J. Korean Math. Soc. **55** (2018), No. 4, pp. 975–986
https://doi.org/10.4134/JKMS.j170573
pISSN: 0304-9914 / eISSN: 2234-3008

TWO VARIABLE HIGHER-ORDER FUBINI POLYNOMIALS

Dae San Kim, Taekyun Kim, Hyuck-In Kwon, and Jin-Woo Park

Abstract. Some new family of Fubini type numbers and polynomials associated with Apostol-Bernoulli numbers and polynomilas were introduced recently by Kilar and Simsek ([5]) and we study the two variable Fubini polynomials as Appell polynomials whose coefficients are the Fubini polynomials. In this paper, we would like to utilize umbral calculus in order to study two variable higher-order Fubini polynomials. We derive some of their properties, explicit expressions and recurrence relations. In addition, we express the two variable higher-order Fubini polynomials in terms of some families of special polynomials and vice versa.

1. Review on umbral calculus

The aim of this paper is to apply umbral calculus in order to study two variable higher-order Fubini polynomials. For this, we need to go over some of the basic facts about umbral calculus. For a complete treatment, one may want to see [10].

Let $\mathbb{C}$ be the field of complex numbers. By $\mathcal{F}$ we denote the algebra of all formal power series in the variable t with the coefficients in $\mathbb{C}$:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$. Here $\mathbb{P} = \mathbb{C}[x]$ is the ring of polynomials in x with the coefficients in $\mathbb{C}$. For each $L \in \mathbb{P}^*$, and each $p(x) \in \mathbb{P}$, by $\langle L | \, p(x) \rangle$ we denote the action of the linear functional L on $p(x)$. For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, we let $\langle f(t) | \cdot \rangle$ denote the linear functional on $\mathbb{P}$ given by

$$\langle f(t) | \, x^n \rangle = a_n, \ (n \geq 0), \ (\text{see } [2,6,8,11]).$$

For $L \in \mathbb{P}^*$, let $f_L(t) = \sum_{k=0}^{\infty} \langle L | \, x^k \rangle \frac{t^k}{k!} \in \mathcal{F}$. Then $\langle f_L(t) | \, x^n \rangle = \langle L | \, x^n \rangle$ for all $n \geq 0$, and the map $C \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathcal{F}$. Then $\mathcal{F}$ may be viewed as the vector space of all linear functionals on $\mathbb{P}$ as well as the algebra of formal power series in t. Hence an element $f(t) \in \mathcal{F}$ will

Received September 3, 2017; Accepted October 25, 2017.
2010 *Mathematics Subject Classification.* 11B68, 11B83, 42A16.
Key words and phrases. umbral calculus, two variable higher-order Fubini polynomials.

be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathcal{F}$ is called the *umbral algebra*, the study of which is the umbral calculus.

The order $o(f(t))$ of $0 \neq f(t) \in \mathcal{F}$ is the smallest integer k such that the coefficient of t^k does not vanish. Let $f(t),\ g(t) \in \mathcal{F}$, with $o(g(t)) = 0$, $o(f(t)) = 1$. Then it is known that there exists a unique sequence of polynomials $S_n(x)$ ($\deg S_n(x) = n$) such that

$$\left\langle g(t)f(t)^k \middle| S_n(x) \right\rangle = n!\delta_{n,k} \quad \text{for } n, k \geq 0. \tag{1.1}$$

Such a sequence is called the *Sheffer sequence* for the Sheffer pair $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$.

It is a basic fact that $S_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g\left(\bar{f}(t)\right)} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!}, \tag{1.2}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

For $S_n(x) \sim (g(t), f(t))$, we have the Sheffer identity:

$$S_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} S_k(x) P_{n-k}(y), \ \text{(see [2,8,11])}, \tag{1.3}$$

where $P_n(x) = g(t)S_n(x) \sim (1, f(t))$.

The following recurrence formula holds:
for $S_n(x) \sim (g(t), f(t))$,

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} S_n(x). \tag{1.4}$$

For any $h(t) \in \mathcal{F}$, $p(x) \in \mathbb{P}$,

$$\langle h(t) |\, xp(x) \rangle = \langle \partial_t h(t) |\, p(x) \rangle. \tag{1.5}$$

The last thing we need is the following: for $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$,

$$S_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x), \tag{1.6}$$

where

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)} \left(l\left(\bar{f}(t)\right)\right)^k \middle| x^n \right\rangle. \tag{1.7}$$

2. Introduction

The *two variable Fubini polynomials* $F_n^{(r)}(x; y)$ *of order* r are defined by

$$\frac{e^{xt}}{\left(1 - y(e^t - 1)\right)^r} = \sum_{n=0}^{\infty} F_n^{(r)}(x; y) \frac{t^n}{n!}, \tag{2.1}$$

where r is a positive integer. Here, in this paper y will be an arbitrary but fixed real number so that $F_n^{(r)}(x;y)$ are polynomials in x for each fixed y. Note here that

$$F_n^{(r)}(x;y) \sim \left((1-y(e^t-1))^r, t\right). \tag{2.2}$$

In particular, if $r=1$, then $F_n(x;y) = F_n^{(1)}(x;y)$ are called *two variable Fubini polynomials* and they were introduced by Kargin in [4].

For $x=0$, $F_n^{(r)}(y) = F_n^{(r)}(0;y)$ and $F_n^{(r)} = F_n^{(r)}(1) = F_n^{(r)}(0;1)$ are respectively called the *Fubini polynomials of order* r and the *Fubini numbers of order* r (see [3, 7–10]). Further, in the special case of $y=1$, $F_n^{(r)}(x;1)$ are the ordered Bell polynomials of order r and they are denoted by $Ob_n^{(r)}(x)$; $F_n^{(r)}(1) = F_n^{(r)}(0;1)$ are also called the ordered Bell numbers of order r and they are also denoted by $Ob_n^{(r)}$. So $Ob_n^{(r)}(x)$ and $Ob_n^{(r)}$ are respectively given by

$$\frac{e^{xt}}{(2-e^t)^r} = \sum_{n=0}^{\infty} Ob_n^{(r)}(x)\frac{t^n}{n!}, \tag{2.3}$$

$$\frac{1}{(2-e^t)^r} = \sum_{n=0}^{\infty} Ob_n^{(r)}\frac{t^n}{n!}, \text{ (see [3])}. \tag{2.4}$$

In this paper, by using umbral calculus we will consider the two variable higher-order Fubini polynomials and derive their properties, recurrence relations and some identities. In particular, we will express the two variable higher-order Fubini polynomials as linear combinations of some well-known families of special polynomials and vice versa.

3. Some properties

Let us first consider the higher-order Fubini polynomials $F_n^{(r)}(y)$.

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n^{(r)}(y)\frac{t^n}{n!} &= \left(1-y(e^t-1)\right)^{-r} \\
&= \sum_{k=0}^{\infty}(r+k-1)_k y^k \frac{1}{k!}(e^t-1)^k \\
&= \sum_{k=0}^{\infty}(r+k-1)_k y^k \sum_{k=0}^{\infty} S_2(n,k)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k-1)_k S_2(n,k) y^k\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus we have

$$F_n^{(r)}(y) = \sum_{k=0}^{n} (r+k-1)_k S_2(n,k) y^k, \tag{3.1}$$

and hence

$$Ob_n^{(r)} = \sum_{k=0}^{n} (r+k-1)_k S_2(n,k). \tag{3.2}$$

As was shown in [8], we have

$$\frac{1}{(1-y)^r} F_n^{(r)}\left(\frac{y}{1-y}\right) = \sum_{k=0}^{\infty} \binom{r+k-1}{k} k^n y^k.$$

In particular, $y = \frac{1}{2}$ gives

$$Ob_n^{(r)} = F_n^{(r)}(1) = \frac{1}{2^r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} \frac{k^n}{2^k}. \tag{3.3}$$

From (1.3), (2.2), and (3.1), we obtain

$$\begin{aligned} F_n^{(r)}(x;y) &= \sum_{m=0}^{n} \binom{n}{m} F_m^{(r)}(y) x^{n-m} \\ &= \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} (r+k-1)_k S_2(m,k) x^{n-m} y^k, \end{aligned} \tag{3.4}$$

and

$$F_n^{(r)}(x_1+x_2;y) = \sum_{m=0}^{n} \binom{n}{m} F_m^{(r)}(x_1;y) x_2^{n-m}. \tag{3.5}$$

From (3.1)-(3.4), we get the following theorem.

Theorem 3.1. *For $n \geq 0$, we have the following*:

$$F_n^{(r)}(x;y) = \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} (r+k-1)_k S_2(m,k) x^{n-m} y^k.$$

In particular,

$$F_n^{(r)}(y) = \sum_{k=0}^{n} (r+k-1)_k S_2(n,k) y^k.$$

Also, the ordered Bell numbers $Ob_n^{(r)}$ of order r can be expressed by

$$Ob_n^{(r)} = \sum_{k=0}^{n} (r+k-1)_k S_2(n,k) = \frac{1}{2^r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} \frac{k^n}{2^k}.$$

For $r \geq 2$, writing $\frac{1}{(1-y(e^t-1))^r}$ as

$$\frac{1}{(1-y(e^t-1))^r} = \frac{1}{(1-y(e^t-1))^{r-1}} \frac{1}{(1-y(e^t-1))},$$

we easily obtain

$$F_n^{(r)}(y) = \sum_{m=0}^{n} \binom{n}{m} F_m^{(r-1)}(y) F_{n-m}(y). \tag{3.6}$$

Before we turn to our next result, we recall that the Frobenius-Euler polynomials $H_n^{(r)}(u;x)$ of order r are given by

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n(u;x) \frac{t^n}{n!}, \quad \text{(see [1,3,5–7])}, \tag{3.7}$$

where $u \neq 1$.

For $y \neq 0$, we see that

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(r)}(x;y) \frac{t^n}{n!} = \frac{1}{(1-y(e^t-1))^r} e^{xt} &= \left(\frac{1-\frac{1+y}{y}}{e^t-\frac{1+y}{y}}\right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} H_n^{(r)}\left(\frac{1+y}{y}\middle| x\right) \frac{t^n}{n!}.\end{aligned}$$

Hence we have

$$F_n^{(r)}(x;y) = H_n^{(r)}\left(\frac{1+y}{y}\middle| x\right), \ (y \neq 0). \tag{3.8}$$

From (3.6) and (3.8), we have the following result.

Theorem 3.2. *For $n \geq 2$, we have*

$$F_n^{(r)}(y) = \sum_{m=0}^{n} \binom{n}{m} F_m^{(r-1)}(y) F_{n-m}(y), \ (r \geq 2),$$

and

$$F_n^{(r)}(x;y) = H_n^{(r)}\left(\frac{1+y}{y}\middle| x\right), \ (y \neq 0).$$

For the next discussion, we first observe the following.

$$\begin{aligned}\left(1-y(e^t-1)\right)^r &= \sum_{l=0}^{\infty} (r)_l (-y)^l \frac{1}{l!} (e^t-1)^l \\ &= \sum_{l=0}^{\infty} (r)_l (-y)^l \sum_{k=l}^{\infty} S_2(k,l) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} (r)_l S_2(k,l)(-y)^l\right) \frac{t^k}{k!}.\end{aligned} \tag{3.9}$$

Now, from (2.1) and (3.9), we get

$$\begin{aligned}\sum_{n=0}^{\infty} x^n \frac{t^n}{n!} &= \sum_{k=0}^{\infty}\Big(\sum_{l=0}^{k}(r)_l S_2(k,l)(-y)^l\Big)\frac{t^k}{k!}\sum_{m=0}^{\infty} F_m^{(r)}(x;y)\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}(r)_l S_2(k,l)(-y)^l F_{n-k}^{(r)}(x;y)\right)\frac{t^n}{n!}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}x^n &= \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(r)_l S_2(k,l)(-y)^l F_{n-k}^{(r)}(x;y)\\ &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_2(n-k,l)(-y)^l F_{k}^{(r)}(x;y).\end{aligned} \tag{3.10}$$

Letting $x=0$, we get

$$\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_2(n-k,l)(-y)^l F_{k}^{(r)}(y) = \begin{cases} 0, & n\geq 1,\\ 1, & n=0.\end{cases} \tag{3.11}$$

Equivalently, (3.11) can be stated as

$$\begin{aligned}&F_0^{(r)}(y)=1,\\ &F_n^{(r)}(y) = -\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_2(n-k,l)(-y)^l F_{k}^{(r)}(y) \quad \text{for } n\geq 1.\end{aligned} \tag{3.12}$$

The next theorem follows from (3.10) and (3.12).

Theorem 3.3. *For $n\geq 0$, we have*

$$x^n = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_2(n-k,l)(-y)^l F_{k}^{(r)}(x;y),$$

and

$$F_n^{(r)}(y) = -\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\binom{n}{k}(r)_l S_2(n-k,l)(-y)^l F_{k}^{(r)}(y) \quad \textit{for } n\geq 1,$$

with $F_0^{(r)}(y)=1$.

Invoking (1.5), for $n\geq 1$, we have

$$\begin{aligned}&F_n^{(r)}(z;y) = \left\langle \frac{e^{zt}}{(1-y(e^t-1))^r}\,\middle|\, x^n\right\rangle\\ &= \left\langle \left(\partial_t \frac{1}{(1-y(e^t-1))^r}\right)e^{zt}\,\middle|\, x^{n-1}\right\rangle + \left\langle \frac{1}{(1-y(e^t-1))^r}\left(\partial_t e^{zt}\right)\,\middle|\, x^{n-1}\right\rangle.\end{aligned} \tag{3.13}$$

Clearly, the second term of (3.13) is $zF_{n-1}^{(r)}(z;y)$. On the other hand, the first term of (3.13) is

$$ry\left\langle \frac{1}{(1-y(e^t-1))^{r+1}}e^{(z+1)t}\middle| x^n\right\rangle = ryF_{n-1}^{(r+1)}(z+1;y).$$

Thus we have derived the next result.

Theorem 3.4. *For $n \geq 0$, we have*

$$F_{n+1}^{(r)}(x;y) = xF_n^{(r)}(x;y) + ryF_n^{(r+1)}(x+1;y),$$

and

$$F_{n+1}^{(r)}(y) = ryF_n^{(r+1)}(1;y).$$

Using (1.4) and (2.2), we obtain

$$F_{n+1}^{(r)}(x;y) = \left(x - \frac{g'(t)}{g(t)}\right) F_n^{(r)}(x;y), \tag{3.14}$$

with $g(t) = (1-y(e^t-1))^r$.

Since $\frac{g'(t)}{g(t)} = \frac{-rye^t}{1-y(e^t-1)}$, (3.14) is equal to

$$\begin{aligned}
F_{n+1}^{(r)}(x;y) &= \left(x + \frac{rye^t}{1-y(e^t-1)}\right) F_n^{(r)}(x;y) \\
&= xF_n^{(r)}(x;y) + rye^t \frac{1}{1-y(e^t-1)} \sum_{m=0}^{n} \binom{n}{m} F_m^{(r)}(y)x^{n-m} \\
&= xF_n^{(r)}(x;y) + ry \sum_{m=0}^{n} \binom{n}{m} F_m^{(r)}(y) F_{n-m}(x+1;y) \\
&= xF_n^{(r)}(x;y) + ryF_n^{(r+1)}(x+1;y).
\end{aligned}$$

This gives another way of obtaining the results in Theorem 3.4.

4. $F_n^{(r)}(x;y)$ in terms of some special polynomials

Here we will express the two variable higher-order Fubini polynomials $F_n^{(r)}(x;y)$ as linear combinations of some well-known families of special polynomials.

We first recall from (2.2) that $F_n^{(r)}(x;y) \sim ((1-y(e^t-1))^r, t)$. Let

$$F_n^{(r)}(x;y) = \sum_{m=0}^{n} C_{n,m} S_m(x), \tag{4.1}$$

with $S_n(x) \sim (h(t), l(t))$. Then, from (1.7) we note that

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle h(t)(l(t))^m \middle| \frac{1}{(1-y(e^t-1))^r} x^n \right\rangle \\
&= \frac{1}{m!}\left\langle h(t)(l(t))^m \middle| F_n^{(r)}(x;y) \right\rangle.
\end{aligned} \tag{4.2}$$

Throughout this section, we will use (4.2).

Let $B_n(x)$ be the Bernoulli polynomials with $S_n(x) = B_n(x) \sim \left(\frac{e^t-1}{t}, t\right)$. Then

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{e^t-1}{t}t^m \middle| F_n^{(r)}(x;y)\right\rangle \\
&= \frac{1}{m!}\left\langle \frac{e^t-1}{t} \middle| t^m F_n^{(r)}(x;y)\right\rangle \\
&= \binom{n}{m}\left\langle \frac{e^t-1}{t} \middle| F_{n-m}^{(r)}(x;y)\right\rangle \\
&= \binom{n}{m}\int_0^1 F_{n-m}^{(r)}(u;y)du \\
&= \binom{n}{m}\frac{1}{n-m+1}\left[F_{n-m+1}^{(r)}(u;y)\right]_0^1 \\
&= \frac{1}{n+1}\binom{n+1}{m}\left(F_{n-m+1}^{(r)}(1;y) - F_{n-m+1}^{(r)}(y)\right).
\end{aligned}$$

Thus we obtain the following result.

Theorem 4.1. *For $n \geq 0$, we have*

$$F_n^{(r)}(x;y) = \frac{1}{n+1}\sum_{m=0}^{n}\binom{n+1}{m}\left(F_{n-m+1}^{(r)}(1;y) - F_{n-m+1}^{(r)}(y)\right)B_m(x).$$

Let $H_n(u|x)$ be the Frobenius-Euler polynomials with $S_n(x) = H_n(u|x) \sim \left(\frac{e^t-u}{1-u}, t\right)$. Then

$$\begin{aligned}
C_{n,m} &= \binom{n}{m}\left\langle \frac{e^t-u}{1-u} \middle| F_{n-m}^{(r)}(x;y)\right\rangle \\
&= \frac{1}{1-u}\binom{n}{m}\left(F_{n-m}^{(r)}(1;y) - uF_{n-m}^{(r)}(y)\right).
\end{aligned}$$

Hence we get the following theorem.

Theorem 4.2. *For $n \geq 0$, we have*

$$F_n^{(r)}(x;y) = \frac{1}{1-u}\sum_{m=0}^{n}\binom{n}{m}\left(F_{n-m}^{(r)}(1;y) - uF_{n-m}^{(r)}(y)\right)H_m(u|x).$$

Here we let $(x)_n$ be the falling factorial polynomials with $S_n(x) = (x)_n \sim (1, e^t - 1)$. Then

$$\begin{aligned}
C_{n,m} &= \left\langle \frac{1}{m!}(e^t-1)^m \middle| F_n^{(r)}(x;y)\right\rangle \\
&= \left\langle \sum_{k=m}^{\infty} S_2(k,m)\frac{t^k}{k!} \middle| F_n^{(r)}(x;y)\right\rangle
\end{aligned}$$

$$= \sum_{k=m}^{n} \frac{1}{k!} S_2(k,m) \left\langle t^k \middle| F_n^{(r)}(x;y) \right\rangle$$
$$= \sum_{k=m}^{n} \binom{n}{k} S_2(k,m) F_{n-k}^{(r)}(y).$$

So we have the following result.

Theorem 4.3. *For $n \geq 0$, we have*

$$F_n^{(r)}(x;y) = \sum_{m=0}^{n} \left(\sum_{k=m}^{n} \binom{n}{k} S_2(k,m) F_{n-k}^{(r)}(y) \right) (x)_m.$$

Finally, let $Bel_n(x)$ be the Bell polynomials with $S_n(x) = Bel_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k \sim (1, \log(1+t))$. Then it is easy to see that

$$C_{n,m} = \left\langle \frac{1}{m!} (\log(1+t))^m \middle| F_n^{(r)}(x;y) \right\rangle$$
$$= \sum_{k=m}^{n} \binom{n}{k} S_1(k,m) F_{n-k}^{(r)}(y).$$

Thus we have the following theorem.

Theorem 4.4. *For $n \geq 0$, we have*

$$F_n^{(r)}(x;y) = \sum_{m=0}^{n} \left(\sum_{k=m}^{n} \binom{n}{k} S_1(k,m) F_{n-k}^{(r)}(y) \right) Bel_m(x).$$

5. Some special polynomials in terms of $F_n^{(r)}(x;y)$

In this section, we will express some families of special polynomials as linear combinations of the two variable higher-order Fubini polynomials $F_n^{(r)}(x;y)$. For this, it is more convenient to use (1.1) than (1.7).

Let $p(x) \in \mathbb{C}[x]$ be a polynomial of degree $\leq n$. Then we can write

$$p(x) = \sum_{m=0}^{n} a_m F_m^{(r)}(x;y)$$

for unique $a_m \in \mathbb{C}(y)$.

Now, from (1.1) and (2.2), we note that

$$\begin{aligned} \left\langle \left(1 - y(e^t - 1)\right)^r t^m \middle| p(x) \right\rangle &= \sum_{l=0}^{n} a_l \left\langle \left(1 - y(e^t - 1)\right)^r t^m \middle| F_l^{(r)}(x;y) \right\rangle \\ &= \sum_{l=0}^{n} a_l l! \delta_{m,l} \\ &= m! a_m. \end{aligned} \tag{5.1}$$

Thus, from (3.9) and (5.1), we further observe that

$$\begin{aligned} a_m &= \frac{1}{m!}\left\langle \left(1-y(e^t-1)\right)^r t^m \middle| p(x)\right\rangle \\ &= \frac{1}{m!}\left\langle \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{l=0}^{k}(r)_l S_2(k,l)(-y)^l \middle| t^{m+k}p(x)\right\rangle \\ &= \frac{1}{m!}\sum_{k=0}^{n-m}\frac{1}{k!}\sum_{l=0}^{k}(r)_l S_2(k,l)(-y)^l \left\langle 1 \middle| t^{m+k}p(x)\right\rangle . \end{aligned} \tag{5.2}$$

Throughout this section, we will use (5.2).

For $p(x)=B_n(x)$,

$$\begin{aligned} a_m &= \frac{1}{m!}\sum_{k=0}^{n-m}\frac{1}{k!}\sum_{l=0}^{k}(r)_l S_2(k,l)(-y)^l (n)_{m+k}B_{n-m-k} \\ &= \sum_{k=0}^{n-m}\sum_{l=0}^{k}\binom{n}{m}\binom{n-m}{k}(r)_l S_2(k,l)B_{n-m-k}(-y)^l. \end{aligned} \tag{5.3}$$

Similarly, for $p(x)=H_n(u|x)$,

$$a_m = \sum_{k=0}^{n-m}\sum_{l=0}^{k}\binom{n}{m}\binom{n-m}{k}(r)_l S_2(k,l)H_{n-m-k}(u)(-y)^l. \tag{5.4}$$

Here $H_n(u)=H_n(u|0)$ are the Frobenius-Euler numbers.

On the other hand, for $p(x)=x^n$,

$$\begin{aligned} a_m &= \sum_{k=0}^{n-m}\sum_{l=0}^{k}\binom{n}{m}\binom{n-m}{k}(r)_l S_2(k,l)(-y)^l \left\langle 1 \middle| x^{n-m-k}\right\rangle \\ &= \sum_{k=0}^{n-m}\sum_{l=0}^{k}\binom{n}{m}\binom{n-m}{k}(r)_l S_2(k,l)(-y)^l \delta_{n-m,k} \\ &= \sum_{l=0}^{n-m}\binom{n}{m}(r)_l S_2(n-m,l)(-y)^l. \end{aligned} \tag{5.5}$$

Collecting our results in (5.3), (5.4) and (5.5), we have the next theorem.

Theorem 5.1. *For $n \geq 0$, we have*

$$B_n(x) = \sum_{m=0}^{n}\left(\sum_{k=0}^{n-m}\sum_{l=0}^{k}\binom{n}{m}\binom{n-m}{k}(r)_l S_2(k,l)B_{n-m-k}(-y)^l\right)F_m^{(r)}(x;y),$$

$$H_n(u|x) = \sum_{m=0}^{n}\left(\sum_{k=0}^{n-m}\sum_{l=0}^{k}\binom{n}{m}\binom{n-m}{k}(r)_l S_2(k,l)H_{n-m-k}(u)(-y)^l\right)F_m^{(r)}(x;y),$$

and

$$x^n = \sum_{m=0}^{n} \left(\sum_{l=0}^{n-m} \binom{n}{m} (r)_l S_2(n-m,l)(-y)^l \right) F_m^{(r)}(x;y).$$

Now, applying (5.2) to $p(x) = Bel_n(x) = \sum_{j=0}^{n} S_2(n,j)x^j$, we get

$$a_m = \frac{1}{m!} \sum_{k=0}^{n-m} \frac{1}{k!} \sum_{l=0}^{k} (r)_l S_2(k,l)(-y)^l \left\langle 1 \left| t^{m+k} Bel_n(x) \right. \right\rangle . \tag{5.6}$$

Here

$$\begin{aligned} \left\langle 1 \left| t^{m+k} Bel_n(x) \right. \right\rangle &= \sum_{j=m+k}^{n} S_2(n,j) \left\langle 1 \left| t^{m+k} x^j \right. \right\rangle \\ &= \sum_{j=m+k}^{n} S_2(n,j)(j)_{m+k} \delta_{j,m+k} \\ &= S_2(n,m+k)(m+k)!. \end{aligned} \tag{5.7}$$

Combining (5.6) and (5.7), we obtain

$$a_m = \sum_{k=0}^{n-m} \sum_{l=0}^{k} \binom{m+k}{m} (r)_l S_2(k,l) S_2(n,m+k)(-y)^l. \tag{5.8}$$

Similarly, application of (5.2) to $p(x) = (x)_n = \sum_{j=0}^{n} S_1(n,j)x^j$ gives

$$a_m = \sum_{k=0}^{n-m} \sum_{l=0}^{k} \binom{m+k}{m} (r)_l S_2(k,l) S_1(n,m+k)(-y)^l. \tag{5.9}$$

Finally, from (5.8) and (5.9), we have the following theorem.

Theorem 5.2. *For $n \geq 0$, we have*

$$Bel_n(x) = \sum_{m=0}^{n} \left(\sum_{k=0}^{n-m} \sum_{l=0}^{k} \binom{m+k}{m} (r)_l S_2(k,l) S_2(n,m+k)(-y)^l \right) F_m^{(r)}(x;y),$$

and

$$(x)_n = \sum_{m=0}^{n} \left(\sum_{k=0}^{n-m} \sum_{l=0}^{k} \binom{m+k}{m} (r)_l S_2(k,l) S_1(n,m+k)(-y)^l \right) F_m^{(r)}(x;y).$$

References

[1] L. Carlitz, *Some polynomials related to the Bernoulli and Euler polynomials*, Utilitas Math. **19** (1981), 81–127.

[2] R. Dere and Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 3, 433–438.

[3] G.-W. Jang and T. Kim, *Some identities of ordred Bell numbers arising from differential equations*, Adv. Stud. Contemp. Math. (Kyungshang) **27** (2017), no. 3, 385–397.

[4] L. Kargin, *Some formulae for products of Fubini polynomials with applications*, arXiv: 1701.01023 v1 [math. CA], 23, Dec. 2016.

[5] N. Kilar and Y. Simsek, *A new family of Fubini type numbers and polynomials associated with Apostol-Bernoulli numbers and polynomilas*, J. Korean Math. Soc. **54** (2017), no. 5, 1605–1621.
[6] T. Kim, *Identities involving Laguerre polynomials derived from umbral calculus*, Russ. J. Math. Phys. **21** (2014), no. 1, 36–45.
[7] ———, *Degenerate ordered Bell numbers and polynomials*, Proc. Jangjeon Math. Soc. **20** (2017), no. 2, 137–144.
[8] T. Kim and D. S. Kim, *On λ-Bell polynomials associated with umbral calculus*, Russ. J. Math. Phys. **24** (2017), no. 1, 69–78.
[9] T. Kim, D. S. Kim, and G.-W. Jang, *Extended Stirling polynomials of the second kind and extended Bell polynomials*, Proc. Jangjeon Math. Soc. **20** (2017), no. 3, 365–376.
[10] M. Mursan and Gh. Toader, *A generalization of Fubini's numbers*, Studia Univ. Babeş-Bolyai Math. **31** (1986), no. 1, 60–65.
[11] S. Roman, *The Umbral Calculus*, Pure and Applied Mathematics, **111**, Academic Press, Inc., New York, 1984.

Dae San Kim
Department of Mathematics
Sogang University
Seoul 04107, Korea
Email address: dskim@sogang.ac.kr

Taekyun Kim
Department of Mathematics
Kwangwoon University
Seoul 01897, Korea
Email address: tkkim@kw.ac.kr

Hyuck-In Kwon
Hanrimwon
Kwangwoon University
Seoul 01897, Korea
Email address: dvdolgy@gmail.com

Jin-Woo Park
Department of Mathematics Education
Daegu University
Gyeongsan-si 38453, Korea
Email address: a0417001@knu.ac.kr