbb{D}, \{\mathbf{h}_0^{\mathbb{D}}\})$, where $\mathbf{h}_0^{\mathbb{D},o}$ is the basis of $H_0(\mathbb{D})$ such that $\mathbb{T}(\mathbb{D}, \{\mathbf{h}_0^{\mathbb{D},o}\}) = 1$. By equation (14) in Proposition 3.2, we get $|T(\mathbb{S}^2, \{\mathbf{h}_0^{\mathbb{S}^2}, 0, \mathbf{h}_2^{\mathbb{S}^2}\})| = (-1)^{n_{\mathbb{S}^2}}\ (\mathbf{h}_0^{\mathbb{S}^2}, \mathbf{h}_2^{\mathbb{S}^2})_{0,2}$. Here, $n_{\mathbb{S}^2}$ is the cardinality of the set $\{i; \mathbf{h}_i^{\mathbb{S}^2}$ and $\mathbf{h}_i^{\mathbb{S}^2,o}$ are not in the same orientation class $\}$ and $\mathbf{h}_i^{\mathbb{S}^2,o}$ is a basis of $H_i(\mathbb{S}^2)$ for which $T(\mathbb{S}^2, \{\mathbf{h}_i^{\mathbb{S}^2,o}\}_i) = 1$.

Thus, combining these formulas, we finish the proof of Proposition 3.4. □

3.2. Reidemeister torsion of once punctured unit disk

Let $\overset{\circ}{\mathbb{D}}$ denote the punctured unit disk $\mathbb{D} \setminus \{p\}$, where p is an interior point of $\mathbb{D}$. Consider the natural chain complex

$$0 \to C_*^p \hookrightarrow C_*(\mathbb{D}) \twoheadrightarrow C_*(\overset{\circ}{\mathbb{D}}) \to 0, \tag{37}$$

corresponding Mayer-Vietoris long-exact sequence. Here, $C_*^p : 0 \to \langle p \rangle \to 0$ and $C_*(\overset{\circ}{\mathbb{D}}) = C_*(\mathbb{D})/C_*^p$.

Following the similar arguments given in Proposition 3.4, we have the following formula for computing Reidemeister torsion of once-punctured unit disk.

Proposition 3.5. *Let $\overset{\circ}{\mathbb{D}}$ be the punctured unit disk $\mathbb{D} \setminus \{p\}$, where p is an interior point of $\mathbb{D}$. For $i = 0, 1$, let $\mathbf{h}_i^{\overset{\circ}{\mathbb{D}}}$ be a basis of $H_i(\overset{\circ}{\mathbb{D}})$. Let us consider the chain complex $C_*^p : 0 \to \langle p \rangle \to 0$. Assume that $f : H_1(\overset{\circ}{\mathbb{D}}) \to H_0(C_*^p)$ and $h : H_0(\mathbb{D}) \to H_0(\overset{\circ}{\mathbb{D}})$ are the isomorphisms obtained by Mayer-Vietoris long-exact sequence associated to the short-exact sequence of chain complexes (37). Let $\mathbf{h}_0^{\mathbb{D}}$ and $\mathbf{h}_0^{C_*^p}$ be the basis $h^{-1}(\mathbf{h}_0^{\overset{\circ}{\mathbb{D}}})$ and $f(\mathbf{h}_1^{\overset{\circ}{\mathbb{D}}})$ of $H_0(\mathbb{D})$ and $H_0(C_*^p)$, respectively. Then, the following formula holds*

$$\mathbb{T}(\overset{\circ}{\mathbb{D}}, \{\mathbf{h}_i^{\overset{\circ}{\mathbb{D}}}\}_0^1) = \mu^{-1}(\mathbb{T}(\mathbb{D}, \{\mathbf{h}_0^{\mathbb{D}}\})).$$

Here, μ is equal to $\mathbb{T}(C_*^p, \{\mathbf{h}_0^{C_*^p}\})$.

3.3. Reidemeister torsion of orientable surface $\Sigma_{g,n,b}$

Throughout this subsection, $\Sigma_{g,n,b}$ is the orientable surface obtained from the orientable closed surface Σ_g of genus $g \geq 2$ by deleting the interior of $n \geq 1$ distinct topological disks $D_1', \ldots, D_n' \subset \Sigma_g$ and $b \geq 1$ the points $\{p_1, \ldots, p_b\} \subset \Sigma_g$ points. Let us denote by S_i' the boundary circle $\partial D_i'$. For $i = 1, \ldots, b$, let $D_i \subset \Sigma_g$ with $\partial D_i = S_i$ be a sufficiently small open disk around p_i and let $\overset{\circ}{D}_i$ be the corresponding once punctured disk with puncture p_i.

By using the similar arguments given in Proposition 3.4, we obtain a formula to compute Reidemeister torsion of orientable surface $\Sigma_{g,n,b}$.

Proposition 3.6. *Consider the surface $\Sigma_{g,n,b}$ obtained by gluing the surfaces $\Sigma_{g,n+1,b-1}$ and $\overset{\circ}{D}_b$ along the common boundary circle S_b. Consider also the natural short-exact sequence of chain complexes*

$$0 \to C_*(S_b) \to C_*(\Sigma_{g,n+1,b-1}) \oplus C_*(\overset{\circ}{D}_b) \to C_*(\Sigma_{g,n,b}) \to 0 \tag{38}$$

and the corresponding Mayer-Vietoris long-exact sequence

$$\begin{aligned}\mathcal{H}_* : 0 \to H_1(S_b) &\xrightarrow{f} H_1(\Sigma_{g,n+1,b-1}) \oplus H_1(\overset{\circ}{D}_b) \xrightarrow{g} H_1(\Sigma_{g,n,b}) \\ &\xrightarrow{h} H_0(S_b) \xrightarrow{i} H_0(\Sigma_{g,n+1,b-1}) \oplus H_0(\overset{\circ}{D}_b) \xrightarrow{j} H_0(\Sigma_{g,n,b}) \xrightarrow{k} 0.\end{aligned}$$

Let $\mathbf{h}_i^{\Sigma_{g,n,b}}$ be a basis of $H_i(\Sigma_{g,n,b})$ and let $\mathbf{h}_i^{S_b}$ be an arbitrary basis of $H_i(S_b)$, $i = 0, 1$. Then, for $\nu = 0, 1$, there are bases $\mathbf{h}_\nu^{\Sigma_{g,n+1,b-1}}$ and $\mathbf{h}_\nu^{\overset{\circ}{D}_b}$ of $H_\nu(\Sigma_{g,n+1,b-1})$ and $H_\nu(\overset{\circ}{D}_b)$, respectively such that Reidemeister torsion of $\mathcal{H}_*$ in these bases is 1 and the following formula is valid

$$\mathbb{T}(\Sigma_{g,n,b}, \{\mathbf{h}_i^{\Sigma_{g,n,b}}\}_0^1) = \mathbb{T}(\Sigma_{g,n+1,b-1}, \{\mathbf{h}_\nu^{\Sigma_{g,n+1,b-1}}\}_0^1) \, \frac{\mathbb{T}(\overset{\circ}{D}_b, \{\mathbf{h}_\nu^{\overset{\circ}{D}_b}\}_{\nu=0}^1)}{\mathbb{T}(S_b, \{\mathbf{h}_i^{S_b}\}_{i=0}^1)}.$$

Applying Proposition 3.6 inductively, we have the following result:

Proposition 3.7. *Let $\Sigma_{g,n,b}$ be the surface obtained by $\Sigma_{g,n,0}$ punctured the points $\{p_1,\dots,p_b\}\subset\Sigma_{g,n,0}$. For $\nu=1,\dots,b$, let $D_\nu\subset\Sigma_{g,n,0}$ with $\partial D_\nu=S_\nu$ be a sufficiently small open disk around p_ν and let $\overset{\circ}{D}_\nu$ be the corresponding once punctured disk with puncture p_ν. Let $\mathbf{h}_\eta^{\Sigma_{g,n,b}}$ and $\mathbf{h}_\eta^{S_\nu}$ be a basis of $H_\eta(\Sigma_{g,n,b})$ and $H_\eta(S_\nu)$, $\nu=1,\dots,b$, $\eta=0,1$. Assume that $\mathbf{h}_\eta^{\Sigma_{g,n+1,\nu-1}}$ and $\mathbf{h}_\eta^{\overset{\circ}{D}_\nu}$ are respectively bases of $H_\eta(\Sigma_{g,n+1,\nu-1})$ and $H_\eta(\overset{\circ}{D}_\nu)$ obtained by using the method in Proposition 3.6. Then, the formula holds*

$$\mathbb{T}(\Sigma_{g,n,b},\{\mathbf{h}_i^{\Sigma_{g,n,b}}\}_0^1)=\mathbb{T}(\Sigma_{g,n+b,0},\{\mathbf{h}_i^{\Sigma_{g,n+b,0}}\}_0^1)\prod_{\nu=1}^{b}\frac{\mathbb{T}(\overset{\circ}{D}_\nu,\{\mathbf{h}_i^{\overset{\circ}{D}_\nu}\}_{i=0}^1)}{\mathbb{T}(S_\nu,\{\mathbf{h}_i^{S_\nu}\}_{i=0}^1)}.$$

Remark 3.8. Let $\Sigma_{g,n,b}$, D_ν, S_ν be as in Proposition 3.7, for $\nu=1,\dots,b$. Let $\overset{\circ}{D}_\nu$ be the corresponding once punctured disk with puncture p_ν. Consider the homeomorphisms $\varphi_\nu:\overset{\circ}{D}_\nu\to\overset{\circ}{\mathbb{D}}_\nu$ and $\phi_\nu:S_\nu\to\mathbb{S}_\nu^1$ obtained by the local patch of $\Sigma_{g,n,0}$ around the point p_ν, $\nu=1,\dots,b$, where $\overset{\circ}{\mathbb{D}}_\nu$ is the punctured unit disk $\mathbb{D}\setminus\{0\}$ in the plane and $\mathbb{S}_\nu^1=\partial\mathbb{D}_\nu$ is the unit circle. Clearly, at the level of homology there are isomorphisms $H_\eta[\varphi_\nu]:H_\eta(\overset{\circ}{D}_\nu)\to H_\eta(\overset{\circ}{\mathbb{D}}_\nu)$ and $H_\eta[\phi_\nu]:H_\eta(S_\nu)\to H_\eta(\mathbb{S}_\nu)$. Let $\mathbf{h}_\eta^{\overset{\circ}{D}_\nu}=H_\eta[\varphi_\nu](\mathbf{h}_\eta^{\overset{\circ}{\mathbb{D}}_\nu})$ and $\mathbf{h}_\eta^{S_\nu}=H_\eta[\phi_\nu](\mathbf{h}_\eta^{\mathbb{S}_\nu^1})$ be respectively basis of $H_\eta(\mathbb{D}_\nu^\circ)$ and $H_\eta(\mathbb{S}_\nu^1)$, $\eta=0,1$, $\nu=1,\dots,b$. Using the fact that Reidemeister torsion is a topological invariant, applying Proposition 3.7, we obtain

$$\mathbb{T}(\Sigma_{g,n,b},\{\mathbf{h}_\eta^{\Sigma_{g,n,b}}\}_0^1)=\mathbb{T}(\Sigma_{g,n+b,0},\{\mathbf{h}_\eta^{\Sigma_{g,n+b,0}}\}_0^1)\prod_{\nu=1}^{b}\frac{\mathbb{T}(\overset{\circ}{\mathbb{D}}_\nu,\{\mathbf{h}_\eta^{\overset{\circ}{\mathbb{D}}_\nu}\}_{\eta=0}^1)}{\mathbb{T}(\mathbb{S}_\nu^1,\{\mathbf{h}_\eta^{\mathbb{S}_\nu^1}\}_{\eta=0}^1)}.$$

Recall that $\Sigma_{g,n,0}$ is the orientable surface obtained from the closed orientable surface Σ_g of genus $g\geq 2$ by deleting the interior of $n\geq 1$ distinct topological disks D_i' with $\partial D_i'=S_i'$, $i=1,\dots,n$. Considering the natural exact sequences obtained by gluing the surfaces $\Sigma_{g,n-i,0}$ and D_i' along the boundary circle S_i', $i=1,\dots,n$ and applying the arguments as in Proposition 3.6, we prove:

Proposition 3.9. *Let $\Sigma_{g,n,0}$ be the orientable surface obtained from the closed orientable surface Σ_g with genus $g\geq 2$ by deleting the the interior of $n\geq 1$ distinct topological disks D_i' with $\partial D_i'=S_i'$, $i=1,\dots,n$. Let $\mathbf{h}_\eta^{\Sigma_{g,n}}$, $\mathbf{h}_0^{D_i'}$ be a basis of $H_\eta(\Sigma_{g,n})$, $H_0(D_i')$, $\eta=0,1$. Then, there exist bases $\mathbf{h}_\eta^{\Sigma_g}$, $\mathbf{h}_\eta^{S_i'}$ of $H_\eta(\Sigma_g)$, $H_\eta(S_i')$, $\eta=0,1$, $i=1,\dots,n$, and the following formula is valid*

$$\mathbb{T}(\Sigma_{g,n},\{\mathbf{h}_i^{\Sigma_{g,n}}\}_0^1)=\mathbb{T}(\Sigma_g,\{\mathbf{h}_i^{\Sigma_g}\}_0^2)\prod_{i=1}^{n}\frac{\mathbb{T}(D_i',\{\mathbf{h}_0^{D_i'}\})}{\mathbb{T}(S_i',\{\mathbf{h}_\eta^{S_i'}\}_{\eta=0}^1)}.$$

Proposition 3.3-Proposition 3.9 and Remark 3.8 yield our main result:

Theorem 3.10. *Let* $\Sigma_{g,n,b}$, D_ν, $\overset{\circ}{D}_\nu$, S_ν, $\nu = 1,\dots,b$ *be as in Theorem 3.7. Let* $\mathbb{D}_\nu$, $\overset{\circ}{\mathbb{D}}_\nu$, $\mathbb{S}^1_\nu$, $\nu = 1,\dots,b$ *be as in Remark 3.8. Let* D'_i, $\mathbb{D}'_i$, S'_i, $\left(\mathbb{S}^1_i\right)'$, $i = 1,\dots,n$ *be as in Theorem 3.9. Let* $\mathbf{h}_\eta^{\Sigma_{g,n,b}}$ *be basis of* $H_\eta(\Sigma_{g,n,b})$ *and let* $\mathbf{h}_0^{\mathbb{D}'_i}$, $\mathbf{h}_\eta^{\overset{\circ}{\mathbb{D}}_\nu}$ *be respectively bases of* $H_0(\mathbb{D}'_i)$, $H_\eta(\overset{\circ}{\mathbb{D}}_\nu)$, $\eta = 0,1$, $i = 1,\dots,n$. *For* $\nu = 1,\dots,b$, *let* $\mathbf{h}_\eta^{\mathbb{S}_\nu}$ *be an arbitrary basis of* $H_\eta(\mathbb{S}_\nu)$, $\eta = 0,1$. *Then, there exist bases* $\mathbf{h}_k^{\Sigma_g}$, $\mathbf{h}_\ell^{(\mathbb{S}^1)'_i}$ *of* $H_k(\Sigma_g)$, $H_\ell((\mathbb{S}^1)'_i)$, $k = 0,1,2$, $\ell = 0,1$, $i = 1,\dots,n$ *and the following formula holds*

$$\begin{aligned}\mathbb{T}(\Sigma_{g,n,b}, \{\mathbf{h}_\eta^{\Sigma_{g,n,b}}\}_0^1) &= \mathbb{T}(\Sigma_g, \{\mathbf{h}_k^{\Sigma_g}\}_0^2) \\ &\times \prod_{i=1}^n \frac{\mathbb{T}(\mathbb{D}'_i, \{\mathbf{h}_0^{\mathbb{D}'_i}\})}{\mathbb{T}((\mathbb{S}^1)'_i, \{\mathbf{h}_\ell^{(\mathbb{S}^1)'_i}\}_{\ell=0}^1)} \prod_{\nu=1}^b \frac{\mathbb{T}(\overset{\circ}{\mathbb{D}}_\nu, \{\mathbf{h}_\mu^{\overset{\circ}{\mathbb{D}}_\nu}\}_{\mu=0}^1)}{\mathbb{T}(\mathbb{S}^1_\nu, \{\mathbf{h}_\mu^{\mathbb{S}^1_\nu}\}_{\mu=0}^1)}.\end{aligned}$$

Considering Remark 2.6 and Theorem 3.10, we have:

Corollary 3.11. *Let* $\Sigma_{g,n,b}$, Σ_g, D_ν, D_ν, $\overset{\circ}{D}_\nu$, S_ν, $\mathbb{D}_\nu$, $\overset{\circ}{\mathbb{D}}_\nu$, $\mathbb{S}^1_\nu$, D'_i, $\mathbb{D}'_i$, S'_i, $\left(\mathbb{S}^1_i\right)'$, $\mathbf{h}_\eta^{\Sigma_{g,n,b}}$, $\mathbf{h}_i^{\Sigma_g}$, $\mathbf{h}_0^{\mathbb{D}'_i}$, $\mathbf{h}_\eta^{\overset{\circ}{\mathbb{D}}_\nu}$ $\mathbf{h}_\eta^{\mathbb{S}_\nu}$, $\mathbf{h}_k^{\Sigma_{g,0}}$, $\mathbf{h}_\ell^{(\mathbb{S}^1)'_i}$ *be as in Theorem 3.10. Then,*

$$\left|\mathbb{T}(\Sigma_{g,n,b}, \{\mathbf{h}_\eta^{\Sigma_{g,n,b}}\}_0^1)\right| = \left|\mathbb{T}(\Sigma_g, \{\mathbf{h}_i^{\Sigma_g}\}_0^2)\right| \prod_{i=1}^n \left|\mathbb{T}(\mathbb{D}_i, \{\mathbf{h}_0^{\mathbb{D}_i}\})\right| \prod_{\nu=1}^b \left|\mathbb{T}(\overset{\circ}{\mathbb{D}}_\nu, \{\mathbf{h}_\eta^{\overset{\circ}{\mathbb{D}}_\nu}\}_0^1)\right|.$$

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