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A NEW TYPE OF TUBULAR SURFACE HAVING POINTWISE 1-TYPE GAUSS MAP IN EUCLIDEAN 4-SPACE $\mathbb{E}^4$

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Abstract. In this paper, we handle the Gauss map of a tubular surface which is constructed according to the parallel transport frame of its spine curve. We show that there is no tubular surface having harmonic Gauss map. Moreover, we give a complete classification of this kind of tubular surface having pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$.

1. Introduction

The term of finite type immersions is presented by Chen, and then the same author writes some papers related to this topic [15, 16]. If a submanifold M is given in Euclidean m-space $\mathbb{E}^m$, and if an isometric immersion $x : M \to \mathbb{E}^m$, also known as the position vector field of M, is written as a finite sum of eigenvectors of the Laplacian Δ of M for a constant map x_0, and non-constant maps $x_1, x_2, \ldots, x_k$, i.e.,

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

then x is called as a finite type. Here, $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If the numbers λ_i's are different from each other, then the submanifold is called as k-type [14].

This term is extended to the Gauss map of M as

$$\Delta G = a(G + C) \tag{1}$$

for a real number a and a constant vector C by Chen and Piccinni in [18]. In this respect, a submanifold satisfying (1) is said to have 1-type Gauss map G. Then many papers have been written about submanifolds having 1-type Gauss map G [7–9, 26].

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Afterwards, in (1), the real number a is replaced with a non-constant function λ. That is, the equation (1) becomes

$$\Delta G = \lambda (G + C). \tag{2}$$

A submanifold satisfying (2) is said to have pointwise 1-type Gauss map G. If the function λ is non-constant, the pointwise 1-type Gauss map is called as proper. Also, if the vector C is zero, the pointwise 1-type Gauss map is called as the first kind. Otherwise, second kind [17].

Surfaces satisfying (2) have been the subject of many studies such as [1–4, 19–21, 27, 28, 32, 35]. In recent years, authors deal with the meridian surfaces with pointwise 1-type Gauss map in some spaces in [5, 6]. Also, authors study the tubular surfaces with pointwise 1-type Gauss map according to the Frenet frame in Euclidean 4-space in [30].

When a space curve $\gamma(u)$, a spine curve, is given, we can define a canal surface as the envelope of a one-parameter family of spheres whose centers are the points of the spine curve $\gamma(u)$ and whose radii $r(u)$ are varying. If the radius function $r(u)$ is constant, the canal surface is called as a tubular (tube) or a pipe surface. Actually, the notion of a canal surface is a generalization of an offset of a plane curve. In [22] and [23], the analysis and algebraic features of offset curves are discussed thoroughly. In [12, 33], authors consider canal surfaces in Euclidean spaces. Also in [31], authors study canal surfaces with parallel transport frame in $\mathbb{E}^4$.

In this paper, we handle tubular surface, constructed to parallel transport frame of its spine curve, with respect to its Gauss map in $\mathbb{E}^4$. We show that there is no tubular surface having harmonic Gauss map, and we give a complete classification of tubular surface having pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$.

2. Basic concepts

Let $\gamma = \gamma(u) : I \to \mathbb{E}^4$ be a unit speed curve in the Euclidean space $\mathbb{E}^4$ for an interval I in $\mathbb{R}$. Then the derivatives of the Frenet frame vectors of γ (Frenet-Serret formula) are as follows:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where $\{T, N, B_1, B_2\}$ is the Frenet frame of γ, and κ, τ, and σ are principal curvature functions according to Frenet frame of the curve γ, respectively.

Definition ([34])**.** A family of curves with constant curvature but non-constant torsion is called Salkowski curves.

Frenet-Serret frame gives way to the study of curves in classical differential geometry in Euclidean space. However, the Frenet frame cannot be constructed at the points in which curvature vanishes. Hence, an alternative frame is needed. In [10], Bishop defines a new frame for a curve and calls it Bishop frame, which is well defined even if the curve's second derivative vanishes in 3-dimensional Euclidean space. In [10,25] the advantages of the Bishop frame and the comparison of Bishop frame with the Frenet frame are given in Euclidean 3-space.

Euclidean 4-space $\mathbb{E}^4$ has the same problem as Euclidean 3-space. That is, one of the i-th $(1 < i < 4)$ derivatives of the curve may be zero. In [24], using the similar idea, authors consider such curves and construct an alternative frame. They give parallel transport frame of a curve and introduce the relations between the frame and Frenet frame of the curve in $\mathbb{E}^4$. They generalize the notion which is well known in Euclidean 3-space for 4-dimensional Euclidean space $\mathbb{E}^4$.

In [24], authors use the tangent vector $T(u)$ and three relatively parallel vector fields $M_1(u)$, $M_2(u)$, and $M_3(u)$ to construct an alternative frame. They call this frame a parallel transport frame along the curve γ. Then they give the following theorem for the parallel transport frame:

Theorem 2.1 ([24]). *Let $\{T, N, B_1, B_2\}$ be a Frenet frame along a unit speed curve $\gamma = \gamma(u) : I \to \mathbb{E}^4$ and $\{T, M_1, M_2, M_3\}$ denotes the parallel transport frame of the curve γ. The relation may be expressed as*

$$\begin{aligned}
T &= T(u),\\
N &= \cos\theta(u)\cos\psi(u)M_1 + (-\cos\phi(u)\sin\psi(u) + \sin\phi(u)\sin\theta(u)\cos\psi(u))M_2\\
&\quad + (\sin\phi(u)\sin\psi(u) + \cos\phi(u)\sin\theta(u)\cos\psi(u))M_3,\\
B_1 &= \cos\theta(u)\sin\psi(u)M_1 + (\cos\phi(u)\cos\psi(u) + \sin\phi(u)\sin\theta(u)\sin\psi(u))M_2\\
&\quad + (-\sin\phi(u)\cos\psi(u) + \cos\phi(u)\sin\theta(u)\sin\psi(u))M_3,\\
B_2 &= -\sin\theta(u)M_1 + \sin\phi(u)\cos\theta(u)M_2 + \cos\phi(u)\cos\theta(u)M_3.
\end{aligned}$$

The alternative parallel frame equations are

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix},$$

where k_1, k_2, k_3 are principal curvature functions according to parallel transport frame of the curve γ and their expressions as follows:

$$\begin{aligned}
k_1 &= \kappa\cos\theta\cos\psi,\\
k_2 &= \kappa(-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi),\\
k_3 &= \kappa(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi),
\end{aligned}$$

where $\theta' = \frac{\sigma}{\sqrt{\kappa^2+\tau^2}}$, $\psi' = -\tau - \sigma\frac{\sqrt{\sigma^2-\theta'^2}}{\sqrt{\kappa^2+\tau^2}}$, $\phi' = -\frac{\sqrt{\sigma^2-\theta'^2}}{\cos\theta}$ *and the following equalities*

$$\begin{aligned}\kappa(u) &= \sqrt{k_1^2+k_2^2+k_3^2},\\ \tau(u) &= -\psi' + \phi'\sin\theta,\\ \sigma(u) &= \frac{\theta'}{\sin\psi},\\ \phi'\cos\theta + \theta'\cot\psi &= 0\end{aligned}$$

hold.

Given a regular surface M in $\mathbb{E}^4$ with the parametrization $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$, at any point $p = X(u,v)$, the vectors X_u and X_v span the tangent space of M. Then the first fundamental form's coefficients are computed as

$$E = \langle X_u, X_u\rangle, F = \langle X_u, X_v\rangle, G = \langle X_v, X_v\rangle. \tag{3}$$

Here, $\langle , \rangle$ is the Euclidean dot product. For the regularity of the surface patch $X(u,v)$, $W^2 = EG - F^2 \neq 0$.

At any point p in M, there is a decomposition $T_p\mathbb{E}^4 = T_pM \oplus T_p^{\perp}M$, where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^4$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Then the induced Riemannian connection on M for any given local vector fields X, Y tangent to M is defined as

$$\nabla_X Y = (\widetilde{\nabla}_X Y)^T,$$

where T represents the tangential component.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the spaces of the smooth vector fields tangent and normal to M, respectively. The second fundamental map is defined as follows:

$$h : \chi(M) \times \chi(M) \to \chi^{\perp}(M), \quad h(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y. \tag{4}$$

This map is well-defined, bilinear, and symmetric. The equation (4) is known as the Gauss equation.

For each $X \in \chi(M)$ and $\xi \in \chi^{\perp}(M)$, the shape operator of M is defined as

$$A : \chi^{\perp}(M) \times \chi(M) \to \chi(M)$$

$$A_\xi X = -(\widetilde{\nabla}_X \xi)^T = -\widetilde{\nabla}_X \xi + \nabla_X^{\perp}\xi, \tag{5}$$

where A_ξ is the shape operator tensor and $\nabla^{\perp}$ is the normal connection belonging to $\chi^{\perp}(M)$. For any $X, Y \in \chi(M)$,

$$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle \tag{6}$$

holds. The operator A_ξ is self-adjoint and bilinear. The equation (5) is known as the Weingarten equation [13]. Thus, the coefficients of the second fundamental forms of M can be defined as follows:

$$h_{ij}^k = \langle h(X_i, X_j), N_k\rangle, \ 1 \leq k \leq 2, \tag{7}$$

where X_i and X_j are the orthonormal vectors of T_pM [13].

The shape operator matrix corresponded to normal vector N_k of M is given as

$$A_{N_k} = \begin{bmatrix} h_{11}^k & h_{12}^k \\ h_{12}^k & h_{22}^k \end{bmatrix}$$

[11]. The Gaussian curvature and the mean curvature vector of M are given as

$$K = \det(A_{N_1}) + \det(A_{N_2}) \tag{8}$$

and

$$\overrightarrow{H} = \frac{1}{2}\left\{iz(A_{N_1})N_1 + iz(A_{N_2})N_2\right\}, \tag{9}$$

respectively [13].

Now, let us recall some basic concepts of the Gauss map of M in $\mathbb{E}^m$. The Grassmannian manifold, $G(n,m)$, consists of all oriented n-planes through the origin of $\mathbb{E}^m$ and the vector space $\wedge^n\mathbb{E}^m$, obtained by the exterior product of n-vectors in $\mathbb{E}^m$, can be defined as an Euclidean space $\mathbb{E}^N$, where $N = \binom{m}{n}$. In the light of this information, we can define the Gauss map. Let $\{e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ be an orthonormal frame field in $\mathbb{E}^m$ such that the first n vectors are tangent and the others are normal to M, respectively. The map $G : M \to G(n,m) \subset \mathbb{E}^N$, which is defined as $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$, is called as Gauss map of M. This map is smooth and assigns a point p into an oriented n-plane in $\mathbb{E}^m$ by parallel translating the tangent space of M at p in $\mathbb{E}^m$.

The Laplacian $\Delta\varphi$ of a real function φ on M is defined as

$$\Delta\varphi = -\sum_i \left(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}\varphi - \widetilde{\nabla}_{\nabla_{e_i}e_i}\varphi\right).$$

3. Tubular surfaces having pointwise 1-type Gauss map in $\mathbb{E}^4$

Let $\gamma(u) = (f_1(u), f_2(u), f_3(u), f_4(u))$ be a curve parametrized by arclength, and $\{T, M_1, M_2, M_3\}$ is the parallel transport frame of the curve. Then the canal surface according to the parallel transport frame is given as

$$M : X(u,v) = \gamma(u) + r(u)\left(M_2(u)\cos v + M_3(u)\sin v\right). \tag{10}$$

Now, we consider the surface M in (10) as a tubular surface. Then its parametrization is given as

$$M : X(u,v) = \gamma(u) + r\left(M_2(u)\cos v + M_3(u)\sin v\right), \tag{11}$$

where r is a real constant. For the vector fields X_1, X_2 are tangent and N_1, N_2 are normal to M, we can choose an orthonormal frame $\{X_1, X_2, N_1, N_2\}$ given as follows:

$$X_1 = \frac{X_u}{\|X_u\|} = T,$$

$$X_2 = \frac{X_v}{\|X_v\|} = -(\sin v) M_2 + (\cos v) M_3, \tag{12}$$
$$N_1 = \frac{M_1 + (\cos v) M_2 + (\sin v) M_3}{\sqrt{2}},$$
$$N_2 = \frac{M_1 - (\cos v) M_2 - (\sin v) M_3}{\sqrt{2}}.$$

Moreover, by differentiating (12) covariantly with respect to X_1 and X_2, we obtain the following derivative formulas:

$$\begin{aligned}
\widetilde{\nabla}_{X_1} X_1 &= a_1 X_2 + a_2 N_1 + a_3 N_2, \\
\widetilde{\nabla}_{X_1} X_2 &= -a_1 X_1, \\
\widetilde{\nabla}_{X_1} N_1 &= -a_2 X_1, \\
\widetilde{\nabla}_{X_1} N_2 &= -a_3 X_1, \\
\widetilde{\nabla}_{X_2} X_1 &= 0, \\
\widetilde{\nabla}_{X_2} X_2 &= -\frac{1}{\sqrt{2}r} N_1 + \frac{1}{\sqrt{2}r} N_2, \\
\widetilde{\nabla}_{X_2} N_1 &= \frac{1}{\sqrt{2}r} X_2, \\
\widetilde{\nabla}_{X_2} N_2 &= -\frac{1}{\sqrt{2}r} X_2.
\end{aligned} \tag{13}$$

Here,

$$\begin{aligned}
a_1(u,v) &= \frac{k_3(u)\cos v - k_2(u)\sin v}{f(u,v)}, \\
a_2(u,v) &= \frac{k_1(u) + k_2(u)\cos v + k_3(u)\sin v}{\sqrt{2} f(u,v)}, \\
a_3(u,v) &= \frac{k_1(u) - k_2(u)\cos v - k_3(u)\sin v}{\sqrt{2} f(u,v)},
\end{aligned} \tag{14}$$

and $f = f(u,v) = 1 - k_2(u) r \cos v - k_3(u) r \sin v$ are differentiable functions.

From the equation (6) and (7), the second fundamental form's coefficients become

$$h_{ij}^k = \langle h(X_i, X_j), N_k \rangle = \langle A_{N_k} X_i, X_j \rangle, \ 1 \le i,j,k \le 2. \tag{15}$$

By considering the equations (5), (13), and (15), we obtain the coefficients

$$h_{11}^1 = a_2, \ h_{12}^1 = 0, \ h_{22}^1 = -\frac{1}{\sqrt{2}r}, \tag{16}$$
$$h_{11}^2 = a_3, \ h_{12}^2 = 0, \ h_{22}^2 = \frac{1}{\sqrt{2}r}.$$

By means of (16), we can give the following lemma, which is a special case of Proposition 9 in the paper [29].

Lemma 3.1 ([29]). *Let M be a tubular surface given with the parametrization* (11) *in $\mathbb{E}^4$. Then the shape operator matrices are given as follows*:

$$A_{N_1} = \begin{bmatrix} a_2 & 0 \\ 0 & -\frac{1}{\sqrt{2}r} \end{bmatrix}, \qquad A_{N_2} = \begin{bmatrix} a_3 & 0 \\ 0 & \frac{1}{\sqrt{2}r} \end{bmatrix}.$$

In the following corollary, we obtain the results which are given in [29] with a different calculation method.

Corollary 3.2 ([29]). *Let M be a tubular surface given with the parametrization* (11) *in $\mathbb{E}^4$. The Gaussian and the mean curvatures of M are respectively given as*

$$K = \frac{f-1}{fr^2}$$

and

$$H = \frac{1}{2fr}\left(4f^2 - 4f + r^2k_1^2 + 1\right)^{\frac{1}{2}}.$$

Proof. Considering $f(u,v) = 1 - k_2 r\cos v - k_3 r\sin v$ and (14), we can write

$$a_2(u,v) = \frac{k_1r + 1 - f}{\sqrt{2}fr} \text{ and } a_3(u,v) = \frac{k_1r - 1 + f}{\sqrt{2}fr}. \tag{17}$$

From the formulas (8) and (9),

$$\begin{aligned} K &= \det(A_{N_1}) + \det(A_{N_2}) \\ &= -\frac{a_2}{\sqrt{2}r} + \frac{a_3}{\sqrt{2}r} \\ &= -\frac{k_1r + 1 - f}{2fr^2} + \frac{k_1r - 1 + f}{2fr^2} \\ &= \frac{f-1}{fr^2}, \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{H} &= \frac{1}{2}\left\{iz(A_{N_1})N_1 + iz(A_{N_2})N_2\right\} \\ &= \frac{1}{2}\left\{\left(\frac{k_1r + 1 - f}{\sqrt{2}fr} - \frac{1}{\sqrt{2}r}\right)N_1 + \left(\frac{k_1r - 1 + f}{\sqrt{2}fr} + \frac{1}{\sqrt{2}r}\right)N_2\right\} \\ &= \frac{1}{2\sqrt{2}fr}\left\{(k_1r + 1 - 2f)N_1 + (k_1r - 1 + 2f)N_2\right\}. \end{aligned}$$

Taking the norm of the last equation, we get the expected mean curvature. □

The Gauss map of a given surface M in $\mathbb{E}^4$ is defined as $G = X_1 \wedge X_2$. Using the equations (13) and a direct computation, we get the Laplacian of the Gauss map as follows:

$$\Delta G = \left(a_2^2 + a_3^2 + \frac{1}{r^2}\right) X_1 \wedge X_2$$

$$
\begin{aligned}
&+\left(-a_1a_2-\frac{a_1}{\sqrt{2}r}\right)X_1\wedge N_1\\
&+\left(-a_1a_3+\frac{a_1}{\sqrt{2}r}\right)X_1\wedge N_2\\
&+\left(X_1\left[a_2\right]\right)X_2\wedge N_1\\
&+\left(X_1\left[a_3\right]\right)X_2\wedge N_2,
\end{aligned}
\tag{18}
$$

where $X_1\left[a_i\right]$ $(i=2,3)$ is the covariant derivative with respect to X_1.

From the equation (18), we can give the following result:

Corollary 3.3. *There is no tubular surface having harmonic Gauss map.*

Proof. Suppose the surface M has harmonic Gauss map, i.e., $\Delta G=0$. From (18), we have $a_2^2+a_3^2+\frac{1}{r^2}=0$, a contradiction. Therefore, there is no tubular surface having harmonic Gauss map. □

Now, we assume that the tubular surface M, given with the parametrization (11), has pointwise 1-type Gauss map. From (2) and (18), the equations

$$
\begin{aligned}
\lambda+\lambda\left\langle C,X_1\wedge X_2\right\rangle&=a_2^2+a_3^2+\frac{1}{r^2},\\
\lambda\left\langle C,X_1\wedge N_1\right\rangle&=-a_1a_2-\frac{a_1}{\sqrt{2}r},\\
\lambda\left\langle C,X_1\wedge N_2\right\rangle&=-a_1a_3+\frac{a_1}{\sqrt{2}r},\\
\lambda\left\langle C,X_2\wedge N_1\right\rangle&=X_1\left[a_2\right],\\
\lambda\left\langle C,X_2\wedge N_2\right\rangle&=X_1\left[a_3\right]
\end{aligned}
$$

hold. Here, λ is a non-zero smooth function. Thus, we obtain

$$
\lambda\left\langle C,N_1\wedge N_2\right\rangle=0. \tag{19}
$$

Differentiating (19) covariantly with respect to X_1 and X_2, we get

$$
\begin{aligned}
-\frac{a_1}{\sqrt{2}r\lambda}\left(a_2+a_3\right)&=0,\\
\frac{X_1\left[a_2\right]+X_1\left[a_3\right]}{\lambda}&=0.
\end{aligned}
\tag{20}
$$

Equations (20) allude to one of the three cases:

i) $a_3=-a_2$ and $a_1\neq 0$,

ii) $a_3=-a_2$ and $a_1=0$,

iii) $X_1\left[a_2+a_3\right]=0$ $(a_3\neq -a_2)$ and $a_1=0$.

From now on, we will cover the above three cases and obtain some results on tubular surfaces satisfying (2).

Case i) Let $a_3=-a_2$ and $a_1\neq 0$. Then $k_1=0$. From (2) and (18), we have

$$
C=\left(\frac{2a_2^2}{\lambda}+\frac{1}{r^2\lambda}-1\right)X_1\wedge X_2
$$

$$
\begin{aligned}
&+\left(-\frac{a_1a_2}{\lambda}-\frac{a_1}{\sqrt{2}r\lambda}\right)X_1\wedge N_1\\
&+\left(\frac{a_1a_2}{\lambda}+\frac{a_1}{\sqrt{2}r\lambda}\right)X_1\wedge N_2\\
&+\left(\frac{X_1\left[a_2\right]}{\lambda}\right)X_2\wedge N_1\\
&+\left(\frac{X_1\left[-a_2\right]}{\lambda}\right)X_2\wedge N_2.
\end{aligned}\tag{21}
$$

a) Assume that M has pointwise 1-type Gauss map of the first kind, i.e., $C=0$. Then we have $\frac{a_1a_2}{\lambda}+\frac{a_1}{\sqrt{2}r\lambda}=0$, which implies $a_1\left(\sqrt{2}ra_2+1\right)=0$. Since $a_1\neq 0$, we get $a_2=-\frac{1}{\sqrt{2}r}$. In the equations (17), using $k_1=0$, we obtain $a_2=\frac{1-f}{\sqrt{2}fr}$, which is a contradiction. Thus, there is no tubular surface having pointwise 1-type Gauss map of the first kind.

Theorem 3.4. *There is no tubular surface M given with the parametrization* (11) *having pointwise* 1*-type Gauss map of the first kind such that $a_3=-a_2$ and $a_1\neq 0$ in $\mathbb{E}^4$.*

b) Assume that M has pointwise 1-type Gauss map of the second kind, i.e., $C\neq 0$. By the use of (13) and (21), we get

$$
\begin{aligned}
\widetilde{\nabla}_{X_1}C=\ &\left\{X_1\left[\tfrac{2a_2^2}{\lambda}+\tfrac{1}{r^2\lambda}-1\right]+2a_2\tfrac{X_1[a_2]}{\lambda}\right\}X_1\wedge X_2\\
&+\left\{X_1\left[-\tfrac{a_1a_2}{\lambda}-\tfrac{a_1}{\sqrt{2}r\lambda}\right]-a_1\tfrac{X_1[a_2]}{\lambda}\right\}X_1\wedge N_1\\
&+\left\{X_1\left[\tfrac{a_1a_2}{\lambda}+\tfrac{a_1}{\sqrt{2}r\lambda}\right]+a_1\tfrac{X_1[a_2]}{\lambda}\right\}X_1\wedge N_2\\
&+\left\{X_1\left[\tfrac{X_1[a_2]}{\lambda}\right]-a_1\left(\tfrac{a_1a_2}{\lambda}+\tfrac{a_1}{\sqrt{2}r\lambda}\right)-a_2\left(\tfrac{2a_2^2}{\lambda}+\tfrac{1}{r^2\lambda}-1\right)\right\}X_2\wedge N_1\\
&+\left\{X_1\left[\tfrac{X_1[-a_2]}{\lambda}\right]+a_1\left(\tfrac{a_1a_2}{\lambda}+\tfrac{a_1}{\sqrt{2}r\lambda}\right)+a_2\left(\tfrac{2a_2^2}{\lambda}+\tfrac{1}{r^2\lambda}-1\right)\right\}X_2\wedge N_2\\
=\ &0
\end{aligned}\tag{22}
$$

and

$$
\begin{aligned}
\widetilde{\nabla}_{X_2}C=\ &\left\{X_2\left[\tfrac{2a_2^2}{\lambda}+\tfrac{1}{r^2\lambda}-1\right]-\tfrac{\sqrt{2}}{r}\left(\tfrac{a_1a_2}{\lambda}+\tfrac{a_1}{\sqrt{2}r\lambda}\right)\right\}X_1\wedge X_2\\
&+\left\{X_2\left[-\tfrac{a_1a_2}{\lambda}-\tfrac{a_1}{\sqrt{2}r\lambda}\right]-\tfrac{1}{\sqrt{2}r}\left(\tfrac{2a_2^2}{\lambda}+\tfrac{1}{r^2\lambda}-1\right)\right\}X_1\wedge N_1\\
&+\left\{X_2\left[\tfrac{a_1a_2}{\lambda}+\tfrac{a_1}{\sqrt{2}r\lambda}\right]+\tfrac{1}{\sqrt{2}r}\left(\tfrac{2a_2^2}{\lambda}+\tfrac{1}{r^2\lambda}-1\right)\right\}X_1\wedge N_2\\
&+\left\{X_2\left[\tfrac{X_1[a_2]}{\lambda}\right]\right\}X_2\wedge N_1\\
&+\left\{X_2\left[\tfrac{X_1[-a_2]}{\lambda}\right]\right\}X_2\wedge N_2\\
=\ &0.
\end{aligned}\tag{23}
$$

From (23), $X_2\left[\frac{X_1[a_2]}{\lambda}\right]=0$, namely

$$-\lambda f_{uv}f^3+f_u\left(\lambda_v f^3+3\lambda f^2 f_v\right)=0. \tag{24}$$

Again considering C is constant, from (23), we can write the differential equation system

$$X_2\left[\frac{2a_2^2}{\lambda}+\frac{1}{r^2\lambda}-1\right]-\frac{\sqrt{2}}{r}\left(\frac{a_1a_2}{\lambda}+\frac{a_1}{\sqrt{2}r\lambda}\right)=0,$$
$$X_2\left[\frac{a_1a_2}{\lambda}+\frac{a_1}{\sqrt{2}r\lambda}\right]+\frac{1}{\sqrt{2}r}\left(\frac{2a_2^2}{\lambda}+\frac{1}{r^2\lambda}-1\right)=0,$$

which has the solution

$$\begin{aligned}\frac{2a_2^2}{\lambda}+\frac{1}{r^2\lambda}-1&=m_1(u)\cos v+m_2(u)\sin v,\\ \frac{a_1a_2}{\lambda}+\frac{a_1}{\sqrt{2}r\lambda}&=m_3(u)\cos v+m_4(u)\sin v,\end{aligned} \tag{25}$$

where $m_i(u)$, $1\leq i\leq 4$ are differentiable functions. From (22), we have

$$\begin{aligned}\frac{X_1[a_2]}{\lambda}&=-\frac{1}{a_1}X_1\left[\frac{a_1a_2}{\lambda}+\frac{a_1}{\sqrt{2}r\lambda}\right],\\ \frac{X_1[a_2]}{\lambda}&=-\frac{1}{2a_2}X_1\left[\frac{2a_2^2}{\lambda}+\frac{1}{r^2\lambda}-1\right].\end{aligned} \tag{26}$$

Further, substituting (25) in (26), we get

$$\left(\frac{m_3'}{a_1}-\frac{m_1'}{2a_2}\right)\cos v+\left(\frac{m_4'}{a_1}-\frac{m_2'}{2a_2}\right)\sin v=0,$$

and so

$$m_3'=\frac{a_1}{2a_2}m_1',\ \text{and}\ m_4'=\frac{a_1}{2a_2}m_2'. \tag{27}$$

Lastly, from the equation (22), we have

$$X_1\left[\frac{X_1[a_2]}{\lambda}\right]=a_1\left(\frac{a_1a_2}{\lambda}+\frac{a_1}{\sqrt{2}r\lambda}\right)+a_2\left(\frac{2a_2^2}{\lambda}+\frac{1}{r^2\lambda}-1\right). \tag{28}$$

Substituting the value of $X_1\left[\frac{X_1[a_2]}{\lambda}\right]$ and (25) in (28), we obtain

$$\begin{aligned}&-f_{uu}f^2\lambda+f_u\left(\lambda_u f^2+3\lambda f f_u\right)\\ &=\sqrt{2}r\left((a_1m_3+a_2m_1)\cos v+(a_1m_4+a_2m_2)\sin v\right).\end{aligned} \tag{29}$$

Then we give Theorem 3.5.

Theorem 3.5. *Let M be a tubular surface given with the parametrization* (11) *in $\mathbb{E}^4$ such that $a_3=-a_2$ and $a_1\neq 0$. Then M has pointwise* 1*-type Gauss map of the second kind if and only if the equations* (24), (25), (27), *and* (29) *hold.*

Case ii) Let $a_3 = -a_2$ and $a_1 = 0$. Then $k_1 = 0$. Since $a_1 = 0$, from the equations (14), we have

$$\frac{k_3(u)}{k_2(u)} = \tan v = c,\ c = \text{constant}, \tag{30}$$

which is satisfied only on the points of the u-parameter curves of M. From (2) and (18), we can write

$$\begin{aligned} C = & \left(\frac{2a_2^2}{\lambda} + \frac{1}{r^2\lambda} - 1\right) X_1 \wedge X_2 \\ & + \left(\frac{X_1[a_2]}{\lambda}\right) X_2 \wedge N_1 \\ & + \left(\frac{X_1[-a_2]}{\lambda}\right) X_2 \wedge N_2. \end{aligned} \tag{31}$$

a) First, assume that M has pointwise 1-type Gauss map of the first kind, i.e., $C = 0$. Then considering the coefficients of $X_2 \wedge N_1$, $X_2 \wedge N_2$ and $a_1 = 0$, we get

$$X_1[a_2] = 0 \Longleftrightarrow k_2' \cos v + k_3' \sin v = 0,$$

which means that the curvature functions $k_2 \neq 0$ and $k_3 \neq 0$ are constants. Moreover, since k_1 is zero, the first curvature κ of the spine curve is constant, where τ and σ do not need to be constant. Thus, the spine curve γ is a Salkowski curve in $\mathbb{E}^4$.

Thus, we can give Theorem 3.6:

Theorem 3.6. *Let M be a tubular surface given with the parametrization* (11) *in $\mathbb{E}^4$ such that $a_3 = -a_2$ and $a_1 = 0$. If M has pointwise* 1*-type Gauss map of the first kind, the spine curve γ is a Salkowski curve. Here, M has a non-proper* 1*-type Gauss map with the constant function $\lambda = 2a_2^2 + \frac{1}{r^2}$.*

b) Now, assume that M has pointwise 1-type Gauss map of the second kind, i.e., $C \neq 0$. Differentiating (31) covariantly with respect to X_1, X_2, and considering C is constant, we get

$$\begin{aligned} \widetilde{\nabla}_{X_1} C = & \left\{ X_1\left[\frac{2a_2^2}{\lambda} + \frac{1}{r^2\lambda} - 1\right] + 2a_2 \frac{X_1[a_2]}{\lambda} \right\} X_1 \wedge X_2 \\ & + \left\{ X_1\left[\frac{X_1[a_2]}{\lambda}\right] - a_2\left(\frac{2a_2^2}{\lambda} + \frac{1}{r^2\lambda} - 1\right) \right\} X_2 \wedge N_1 \\ & + \left\{ X_1\left[\frac{X_1[-a_2]}{\lambda}\right] + a_2\left(\frac{2a_2^2}{\lambda} + \frac{1}{r^2\lambda} - 1\right) \right\} X_2 \wedge N_2 \\ = & \ 0 \end{aligned} \tag{32}$$

and

$$\widetilde{\nabla}_{X_2} C = \left\{ X_2\left[\frac{2a_2^2}{\lambda} + \frac{1}{r^2\lambda} - 1\right] \right\} X_1 \wedge X_2$$

$$\begin{aligned}
&-\frac{1}{\sqrt{2}r}\left(\frac{2a_2^2}{\lambda}+\frac{1}{r^2\lambda}-1\right)X_1\wedge N_1\\
&+\frac{1}{\sqrt{2}r}\left(\frac{2a_2^2}{\lambda}+\frac{1}{r^2\lambda}-1\right)X_1\wedge N_2\\
&+\left\{X_2\left[\frac{X_1[a_2]}{\lambda}\right]\right\}X_2\wedge N_1\\
&+\left\{X_2\left[\frac{X_1[-a_2]}{\lambda}\right]\right\}X_2\wedge N_2\\
=\ &0.
\end{aligned}\tag{33}$$

From (32) and (33), we get

$$\frac{2a_2^2}{\lambda}+\frac{1}{r^2\lambda}-1=0$$

and

$$X_1\left[\frac{X_1[a_2]}{\lambda}\right]=X_2\left[\frac{X_1[a_2]}{\lambda}\right]=a_2\frac{X_1[a_2]}{\lambda}=0.$$

Then we have $X_1[a_2]=0$, where $a_2\neq 0$. Thus,

$$X_1[a_2]=k_2'\cos v+k_3'\sin v=0,$$

which implies M has pointwise 1-type Gauss map of the first kind.

Theorem 3.7. *There is no tubular surface M given with the parametrization (11) having pointwise 1-type Gauss map of the second kind such that $a_3=-a_2$ and $a_1=0$ in $\mathbb{E}^4$.*

Case iii) Let $X_1[a_2+a_3]=0$ $(a_3\neq -a_2)$ and $a_1=0$. As in the second case, this case exists only on the points of the u-parameter curves of M. Moreover, from the assumption, we have

$$X_1[a_2+a_3]=X_1\left[\frac{\sqrt{2}k_1}{f}\right]=0\Longleftrightarrow k_1'f-k_1f_u=0.\tag{34}$$

Using the equations (2) and (18), we can write

$$\begin{aligned}
C=&\left(\frac{a_2^2+a_3^2}{\lambda}+\frac{1}{r^2\lambda}-1\right)X_1\wedge X_2\\
&+\left(\frac{X_1[a_2]}{\lambda}\right)X_2\wedge N_1\\
&+\left(\frac{X_1[a_3]}{\lambda}\right)X_2\wedge N_2.
\end{aligned}\tag{35}$$

a) First, assume that M has pointwise 1-type Gauss map of the first kind, i.e., $C=0$. Then all of the coefficients of C must vanish. It is enough to have a look at the second coefficient of C. Using (17) and (34), we get

$$X_1[a_2]=0\Longleftrightarrow k_1'rf-k_1rf_u-f_u=0$$

$$\begin{aligned}&\Longleftrightarrow f_u = 0\\ &\Longleftrightarrow k_2' \cos v + k_3' \sin v = 0,\end{aligned}$$

which means the curvature functions $k_2 \neq 0$ and $k_3 \neq 0$ are constants. From the equation (34), we get that k_1 is constant, too. Thus, the curvature function κ is constant, where τ and σ do not need to be constant. Hence, the spine curve γ is a Salkowski curve.

To sum up, we can give Theorem 3.8:

Theorem 3.8. *Let M be a tubular surface given with the parametrization* (11) *in $\mathbb{E}^4$ such that $X_1 [a_2 + a_3] = 0$ $(a_3 \neq -a_2)$ and $a_1 = 0$. Then M has pointwise* 1*-type Gauss map of the first kind if and only if the spine curve γ is a Salkowski curve, and the Gauss map is non-proper with the constant function $\lambda = a_2^2 + a_3^2 + \frac{1}{r^2}$.*

b) Now, assume that M has pointwise 1-type Gauss map of the second kind, i.e., $C \neq 0$. As in the other cases, differentiating (35) covariantly with respect to X_1, X_2 and considering C is constant, we get

$$\begin{aligned}\widetilde{\nabla}_{X_1} C = &\left\{ X_1 \left[\frac{a_2^2 + a_3^2}{\lambda} + \frac{1}{r^2\lambda} - 1 \right] + \frac{a_2 X_1 [a_2]}{\lambda} + \frac{a_3 X_1 [a_3]}{\lambda} \right\} X_1 \wedge X_2\\ &+ \left\{ X_1 \left[\frac{X_1 [a_2]}{\lambda} \right] - a_2 \left(\frac{a_2^2 + a_3^2}{\lambda} + \frac{1}{r^2\lambda} - 1 \right) \right\} X_2 \wedge N_1\\ &+ \left\{ X_1 \left[\frac{X_1 [a_3]}{\lambda} \right] - a_3 \left(\frac{a_2^2 + a_3^2}{\lambda} + \frac{1}{r^2\lambda} - 1 \right) \right\} X_2 \wedge N_2\\ = &\, 0\end{aligned} \tag{36}$$

and

$$\begin{aligned}\widetilde{\nabla}_{X_2} C = &\left\{ X_2 \left[\frac{a_2^2 + a_3^2}{\lambda} + \frac{1}{r^2\lambda} - 1 \right] \right\} X_1 \wedge X_2\\ &- \frac{1}{\sqrt{2}r} \left(\frac{a_2^2 + a_3^2}{\lambda} + \frac{1}{r^2\lambda} - 1 \right) X_1 \wedge N_1\\ &+ \frac{1}{\sqrt{2}r} \left(\frac{a_2^2 + a_3^2}{\lambda} + \frac{1}{r^2\lambda} - 1 \right) X_1 \wedge N_2\\ &+ \left\{ X_2 \left[\frac{X_1 [a_2]}{\lambda} \right] \right\} X_2 \wedge N_1\\ &+ \left\{ X_2 \left[\frac{X_1 [a_3]}{\lambda} \right] \right\} X_2 \wedge N_2\\ = &\, 0.\end{aligned} \tag{37}$$

From (36) and (37), we obtain

$$\frac{a_2^2 + a_3^2}{\lambda} + \frac{1}{r^2\lambda} - 1 = 0,$$

$$X_1\left[\frac{X_1[a_2]}{\lambda}\right]=X_2\left[\frac{X_1[a_2]}{\lambda}\right]=-X_1\left[\frac{X_1[a_3]}{\lambda}\right]=-X_2\left[\frac{X_1[a_3]}{\lambda}\right]=0,$$

and

$$\frac{a_2X_1[a_2]}{\lambda}+\frac{a_3X_1[a_3]}{\lambda}=0. \tag{38}$$

From the assumption, we can write (38) as:

$$a_2X_1[a_2]-a_3X_1[a_2]=X_1[a_2](a_2-a_3)=0.$$

Here, either $a_2=a_3$ or $X_1[a_2]=0$. If $a_2=a_3$, considering $X_1[a_2]=-X_1[a_3]$, we get $X_1[a_2]=0$, which means M has a pointwise 1-type Gauss map of the first kind. Thus, there is no tubular surface having pointwise 1-type Gauss map of the second kind.

Theorem 3.9. *There is no tubular surface M given with the parametrization* (11) *having pointwise* 1*-type Gauss map of the second kind such that $X_1[a_2+a_3]=0$ $(a_3\neq -a_2)$ and $a_1=0$ in $\mathbb{E}^4$.*

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