J. Korean Math. Soc. **55** (2018), No. 4, pp. 897–921
https://doi.org/10.4134/JKMS.j170515
pISSN: 0304-9914 / eISSN: 2234-3008

GEOMETRIC ANALYSIS ON THE DIEDERICH–FORNÆSS INDEX

Steven George Krantz, Bingyuan Liu, and Marco Maria Peloso

Abstract. Given bounded pseudoconvex domains in 2-dimensional complex Euclidean space, we derive analytical and geometric conditions which guarantee the Diederich-Fornæss index is 1. The analytical condition is independent of strongly pseudoconvex points and extends Fornæss–Herbig's theorem in 2007. The geometric condition reveals the index reflects topological properties of boundary. The proof uses an idea including differential equations and geometric analysis to find the optimal defining function. We also give a precise domain of which the Diederich–Fornæss index is 1. The index of this domain can not be verified by formerly known theorems.

1. Introduction

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. It is well known that such a domain Ω admits a plurisubharmonic function $-\log(-\delta(z))$, where δ is the signed distance function, that is,

$$\delta(z) := \begin{cases} -\operatorname{dist}(z, \partial\Omega), & z \in \Omega, \\ \operatorname{dist}(z, \partial\Omega), & \text{otherwise}. \end{cases}$$

However, the function $-\log(-\delta(z))$ is unbounded when z approaches the boundary, which makes some analysis on the boundary of the domain intractable. In 1977, Diederich and Fornæss showed in [10], that on any bounded pseudoconvex domain with smooth boundary there exists a bounded, plurisubharmonic exhaustion function. Their idea was to replace $-\log(-\delta(z))$ with $-(-\rho)^\eta$, where ρ is some defining function for Ω and $0 < \eta < 1$. In fact, they proved that, on any smoothly bounded pseudoconvex domain Ω with defining function ρ there exists $0 < \eta \leq 1$ such that $-(-\rho)^\eta$ is a strictly plurisubharmonic exhaustion function. Observe that $-(-\rho)^\eta$ will approach 0 when z goes to boundary, even if it will not be smooth at the boundary.

Received August 2, 2017; Revised November 15, 2017; Accepted January 30, 2018.
2010 *Mathematics Subject Classification.* Primary 32U05; Secondary 53C21.
Key words and phrases. Diederich-Fornæss index, pseudoconvex, domain, plurisubharmonic.

The existence of bounded plurisubharmonic exhaustion functions was later generalized to C^1 boundary by Kerzman and Rosay in [19] and to Lipschitz boundary by Demailly in [8] (see Harrington [15] too). Recently, Harrington generalized the existence theorem to $\mathbb{CP}^n$ in [16]. For discussions in $\mathbb{CP}^n$, the reader is also referred to [25] and [26] by Ohsawa and Sibony.

In this paper, we study properties of a given domain Ω, in connection with the optimization of the exponent in $-(-\rho)^\eta$. We now introduce the Diederich-Fornæss index.

Definition. Let Ω be a bounded, pseudoconvex domain in $\mathbb{C}^n$. The number $0 < \tau_\rho < 1$ is called a *Diederich-Fornæss exponent* if there exists a defining function ρ of Ω so that $-(-\rho)^{\tau_\rho}$ is plurisubharmonic. The index

$$\eta := \sup \tau_\rho \, ,$$

where the supremum is taken over all defining functions of Ω, is called the *Diederich-Fornæss index* of the domain Ω.

As an indication of the importance of the Diederich-Fornæss index of Ω we mentioned that Berndtsson and Charpentier [4] and Kohn [20], with two completely different methods, showed that, if Ω is smooth, bounded and pseudoconvex, then there exists $0 < s_\Omega \leq +\infty$ such that the Bergman projection $P : W^s(\Omega) \to W^s(\Omega)$ is bounded if $0 < s < s_\Omega$, where $W^s(\Omega)$ denotes the classical Sobolev space. Berndtsson and Charpentier showed that $s_\Omega \geq \eta/2$, where η is the Diederich-Fornæss index of Ω. On the other hand, Kohn provided an estimated for s_Ω again in terms of the Diederich-Fornæss index of Ω, although in a less explicit fashion; see also the paper [28].

In an earlier paper, Boas and Straube in [5] proved that if Ω is a smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$ admitting a defining function that is plurisubharmonic on the boundary, then the Bergman projection $P : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ is bounded, that is, Ω *satisfies condition R.*

In [12] and [13] Fornæss–Herbig addressed the question whether a smooth, bounded, pseudoconvex domain in $\mathbb{C}^2$ and $\mathbb{C}^n$, respectively, possessing a defining function that is plurisubharmonic on the boundary has Diederich-Fornæss index equal to 1. They answered this question in the positive. The converse does not hold in general. That is, if a domain has Diederich-Fornæss index 1, it does not necessarily admit a defining function which is plurisubharmonic on the boundary. The latter statement was proved by Behrens in [3] where she gave an example of a bounded domain with real analytic boundary and not having any local defining function that is plurisubharmonic on near a fixed boundary point. Nonetheless, this domain has Diederich-Fornæss index 1. The conclusion follows from another, related work by Diederich and Fornæss [11], where they showed the Diederich-Fornæss index is 1 if the pseudoconvex domain is *regular*, see Definition 1 and Theorem 1 in [11].

The main goal of this paper is to extend Fornæss–Herbig's result. More precisely, we would like to address the following questions:

Questions. (1) Can one find a more general condition than plurisubharmonicity of a defining function on the boundary to guarantee the Diederich–Fornæss index is 1? Possibly, this condition should cover the example of Behrens.

(2) On the other hand, how can one realize the condition from a geometric point of view?

(3) Can one find a bounded pseudoconvex domain admitting Diederich–Fornæss index 1, of which the fact is not discovered by formerly known theorems. In other words, we want to see a new application of the condition we found in Question 1 and this application should be new to us.

The Question 1 is necessary to the Diederich–Fornæss index, because the condition of Fornæss–Herbig is not sharp. We need to find a sufficient condition to cover the example of Behrens at least. Indeed, the following theorem is an extension of Fornæss–Herbig's theorem. The proof will be in Section 3. Please also have a look at Section 2 and Section 3 for basic notations.

Theorem 1. *Let Ω be a bounded domain with smooth boundary in $\mathbb{C}^2$. Let Σ denote the Levi-flat set in $\partial\Omega$. Assume that there exists a defining function ρ of Ω such that, on Σ, we have the condition* $\mathrm{Hess}_r(L, N) = 0$ *where L is the normalized $(1, 0)$-tangential vector field of $\partial\Omega$ and N is the normalized complex normal vector field of $\partial\Omega$. Then the Diederich-Fornæss index of Ω is* 1.

Remark 1. Here, the Levi-flat set can be read as weakly pseudoconvex set. We will use them interchangeably in this paper.

Remark 2. The condition $\mathrm{Hess}_\rho(L, N) = 0$ dates back to Boas-Straube's work in [5] where they showed that this is satisfied when r is plurisubharmonic on the boundary. Moreover, in practice, we do not need to assume that the L and N are normalized vectors. This is because $\mathrm{Hess}_\rho(L, N)$ is tensorial, that is, $\mathrm{Hess}_\rho(fL, gN) = f\bar{g}\,\mathrm{Hess}_\rho(L, N) = 0$ for arbitrary functions f and g.

The preceding theorem not only extends Fornæss–Herbig's theorem, but also relates more geometric informations to the index. This connects the Diederich–Fornæss index to Question 2. For this aim, we have to introduce some of our conventions. Namely, we will call a simple curve a *real curve* if it can be parametrized by a smooth map $\Psi : t \mapsto \mathbb{C}^2$. Also, for the definition and discussion of transversality, see Section 2.

We are ready to answer Question 2 with a series of results as follows. All of these will be discussed in Section 4.

Theorem 2. *Let Ω be a bounded domain with smooth boundary in $\mathbb{C}^2$. Let Σ denote the set of Levi-flat points in $\partial\Omega$. Assume that Σ is a real curve and transversal to the $(1, 0)$-tangent vector of $\partial\Omega$. Then the Diederich-Fornæss index of Ω is* 1.

Remark 3. In particular, as a consequence, we obtain that if Ω is a bounded domain with smooth boundary in $\mathbb{C}^2$, and the set of Levi-flat points in $\partial\Omega$ Σ is a set of isolated points, then the Diederich-Fornæss index of Ω is 1.

In fact, it is not hard to see also that if the set of Levi-flats points consists of finitely many isolated points and finitely many disjoint real curves transversal to the $(1,0)$-vector fields, then the Diederich-Fornæss index is 1.

Moreover, Theorem 2 is a special case of the following proposition. Indeed, Proposition 3 describes the geometry of the weakly pseudoconvex sets by the existence of solution to a type of partial differential equations.

Proposition 3. *Let δ be an arbitrarily defining function of a bounded domain $\Omega \subset \mathbb{C}^2$ with smooth boundary. Let $\Sigma \subset \partial\Omega$ denote the Levi-flat sets of $\partial\Omega$. Suppose there is a real function u which solves*

$$L(u) = -\frac{\text{Hess}_\delta(L, N)}{\|\nabla\delta\|}$$

on Σ. Then the Diederich-Fornæss index of Ω is 1.

Remark 4. The differential equation in the preceding proposition has been implicitly studied by Boas-Straube in [6].

In Section 5, we construct a specific bounded pseudoconvex domain $\tilde{\Omega}$ to answer Question 3. We remind the reader that our example cannot be verified by any known theorems except ours. Finally, Theorem 5.2 gives a satisfactory answer.

Before we proceed to prove our theorems, we briefly mention some history here and from it, one can have a full picture of the other extreme cases in which the Diederich-Fornæss index is away from 1. In 1977, Diederich-Fornæss found a domain called the worm domain in [9] which gives a non-trivial Diederich-Fornæss index (i.e., an index strictly between 0 and 1). In fact, they show that the Diederich-Fornæss exponent can be arbitrarily close to 0, see [9].

In 1992, Barrett showed in [2], that the Bergman projection P on Ω_β does not map the Sobolev space $W^k(\Omega_\beta)$ into $W^k(\Omega_\beta)$ when $k \geq \pi/(2\beta - \pi)$. In 2000, Berndtsson and Charpentier showed, in [4], that the Bergman projection P on Ω_β does map the Sobolev space $W^k(\Omega_\beta)$ into $W^k(\Omega_\beta)$ when $k < \tau/2$ where τ is a Diederich-Fornæss exponent. As a consequence, the Diederich-Fornæss index of Ω_β is less than or equal to $2\pi/(2\beta - \pi)$. The reader can also deduce this result from Krantz and Peloso [21]. Indeed, Theorem 6 in [9] says that if the standard defining function of Ω_β has exponent less than or equal to η, then all other defining functions have exponent less than or equal to η, that is, the Diederich-Fornæss index of Ω_β less than or equal to η. Thus, the calculation in [21] shows that the Ω_β less than or equal to $\pi/(2\beta - \pi)$. Recently Fu and Shaw and Adachi and Brinkschulte proved independently in [14] and [1] respectively that, roughly speaking, if a relatively compact domain in a complex manifold has all boundary points Levi-flat, then the Diederich-Fornæss index is non-trivial. (Here, the non-trivial index means the index is not

1 while the trivial index means the index is 1. Because the Diederich-Fornæss index cannot be 0, we define the trivial index to be the index 1.) Also, two papers of Herbig–McNeal in [17] and [18] include some interesting results.

2. Preliminaries

We begin by fixing some basic notation. Let M be a Hermitian manifold with complex structure J and metric g. For a real tangent vector field X we define

$$Z = \frac{1}{2}(X - \sqrt{-1}JX)$$

to be a $(1,0)$-tangent vector field and

$$\overline{Z} = \frac{1}{2}(X + \sqrt{-1}JX)$$

to be a $(0,1)$-tangent vector field. Recall that, if f is a function defined on M, then

$$Zf = g\left(\nabla f, \bar{Z}\right) = g\left(Z, \nabla f\right).$$

We also define the Hessian of a function f on real tangent vector fields:

$$\mathrm{Hess}_f(X,Y) = g(\nabla_X \nabla f, Y) = Y(Xf) - (\nabla_Y X)f,$$

and for $(1,0)$-tangent vectors we calculate as follows:

$$\mathrm{Hess}_f(Z,W) = g(\nabla_Z \nabla f, W) = Z(\overline{W}f) - \nabla_Z \overline{W} f = \overline{\mathrm{Hess}_f(W,Z)}.$$

We can also write the gradient in complex notation. Namely,

$$\nabla f = 2\left(\frac{\partial f}{\partial z}\frac{\partial}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{z}}\frac{\partial}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial}{\partial \bar{w}} + \frac{\partial f}{\partial \bar{w}}\frac{\partial}{\partial w}\right).$$

If the sectional curvature of M vanishes, then we have that the curvature tensor vanishes which means

$$R_m(Z_1, Z_2, Z_3, Z_4) \equiv 0,$$

where R_m denotes the curvature tensor. That means

$$0 \equiv \nabla_{Z_1}\nabla_{Z_2} Z_3 - \nabla_{Z_2}\nabla_{Z_1} Z_3 - \nabla_{[Z_1,Z_2]} Z_3$$

for arbitrary $(1,0)$-tangent fields Z_1, Z_2, Z_3 of M. For the basic notion of curvatures see [27].

From now on, we work on a domain in $\mathbb{C}^2$ and discuss transversality. Recall that a tangent vector of $\mathbb{C}^2$

$$L = f_1(z,w)\frac{\partial}{\partial z} + f_2(z,w)\frac{\partial}{\partial \bar{z}} + g_1(z,w)\frac{\partial}{\partial w} + g_2(z,w)\frac{\partial}{\partial \bar{w}}$$

indeed defines two real tangent vectors:

$$2\,\mathrm{Re}\, L := \mathrm{Re}(f_1 + f_2)\frac{\partial}{\partial x} + \mathrm{Im}(f_1 - f_2)\frac{\partial}{\partial y} + \mathrm{Re}(g_1 + g_2)\frac{\partial}{\partial u} + \mathrm{Im}(g_1 - g_2)\frac{\partial}{\partial v}$$

and

$$2\,\mathrm{Im}\,L := \mathrm{Im}(f_1+f_2)\frac{\partial}{\partial x} + \mathrm{Re}(f_2-f_1)\frac{\partial}{\partial y} + \mathrm{Im}(g_1+g_2)\frac{\partial}{\partial u} + \mathrm{Re}(g_2-g_1)\frac{\partial}{\partial v},$$

where we let the coordinates be

$$z = x + yi, \text{ and } w = u + vi.$$

Sometimes $\mathrm{Re}\,L$ and $\mathrm{Im}\,L$ are linearly dependent. But, for a nonzero $(1,0)$-tangent vector

$$L = f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w},$$

$\mathrm{Re}\,L$ and $\mathrm{Im}\,L$ are always independent because of the following easy lemma.

Lemma 2.1. *Let*

$$V = f_1(z,w)\frac{\partial}{\partial z} + f_2(z,w)\frac{\partial}{\partial \bar z} + g_1(z,w)\frac{\partial}{\partial w} + g_2(z,w)\frac{\partial}{\partial \bar w}$$

be a complex vector field. If $\mathrm{Re}\,V$ *and* $\mathrm{Im}\,V$ *are linearly dependent, then*

$$|f_1| = |f_2| \quad \text{and} \quad |g_1| = |g_2|.$$

In particular, if $V = L$ *is a* $(1,0)$*-vector field with* $\mathrm{Re}\,L$ *and* $\mathrm{Im}\,L$ *linearly dependent, then* $L = 0$.

Proof. Assume that

$$V = f_1(z,w)\frac{\partial}{\partial z} + f_2(z,w)\frac{\partial}{\partial \bar z} + g_1(z,w)\frac{\partial}{\partial w} + g_2(z,w)\frac{\partial}{\partial \bar w},$$

and that

$$\begin{pmatrix}\mathrm{Re}(f_1+f_2)\\ \mathrm{Im}(f_1-f_2)\\ \mathrm{Re}(g_1+g_2)\\ \mathrm{Im}(g_1-g_2)\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\mathrm{Im}(f_1+f_2)\\ \mathrm{Re}(f_2-f_1)\\ \mathrm{Im}(g_1+g_2)\\ \mathrm{Re}(g_2-g_1)\end{pmatrix}$$

are linearly dependent. Hence, both of the following two determinants

$$\begin{vmatrix}\mathrm{Re}(f_1+f_2) & \mathrm{Im}(f_1+f_2)\\ \mathrm{Im}(f_1-f_2) & \mathrm{Re}(f_2-f_1)\end{vmatrix} \quad \text{or} \quad \begin{vmatrix}\mathrm{Re}(g_1+g_2) & \mathrm{Im}(g_1+g_2)\\ \mathrm{Im}(g_1-g_2) & \mathrm{Re}(g_2-g_1)\end{vmatrix}$$

have to be zero.

By straightforward calculation,

$$\begin{aligned}\begin{vmatrix}\mathrm{Re}(f_1+f_2) & \mathrm{Im}(f_1+f_2)\\ \mathrm{Im}(f_1-f_2) & \mathrm{Re}(f_2-f_1)\end{vmatrix} &= |\,\mathrm{Re}\,f_2|^2 - |\,\mathrm{Re}\,f_1|^2 - |\,\mathrm{Im}\,f_1|^2 + |\,\mathrm{Im}\,f_2|^2\\ &= |f_2|^2 - |f_1|^2\end{aligned}$$

and

$$\begin{aligned}\begin{vmatrix}\mathrm{Re}(g_1+g_2) & \mathrm{Im}(g_1+g_2)\\ \mathrm{Im}(g_1-g_2) & \mathrm{Re}(g_2-g_1)\end{vmatrix} &= |\,\mathrm{Re}\,g_2|^2 - |\,\mathrm{Re}\,g_1|^2 - |\,\mathrm{Im}\,g_1|^2 + |\,\mathrm{Im}\,g_2|^2\\ &= |g_2|^2 - |g_1|^2.\end{aligned}$$

This immediately gives

$$|f_1| = |f_2| \quad \text{and} \quad |g_1| = |g_2|. \qquad \square$$

Lemma 2.1 guarantees that we are able to generalize the notion of *transversality* to nonzero $(1,0)$-tangent vector fields, because $|f_2| = |g_2| = 0$.

Definition 2.1. We say that a real curve $\gamma(t)$ is *transversal* to a nonzero $(1,0)$-tangent vector

$$L = f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w}$$

if $\gamma'(t)$, $\operatorname{Re} L$ and $\operatorname{Im} L$ are linear independent.

It is also easy to see that linear independence is preserved by a diffeomorphism.

3. Calculation of the D-F index

Let r be an arbitrary defining function of Ω. We want to modify the defining function in order to seek the best one for optimizing the Diederich-Fornæss exponent. Put $\rho = re^{\psi}$, where ψ will be determined later.

We first introduce some definitions.

Definition 3.1. Let Ω be a bounded domain with smooth boundary in $\mathbb{C}^2$ defined by a smooth defining function ρ. The vector field

$$L = \frac{1}{\sqrt{|\frac{\partial\rho}{\partial z}|^2 + |\frac{\partial\rho}{\partial w}|^2}}\left(\frac{\partial\rho}{\partial w}\frac{\partial}{\partial z} - \frac{\partial\rho}{\partial z}\frac{\partial}{\partial w}\right)$$

on $\partial\Omega$ is called the *normalized* $(1,0)$*-tangential vector field*, and

$$N = \frac{1}{\sqrt{|\frac{\partial\rho}{\partial z}|^2 + |\frac{\partial\rho}{\partial w}|^2}}\left(\frac{\partial\rho}{\partial \bar{z}}\frac{\partial}{\partial z} + \frac{\partial\rho}{\partial \bar{w}}\frac{\partial}{\partial w}\right)$$

on $\partial\Omega$ is called the *normalized complex normal vector field.*

Note that, due to the fact that L and N are unit vectors, $\mathrm{Hess}_{|z|^2}(L,L) = \mathrm{Hess}_{|z|^2}(N,N) = 1$. Also $\mathrm{Hess}_{|z|^2}(L,N) = 0$ due to the fact they are orthogonal. Here $|z|^2$ should be read as $|(z,w)|^2$, but for concision, we will not write it as $|(z,w)|^2$.

The following lemma is proved by a direct calculation. Since the calculation is tedious, we put it in the appendix.

Lemma 3.1. *Let Ω, r, L and N be as above, and let ψ be a smooth function. Let $\eta, \delta > 0$. Then*

$$\begin{aligned}&\mathrm{Hess}_{-(-re^{\psi})^{\eta}e^{-\delta\eta|z|^2}}(aL+bN, aL+bN)\\ =&\ -\eta e^{-\delta\eta|z|^2}(-re^{\psi})^{\eta-1}\left(|a|^2 I + 2\operatorname{Re}(a\bar{b}II) + |b|^2 III\right),\end{aligned}$$

where, on a sufficiently small neighborhood of $\partial\Omega$ in $\mathbb{C}^2$,

$$I = e^{\psi}\Big((\delta^2\eta)(-r)L(|z|^2)\overline{L}(|z|^2) - \delta(-r) + rL(\psi)\overline{L}(\psi) - \operatorname{Hess}_r(L,L) - r\operatorname{Hess}_\psi(L,L)\Big),$$

$$III < \frac{e^{\psi}}{2(-r)}(\eta-1)|N(r)|^2$$

and on $\partial\Omega$, we have the estimate

$$|II| < e^{\psi}\Big(\delta\eta|L(|z|^2)\overline{N}(r)| + |L(\psi)\overline{N}(r)| + |\operatorname{Hess}_r(L,N)|\Big).$$

We are ready to define $\psi = -C|\operatorname{Hess}_r(L_r,N_r)|^2$, where $C > 0$ is some number to be determined and

$$L_r = \frac{1}{\sqrt{|\frac{\partial r}{\partial z}|^2 + |\frac{\partial r}{\partial w}|^2}}\left(\frac{\partial r}{\partial w}\frac{\partial}{\partial z} - \frac{\partial r}{\partial z}\frac{\partial}{\partial w}\right)$$

and

$$N_r = \frac{1}{\sqrt{|\frac{\partial r}{\partial z}|^2 + |\frac{\partial r}{\partial w}|^2}}\left(\frac{\partial r}{\partial \bar{z}}\frac{\partial}{\partial z} + \frac{\partial r}{\partial \bar{w}}\frac{\partial}{\partial w}\right).$$

This definition of ψ is originally due to Fornæss–Herbig in [12]. More specifically, they proved the following lemma in [12]. Here we rewrite it with a language of differential geometry. For the detail of the proof, please see the Appendix.

Lemma 3.2. *Assume that*

$$\operatorname{Hess}_r(L,N) = 0$$

on Σ, where Σ is a subset of $\partial\Omega$. Let

$$\psi = -C|\operatorname{Hess}_r(L_r,N_r)|^2$$

for arbitrary $C > 0$. Then

$$L_r(\psi) = 0, \tag{1}$$

and

$$\begin{aligned}\operatorname{Hess}_\psi(L,L) &= \operatorname{Hess}_\psi(L_r,L_r)\\ &\le -C|L_r\operatorname{Hess}_r(N_r,L_r)|^2 = -C|N_r\operatorname{Hess}_r(L_r,L_r)|^2\end{aligned} \tag{2}$$

on Σ.

Remark 5. From the preceding lemma, we can also see that on Σ

$$L(\psi) = 0,$$

and

$$\operatorname{Hess}_\psi(L,L) \le -C|L\operatorname{Hess}_r(N,L)|^2 = -C|N\operatorname{Hess}_r(L,L)|^2.$$

The reason is as follows. On $\partial\Omega$, L, N coincide with L_r, N_r. Thus,

$$L \operatorname{Hess}_r(N, L) = L_r \operatorname{Hess}_r(N_r, L_r)$$

(we only need to consider the value on $\partial\Omega$). The identity $L \operatorname{Hess}_r(N, L) = N \operatorname{Hess}_r(L, L)$ is obtained from the following computation. Due to the vanishing curvature tensor in Euclidean spaces, we have that

$$\begin{aligned} 0 &= g(\nabla_L \nabla_N \nabla r, L) - g(\nabla_N \nabla_L \nabla r, L) - g(\nabla_{[L,N]} \nabla r, L) \\ &= Lg(\nabla_N \nabla r, L) - g(\nabla_N \nabla r, \nabla_{\overline{L}} L) - Ng(\nabla_L \nabla r, L) + g(\nabla_L \nabla r, \nabla_{\overline{N}} L) \\ &\quad - g(\nabla_{[L,N]} \nabla r, L). \end{aligned}$$

Since $\nabla_{\overline{N}} L$ and $[N, L]$ is spanned by L and N and $\nabla_{\overline{L}} L$ is parallel to L on $\partial\Omega$,

$$g(\nabla_N \nabla r, \nabla_{\overline{L}} L) = g(\nabla_L \nabla r, \nabla_{\overline{N}} L) = g(\nabla_{[L,N]} \nabla r, L) = 0.$$

Thus, we have that on $\partial\Omega$, $Lg(\nabla_N \nabla r, L) = Ng(\nabla_L \nabla r, L)$. The similar discussion can be found in the proof of Lemma 3.2 in the Appendix.

Then, with the notation of Lemma 3.1, we have on Σ,

$$|II| < e^{\psi} \big(\delta\eta |L(|z|^2) \overline{N}(r)| \big).$$

Moreover, there must be a neighborhood Σ_ϵ, which is dependent on $\epsilon > 0$, of Σ in $\mathbb{C}^2$, and on the neighborhood, we have

$$|II| < e^{\psi} \big(3\delta\eta \max\{|L(|z|^2)|, \epsilon\} |\overline{N}(r)| \big)$$

for some $\epsilon > 0$.

We can now prove Theorem 1. We want to point out that Behren's counterexample that we mentioned in Section 1 will not contradict the converse of Theorem 1. This is because her example has only one Levi-flat point on the boundary and we will show more generally that, if the Levi-flat points form a real curve (see Theorem 2), then it satisfies the condition of Theorem 1.

We prove the theorem by modifying the argument of Fornæss–Herbig in [12]. Our proof has a few new arguments. We need extra estimates on the points with Levi-forms bounded below. We also need to consider the points which have small positive Levi-forms.

Proof of Theorem 1. Let ψ be defined in Lemma 3.2. Firstly, we claim that if the Levi-form is bounded below by a positive number $\alpha > 0$, then in a neighborhood of these boundary points in $\mathbb{C}^2$

$$\operatorname{Hess}_{-(-\rho)^\eta}(aL + bN, aL + bN) > 0$$

holds for any defining function ρ of Ω and any $0 < \eta < 1$. We are going to show this fact in the following paragraph.

It is enough to show that the complex Hessian of $-(-\rho)^\eta$ is positive definite in a neighborhood of any strongly pseudoconvex boundary points. Rewrite the complex Hessian of $-(-\rho)^\eta$ with matrices.

$$\operatorname{Hess}_{-(-\rho)^\eta} = \begin{pmatrix} \operatorname{Hess}_{-(-\rho)^\eta}(L, L) & \operatorname{Hess}_{-(-\rho)^\eta}(L, N) \\ \operatorname{Hess}_{-(-\rho)^\eta}(N, L) & \operatorname{Hess}_{-(-\rho)^\eta}(N, N) \end{pmatrix}$$

$$= \begin{pmatrix} \eta(-\rho)^{\eta-1}\operatorname{Hess}_\rho(L,L) & \eta(-\rho)^{\eta-1}\operatorname{Hess}_\rho(L,N) \\ \eta(-\rho)^{\eta-1}\operatorname{Hess}_\rho(N,L) & \eta(-\rho)^{\eta-1}\left(\operatorname{Hess}_\rho(N,N)+\frac{1-\eta}{-\rho}|N\rho|^2\right)\end{pmatrix}$$

$$= \eta(-\rho)^{\eta-1}\begin{pmatrix} \operatorname{Hess}_\rho(L,L) & \operatorname{Hess}_\rho(L,N) \\ \operatorname{Hess}_\rho(N,L) & \operatorname{Hess}_\rho(N,N)+\frac{1-\eta}{-\rho}|N\rho|^2\end{pmatrix}$$

is positive definite if and only if $\operatorname{Hess}_\rho(L,L)>0$ and

$$\begin{vmatrix} \operatorname{Hess}_\rho(L,L) & \operatorname{Hess}_\rho(L,N) \\ \operatorname{Hess}_\rho(N,L) & \operatorname{Hess}_\rho(N,N)+\frac{1-\eta}{-\rho}|N\rho|^2\end{vmatrix}$$

$$= \operatorname{Hess}_\rho(L,L)\left(\operatorname{Hess}_\rho(N,N)+\frac{1-\eta}{-\rho}|N\rho|^2\right) - |\operatorname{Hess}_\rho(L,N)|^2$$

$$= \operatorname{Hess}_\rho(L,L)\operatorname{Hess}_\rho(N,N) + \operatorname{Hess}_\rho(L,L)\frac{1-\eta}{-\rho}|N\rho|^2 - |\operatorname{Hess}_\rho(L,N)|^2 > 0.$$

But this is clear because of $\operatorname{Hess}_\rho(L,L)\geq\frac{\alpha}{2}$ and

$$\operatorname{Hess}_\rho(L,L)\frac{1-\eta}{-\rho}|N\rho|^2 \geq \frac{\alpha}{2}\frac{1-\eta}{-\rho}|N\rho|^2$$

in a neighborhood of any strongly pseudoconvex boundary points. Indeed, the term $\frac{1-\eta}{-\rho}|N\rho|^2$ can approach $+\infty$ as ρ goes to 0, while

$$\operatorname{Hess}_\rho(L,L)\operatorname{Hess}_\rho(N,N) - |\operatorname{Hess}_\rho(L,N)|^2$$

has to be bounded which completes the proof of the fact.

This implies that

$$\operatorname{Hess}_{-(-re^{\psi})^{\eta}e^{-\delta\eta|z|^2}}(aL+bN,aL+bN)>0.$$

So we just need to prove that

$$\operatorname{Hess}_{-(-re^{\psi})^{\eta}e^{-\delta\eta|z|^2}}(aL+bN,aL+bN)>0$$

on a neighborhood of the Levi-flat points in $\mathbb{C}^2$.

Let Σ be the Levi-flat subset of $\partial\Omega$. We learned in Lemma 3.1 that there exists a neighborhood of $\partial\Omega$ in $\mathbb{C}^2$ such that, on this neighborhood,

$$III < \frac{e^{\psi}}{-2r}(\eta-1)|N(r)|^2.$$

To prove for any $0<\eta<1$, there exists an appropriate δ so that for all $a,b\in\mathbb{C}$

$$\operatorname{Hess}_{-(-re^{\psi})^{\eta}e^{-\delta\eta|z|^2}}(aL+bN,aL+bN)>0$$

on a neighborhood of Σ, we need to show that

$$|a|^2I+|b|^2III < -2|ab||II|$$

on a neighborhood of Σ in $\mathbb{C}^2$. We have seen on the neighborhood Σ_ϵ of Σ in $\mathbb{C}^2$,

$$III < \frac{e^{\psi}}{-2r}(\eta-1)|N(r)|^2$$

and

$$|II| < e^{\psi}\big(3\delta\eta \max\{L(|z|^2), \epsilon\}|\overline{N}(r)|.$$

If we can show that, on a neighborhood of Σ in $\mathbb{C}^2$,

$$I < -\frac{e^{\psi}}{4}\delta(-r),$$

then we are able to see that

$$\mathrm{Hess}_{-(-re^{\psi})^{\eta}e^{-\delta\eta|z|^2}}(aL + bN, aL + bN) > 0.$$

This is because, after shrinking δ further,

$$\begin{aligned}(3)\qquad -|a|^2\frac{1}{4}\delta(-r) - |b|^2\frac{1-\eta}{-2r}|N(r)|^2 &< -|ab||N(r)|\frac{\sqrt{1-\eta}}{2}\sqrt{\delta}\\ &< -6|ab|\delta\eta\max\{L(|z|^2), \epsilon\}|\overline{N}(r)|\end{aligned}$$

for some $\epsilon > 0$.

Thus, if we can show that, on a neighborhood of Σ,

$$\begin{aligned}&(\delta^2\eta)(-r)L(|z|^2)\overline{L}(|z|^2) - \delta(-r) + rL(\psi)\overline{L}(\psi) - \mathrm{Hess}_r(L, L) - r\,\mathrm{Hess}_{\psi}(L, L)\\ &< -\frac{1}{4}\delta(-r),\end{aligned}$$

then we are done.

Indeed, after shrinking δ, we do not need to consider $(\delta^2\eta)(-r)L(|z|^2)\overline{L}(|z|^2)$ because it is $o(\delta^2)$ and we can control it to make

$$(\delta^2\eta)(-r)L(|z|^2)\overline{L}(|z|^2) = (\delta^2\eta)(-r)|L(|z|^2)|^2 < \frac{1}{2}\delta(-r).$$

Hence we just need to show that

$$\begin{aligned}&r(q)L(\psi)|_q\overline{L}(\psi)|_q - \mathrm{Hess}_r(L, L)|_q - r(q)\,\mathrm{Hess}_{\psi}(L, L)|_q\\ &< \frac{1}{4}\delta(-r(q))\,\mathrm{Hess}_{|z|^2}(L, L)|_q\end{aligned}$$

for $q \in \Omega$. And thus it is enough to show that

$$\begin{aligned}&r(q)L(\psi)|_q\overline{L}(\psi)|_q - \mathrm{Hess}_r(L, L)|_q + \mathrm{Hess}_r(L, L)|_p - r(q)\,\mathrm{Hess}_{\psi}(L, L)|_q\\ &< \frac{1}{4}\delta(-r(q))\end{aligned}$$

for q in some neighborhood of $\partial\Omega$ in $\mathbb{C}^2$ and $p \in \partial\Omega$ so that $\mathrm{dist}(q, p) = \mathrm{dist}(q, \partial\Omega)$, because of $\mathrm{Hess}_r(L, L)|_p \geq 0$.

Thus it is enough to show that

$$-L(\psi)|_q\overline{L}(\psi)|_q - \lim_{q\to p}\frac{\mathrm{Hess}_r(L, L)|_q - \mathrm{Hess}_r(L, L)|_p}{-r(q) + r(p)} + \mathrm{Hess}_{\psi}(L, L)|_p < \frac{1}{5}\delta$$

on a neighborhood of Σ in $\partial\Omega$ where the limit means that q approaches p along the shortest path.

But here

$$-\lim_{q\to p}\frac{\mathrm{Hess}_r(L,L)|_q-\mathrm{Hess}_r(L,L)|_p}{-r(q)+r(p)}=\frac{g(\nabla\,\mathrm{Hess}_r(L,L),N+\overline{N})}{g(\nabla r,N+\overline{N})}$$
$$=\frac{2\,\mathrm{Re}(N\,\mathrm{Hess}_r(L,L))}{\|\nabla r\|}$$
$$\le\mathcal{K}\,\mathrm{Re}(N\,\mathrm{Hess}_r(L,L))$$

for some $\mathcal{K}>0$.

It is enough to show that

$$\mathcal{K}|N\,\mathrm{Hess}_r(L,L)|+\mathrm{Hess}_\psi(L,L)<\frac{1}{8}\delta$$

on Σ because

$$L(\psi)|_q\overline{L}(\psi)|_q=|L(\psi)|_q|^2$$

is nonnegative.

We calculate

$$\mathcal{K}|N\,\mathrm{Hess}_r(L,L)|+\mathrm{Hess}_\psi(L,L)\le\mathcal{K}|N\,\mathrm{Hess}_r(L,L)|-C|N\,\mathrm{Hess}_r(L,L)|^2$$
$$<\frac{\mathcal{K}^2}{4C},$$

and we take C so that

$$\frac{\mathcal{K}^2}{4C}\le\frac{1}{8}\delta.$$

Then we have

$$\mathcal{K}|N\,\mathrm{Hess}_r(L,L)|+\mathrm{Hess}_\psi(L,L)<\frac{1}{8}\delta,$$

which completes the proof. □

4. Complex transport equations

In this section, we are going to imitate transport equations in the real sense to the complex sense. It is well known that differential equations are very different in the real and complex contents. Thus, the imitation of transport equation in the complex sense cannot be fully extended.

In this section, our aim is to show that, if L is a $(1,0)$-tangent vector field and h is a smooth complex-valued function, the equation

$$Lu=h$$

is always solvable for real u. We prove the following proposition.

Proposition 4.1. *Let $\epsilon>0$ be arbitrary and Γ be a compact (real) smooth curve parametrized by $\gamma:[0,r]\to\Gamma$. (Here $\gamma(0)=\gamma(r)$ or $\gamma(0)\neq\gamma(r)$.) Assume that γ', $\mathrm{Re}\,L$ and $\mathrm{Im}\,L$ are linearly independent along Γ. Let h be a complex-valued smooth function defined on a neighborhood of Γ in $\partial\Omega$. Then, the following equation*

$$Lu=h\quad\text{on }\Gamma\tag{4}$$

has a real solution, where u is a real-valued smooth function defined in the neighborhood of Γ.

Proof. Assume Γ is parametrized by $\Gamma = \gamma([0,r])$. We can extend Γ smoothly beyond its endpoints if it is not a closed curve in order to give a neighborhood of endpoints to study. In case of a non-closed curve, redefine $\gamma : [-\epsilon, r+\epsilon] \to \partial\Omega$ and still assume that γ is a smooth real curve and γ', $\operatorname{Re} L$ and $\operatorname{Im} L$ are independent. (This is feasible if $\epsilon > 0$ is small enough.) We prove (4). Let $p \in \Gamma$. We find a neighborhood U_p of p and assume $\phi_p : U_p \to \mathbb{R}^3$ is a diffeomorphism. Then we have in the coordinates, the $(1,0)$-tangent vector

$$L = f_1(z,t)\frac{\partial}{\partial z} + f_2(z,t)\frac{\partial}{\partial \bar{z}} + g(z,t)\frac{\partial}{\partial t}$$

on the neighborhood. This gives two vectors $\operatorname{Re} L$ and $\operatorname{Im} L$. Possibly after shrinking U_p, we can always assume that $\phi_p \circ \gamma(t) = (0,0,t)$ and $\phi_p(\Gamma)$ is contained in the t-axis of $\mathbb{R}^3$. We first consider the following equation which admits a smooth real solution

$$(x(s_1,s_2,s_3), y(s_1,s_2,s_3), t(s_1,s_2,s_3)) :$$

$$\begin{cases} \frac{\partial x}{\partial s_1} = \operatorname{Re}(f_1(0,0,s_3) + f_2(0,0,s_3)) \\ \frac{\partial y}{\partial s_1} = \operatorname{Im}(f_1(0,0,s_3) - f_2(0,0,s_3)) \\ \frac{\partial t}{\partial s_1} = \operatorname{Re} g(0,0,s_3), \end{cases} \tag{5}$$

$$\begin{cases} \frac{\partial x}{\partial s_2} = \operatorname{Im}(f_1(0,0,s_3) + f_2(0,0,s_3)) \\ \frac{\partial y}{\partial s_2} = \operatorname{Re}(f_2(0,0,s_3) - f_1(0,0,s_3)) \\ \frac{\partial t}{\partial s_2} = \operatorname{Im} g(0,0,s_3) \end{cases} \tag{6}$$

and $(x(s_1,s_2,s_3), x(s_1,s_2,s_3), t(s_1,s_2,s_3))$ also satisfies the initial conditions:

$$(x(0,0,s_3), y(0,0,s_3), t(0,0,s_3)) = \phi_p \circ \gamma(s_3) = (0,0,s_3).$$

To solve (5) and (6) in $\phi_p(U_p) \subset \mathbb{R}^3$, we let

$$x(s_1,s_2,s_3) = \operatorname{Re}(f_1(0,0,s_3) + f_2(0,0,s_3))s_1 + \operatorname{Im}(f_1(0,0,s_3) + f_2(0,0,s_3))s_2,$$

$$y(s_1,s_2,s_3) = \operatorname{Im}(f_1(0,0,s_3) - f_2(0,0,s_3))s_1 + \operatorname{Re}(f_2(0,0,s_3) - f_1(0,0,s_3))s_2$$

and

$$t(s_1,s_2,s_3) = s_3 + (\operatorname{Re} g(0,0,s_3))s_1 + (\operatorname{Im} g(0,0,s_3))s_2.$$

We check also the initial condition $(x(0,0,s_3), y(0,0,s_3)) = (0,0)$ and $t(0,0,s_3) = s_3$. Hence

$$(x(0,0,s_3), y(0,0,s_3), t(0,0,s_3)) = (0,0,s_3).$$

Without loss of generality, we consider $x(s_1,s_2,s_3)$ solving

$$\begin{cases} \frac{\partial x}{\partial s_1} = \operatorname{Re}(f_1(0,0,s_3) + f_2(0,0,s_3)) \\ \frac{\partial x}{\partial s_2} = \operatorname{Im}(f_1(0,0,s_3) + f_2(0,0,s_3)) \end{cases} \tag{7}$$

with initial condition $x(0,0,s_3) = 0$. It is clear that the solution is unique. Hence it can be extended to the whole curve $\gamma([0,r])$ uniquely.

We are going to check the condition of the inverse function theorem on the map

$$(x(s_1,s_2,s_3), y(s_1,s_2,s_3), t(s_1,s_2,s_3))$$

at $(0,0,s_3)$. Now

$$\begin{pmatrix} \frac{\partial x}{\partial s_1} & \frac{\partial x}{\partial s_2} & \frac{\partial x}{\partial s_3} \\ \frac{\partial y}{\partial s_1} & \frac{\partial y}{\partial s_2} & \frac{\partial y}{\partial s_3} \\ \frac{\partial t}{\partial s_1} & \frac{\partial t}{\partial s_2} & \frac{\partial t}{\partial s_3} \end{pmatrix}$$

has rank 3 because $\gamma'(s_3)$, $\operatorname{Re} L$ and $\operatorname{Im} L$ are linearly independent. Hence locally we can define

$$h(s_1,s_2,s_3) := h(x(s_1,s_2,s_3), y(s_1,s_2,s_3), t(s_1,s_2,s_3)).$$

Thus, we define

$$u(s_1,s_2,s_3) = h_1(0,0,s_3)s_1 + h_2(0,0,s_3)s_2.$$

We immediately find that

$$\frac{\partial u}{\partial s_1} = h_1(0,0,s_3)$$

and

$$\frac{\partial u}{\partial s_2} = h_2(0,0,s_3)$$

can be solved uniquely by

$$u = h_1(0,0,s_3)s_1 + h_2(0,0,s_3)s_2$$

given with the initial condition

$$u(0,0,s_3) = 0.$$

Here, $h_1 := \operatorname{Re} h$ and $h_2 := \operatorname{Im} h$.

Since

$$\operatorname{Re} L = \operatorname{Re}(f_1 + f_2)\frac{\partial}{\partial x} + \operatorname{Im}(f_1 - f_2)\frac{\partial}{\partial y} + \operatorname{Re} g \frac{\partial}{\partial t}$$

on Γ locally, it can be rewritten as

$$\begin{aligned} \operatorname{Re} L = {} & \operatorname{Re}(f_1(0,0,s_3) + f_2(0,0,s_3))\frac{\partial}{\partial x} + \operatorname{Im}(f_1(0,0,s_3) - f_2(0,0,s_3))\frac{\partial}{\partial y} \\ & + \operatorname{Re} g(0,0,s_3)\frac{\partial}{\partial t} \\ = {} & \frac{\partial x}{\partial s_1}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s_1}\frac{\partial}{\partial y} + \frac{\partial t}{\partial s_1}\frac{\partial}{\partial t} \\ = {} & \frac{\partial}{\partial s_1}. \end{aligned}$$

For the same reason

$$\operatorname{Im} L = \frac{\partial}{\partial s_2}.$$

One can see again that u solves

$$\frac{\partial u}{\partial s_1} = h_1$$

and

$$\frac{\partial u}{\partial s_2} = h_2$$

with initial condition $u(0, 0, s_3) = 0$ uniquely. By the inverse function theorem, we can find $u(x, y, t)$ solving $Lu = h$ on Γ and such a u is defined on a neighborhood of Γ. □

The following extension lemma is classical. One can find it in any book on smooth manifolds, e.g. Lemma 2.26 of [22].

Lemma 4.1 (Extension Lemma for Smooth Functions)**.** *Suppose that M is a smooth manifold with or without boundary, $A \subset M$ is a closed subset, and $f : A \mapsto \mathbb{R}$ is a smooth function. For any open subset U containing A, there exists a smooth function $\tilde{f} : M \mapsto \mathbb{R}$ such that $\tilde{f}|_A = f$ and* $\operatorname{supp}\tilde{f} \subset U$*.*

Proof of Theorem 2. Calculate that

$$\begin{aligned}
\operatorname{Hess}_{\delta e^{\phi}}(L, N) &= g(\nabla_L \nabla(\delta e^{\phi}), N)\\
&= g(\nabla_L(\delta \nabla e^{\phi}), N) + g(\nabla_L(e^{\phi}\nabla\delta), N)\\
&= \delta g(\nabla_L(\nabla e^{\phi}), N) + e^{\phi} L(\delta)\overline{N}(\phi) + e^{\phi} g(\nabla_L(\nabla\delta), N)\\
&\quad + e^{\phi} L(\phi)\overline{N}(\delta).
\end{aligned}$$

On $\partial\Omega$,

$$\operatorname{Hess}_{\delta e^{\phi}}(L, N) = e^{\phi} g(\nabla_L(\nabla\delta), N) + e^{\phi} L(\phi)\overline{N}(\delta) = e^{\phi}(\operatorname{Hess}_{\delta}(L, N) + L(\phi)\overline{N}(\delta)).$$

We let ϕ be the solution of

$$\overline{L}(\phi) = -\frac{\overline{\operatorname{Hess}_{\delta}(L, N)}}{N(\delta)}$$

from Proposition 4.1 defined on a closed neighborhood $\overline{V}$ of Γ in $\partial\Omega$. Then one finds on Γ that

$$\operatorname{Hess}_{\delta e^{\phi}}(L, N) = 0.$$

By the extension lemma for smooth functions, we can find $\tilde{\phi}$ defined on a neighborhood U of Γ in $\mathbb{C}^2$ such that $\tilde{\phi}|_{\overline{V}} = \phi$. We find a smooth function χ defined on a neighborhood of Ω and $\overline{\operatorname{supp}\tilde{\phi}} \subset W$ where W is an open subset of $\mathbb{C}^2$ such that $\Gamma \subset \overline{W} \subset U$. We may assume that χ also satisfies

$$\chi(z, w) = \begin{cases} 1, & \text{if}(z, w) \in W,\\ 0, & \text{if}(z, w) \notin U. \end{cases}$$

Then we define Φ to be

$$\Phi(z,w) = \begin{cases} \chi\tilde{\phi}, & \text{if}(z,w) \in U, \\ 0, & \text{otherwise.} \end{cases}$$

Then δe^{Φ} is a defining function of Ω which satisfies

$$\mathrm{Hess}_{\delta e^{\Phi}}(L,N) = 0$$

on Γ. By Theorem 1, the theorem is proved. ☐

Inspired by the preceding proof, the following theorem can be established.

Theorem 4.1. *Let δ be an arbitrarily defining function of a bounded domain $\Omega \subset \mathbb{C}^2$ with smooth boundary. Let $\Sigma \subset \partial\Omega$ denote the Levi-flat sets of $\partial\Omega$. Then*

$$\mathrm{Hess}_{\rho}(L,N) = 0$$

on Σ for the defining function $\rho = \delta e^{\phi}$ of Ω if and only if there exists a (*real*) *smooth function ϕ so that*

$$L(\phi) = -\frac{\mathrm{Hess}_{\delta}(L,N)}{\overline{N}(\delta)} \qquad \text{on } \Sigma,$$

where L is the normalized $(1,0)$-tangential vector field of $\partial\Omega$ and N is the normalized $(1,0)$-normal vector field of $\partial\Omega$. Specifically, if there is no real function ϕ which solves

$$L(\phi) = -\frac{\mathrm{Hess}_{\delta}(L,N)}{\overline{N}(\delta)},$$

then there exists no defining functions r so that $\mathrm{Hess}_r(L,N) = 0$.

Proof. The first part is true because of the equality:

$$\mathrm{Hess}_{\delta e^{\phi}}(L,N) = e^{\phi}(\mathrm{Hess}_{\delta}(L,N) + L(\phi)\overline{N}(\delta)) \qquad \text{on } \Sigma.$$

The second part holds because it is well known that every defining function ρ can be written as $\rho = \delta \cdot h$ for some smooth $h > 0$. We define $\phi = \log h$ and then the second part follows from the first part. ☐

From the preceding theorem, combining with Theorem 1 we have the following result.

Theorem 4.2. *Let δ be an arbitrarily defining function of a bounded domain $\Omega \subset \mathbb{C}^2$ with smooth boundary. Let $\Sigma \subset \partial\Omega$ denote the Levi-flat sets of $\partial\Omega$. Suppose there is a real function u which solves*

$$L(u) = -\frac{\mathrm{Hess}_{\delta}(L,N)}{\|\nabla\delta\|}$$

on Σ. Then the Diederich-Fornæss index of Ω is 1.

5. Infinite type and Diederich-Fornæss index

In this section, we will answer Question 3 raised in Section 1. We want to see a new example which has the Diederich–Fornæss index 1 but cannot be verified by formerly known theorems. Our example will neither be of finite type nor admit a plurisubharmonic defining function. Thus, to show the Diederich–Fornæss index to be 1, we have to use our theorems in the current article.

It has been known to the experts that there is no equivalence between finite type and trivial Diederich-Fornæss index. Nevertheless, one can show that the domains of finite type have Diederich-Fornæss index 1. (Indeed, we could not find a precise reference. Professor Anne-Katrin Gallagher was kind to teach us the following arguments, so the authors owe her the credit.) This is done similar to the usual construction: $re^{-\phi_M}$ where r is some defining function for the domain, $\{\phi_M\}$ are the functions constructed by Catlin in [7]. That is $0 \leq \phi_M \leq 1$ and its complex Hessian of ϕ_M in a direction ξ is larger than $M|\xi|^2$.

Let us consider a domain of which the Levi-flat points form a real curve transversal to the $(1,0)$-tangent vector fields on the boundary. The reader should be warned that the following domain also admits a defining function which is plurisubharmonic on the boundary.

Example 5.1. Let

$$\Omega = \{(z,w) \in \mathbb{C}^2 : |z|^2 + 2e^{-1/|w|^2} < 1\}$$

be a bounded domain with smooth boundary in $\mathbb{C}^2$. Moreover, it has an infinite type point at $(1,0)$. Here, the defining function at $w = 0$ should be understood as $|z|^2 + 2\lim_{w\to 0} e^{-1/|w|^2} < 1$. The similar treatment will be seen later when the authors compute the Levi-forms.

We are going to verify the Levi-flat points are only at $(e^{i\theta}, 0)$ for $\theta \in [0, 2\pi)$. Since

$$\rho(z,w) = |z|^2 + 2e^{-1/|w|^2} - 1,$$

we obtain that

$$\frac{\partial\rho}{\partial w} = 2\frac{e^{-1/|w|^2}}{w^2\overline{w}},$$

$$\frac{\partial\rho}{\partial z} = \overline{z},$$

$$\frac{\partial^2\rho}{\partial z\partial\bar{z}} = 1$$

and

$$\frac{\partial^2\rho}{\partial w\partial\bar{w}} = 2e^{-1/|w|^2}\left(\frac{1}{|w|^6} - \frac{1}{|w|^4}\right).$$

Moreover, the $(1,0)$-tangent vector field is

$$L = 2\frac{e^{-1/|w|^2}}{w^2\overline{w}}\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial w}$$

and its complex Hessian is

$$\begin{pmatrix} 1 & 0 \\ 0 & 2e^{-1/|w|^2}(\frac{1}{|w|^6} - \frac{1}{|w|^4}) \end{pmatrix}.$$

Thus the Levi form is

$$\begin{aligned} &\begin{pmatrix} 2\frac{e^{-1/|w|^2}}{w^2\overline{w}} & -\overline{z} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2e^{-1/|w|^2}(\frac{1}{|w|^6} - \frac{1}{|w|^4}) \end{pmatrix} \begin{pmatrix} 2\frac{e^{-1/|w|^2}}{w\overline{w}^2} \\ -z \end{pmatrix} \\ &= 4\frac{e^{-2/|w|^2}}{|w|^6} + 2|z|^2 e^{-1/|w|^2}\left(\frac{1}{|w|^6} - \frac{1}{|w|^4}\right) \\ &= 2e^{-1/|w|^2}\left(2\frac{e^{-1/|w|^2}}{|w|^6} + |z|^2\left(\frac{1}{|w|^6} - \frac{1}{|w|^4}\right)\right). \end{aligned}$$

We can see that

$$\{(z,w) \in \mathbb{C}^2 : |z| = 1, w = 0\}$$

is a set of Levi-flat points. (Again, if the formula does not make sense at a point, we use limit to treat this point. The similar treatment has been seen before in the Example 5.1.) To see if all Levi-flat points belong to it, we need to solve the algebraic equation:

$$\begin{cases} 2e^{-1/|w|^2}\left(2\frac{e^{-1/|w|^2}}{|w|^6} + |z|^2\left(\frac{1}{|w|^6} - \frac{1}{|w|^4}\right)\right) = 0, \\ |z|^2 + 2e^{-1/|w|^2} = 1. \end{cases}$$

In case $w \neq 0$, the previous equation is equivalent to the following:

$$\begin{aligned} 0 &= 2\frac{e^{-1/|w|^2}}{|w|^6} + \left(1 - 2e^{-1/|w|^2}\right)\left(\frac{1}{|w|^6} - \frac{1}{|w|^4}\right) \\ &= \frac{1}{|w|^6} - \frac{1}{|w|^4} + 2\frac{e^{-1/|w|^2}}{|w|^4} \\ &= \frac{1}{|w|^4}\left(\frac{1}{|w|^2} - 1 + 2e^{-1/|w|^2}\right). \end{aligned}$$

To solve

$$0 = \frac{1}{|w|^2} - 1 + 2e^{-1/|w|^2},$$

we let $t = -\frac{1}{|w|^2} < 0$ and it converts to

$$0 = -t - 1 + 2e^t$$

which asserts that $t < 0$ has no solution ($2e^t - t - 1$ is a decreasing function for $t < 0$ and $2e^0 - 0 - 1 = 1 > 0$). Thus the complete Levi-flat set is

$$\Gamma := \{(z,w) \in \mathbb{C}^2 : |z| = 1, w = 0\}.$$

The tangent vector on Γ is

$$-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+0\frac{\partial}{\partial u}+0\frac{\partial}{\partial v},$$

where $z=x+iy$ and $w=u+iv$. At Γ,

$$L=-\bar{z}\frac{\partial}{\partial w}.$$

Hence the Diederich-Fornæss index of Ω is 1, because of L is transversal to Γ. The same computation works for a slightly more general domain as follows.

Proposition 5.1. *Assume $z_0\in\mathbb{C}$ and $v_0\in\mathbb{R}$. Let $p=(z_0,v_0)$ and*

$$\Omega_{a,b}(p):=\{(z,w)\in\mathbb{C}^2:a|z-z_0|^2+2e^{-1/|w-v_0|^2}<b\}$$

for arbitrary $a>0$ and $b\in(0,2)$. Then the Levi-flat sets $\mathcal{F}_{a,b}(p)$ is

$$\{(z,w)\in\mathbb{C}^2:|z-z_0|=\sqrt{\frac{b}{a}},w=v_0\}$$

is a real curve and is transversal to $(1,0)$-tangent vector fields. Hence, the Diederich-Fornæss index of $\Omega_{a,b}(p)$ is 1.

Remark 6. Let $z=x+iy$ and $w=u+iv$ and think of the v-axis as the vertical direction. By calculation, we find out the north pole $\mathcal{N}$ of $\Omega_{a,b}(p)$ is at $\left(z_0,v_0+\sqrt{\frac{1}{\ln 2-\ln b}}\right)$ and $\mathcal{F}_{a,b}(p)$ is the equator. We also define $\Omega^+_{a,b}(p)$ to be $\mathbb{B}(\mathcal{N},\epsilon)\cap\Omega_{a,b}(p)$, where $\epsilon>0$ is chosen to be very small so that $\mathcal{F}_{a,b}(p)\cap\mathbb{B}(\mathcal{N},2\epsilon)=\varnothing$. We denote the complement of $\Omega^+_{a,b}(p)$ in $\Omega_{a,b}(p)$ by $\Omega^-_{a,b}(p)$.

The previous example admits a defining function plurisubharmonic on the boundary. Of course, by Fornæss–Herbig's theorem, the Diederich–Fornæss index is 1. For the following paragraphs, we are going to construct a bounded pseudoconvex, infinite type domain $\tilde{\Omega}$ with smooth boundary which does not admit a defining function which is plurisubharmonic on the boundary. Our example is motivated by the method McNeal uses to prove Proposition 2.1 of [23]. More specifically, we will solder one piece of the domain in [3] with our domain $\Omega_{a,b}$ along the strongly pseudoconvex boundary points. The resulting domain should be infinite type because of $\Omega_{a,b}$ and does not admit a defining function which is plurisubharmonic on the boundary because of Behrens' result. Moreover, the domain has Diederich-Fornæss index 1 by Remark 3 because the Levi-flat points are a point united with a real curve transversal to $(1,0)$-tangent vector fields.

Firstly, recall the result of [3].

Theorem 5.1 (Behrens). *There is a $\tau_0>0$ such that $H\cap\mathbb{B}(0,\tau_0)$ is pseudoconvex from the side $\rho_H<0$ and the origin is the only non-strongly pseudoconvex point, where H is a smooth hypersurface defined by the function*

$$\rho_H(z,w)=v+R(z,w),$$

where

$$
\begin{aligned}
P_6(z) &= \frac{1}{2}|z|^6 + 2\,\mathrm{Re}\left(-\frac{1}{20}\bar{z}^5 z + \frac{i}{4}\bar{z}^4 z^2\right),\\
Q_4(z) &= \frac{1}{2}|z|^4 - \frac{i}{6}z^3\bar{z} + \frac{i}{6}\bar{z}^3 z,\\
R(z,w) &= P_6(z) + 2uQ_4(z) + |z|^2u^2 + |z|^2u^4 + |z|^{10} + |z|^6u^2.
\end{aligned}
$$

Recall that $p = (0, v_0)$, and we define

$$T(p,(z,w)) = a|z|^2 + 2e^{-1/|w-v_0|^2} \text{ for } a > 0.$$

Let $a > 0$ big, $0 < \delta < 1$ small, p in the side of $\rho_H < 0$ so that $\partial\Omega_{a,1}(p)$ intersects H transversally, $H \cap \partial\Omega_{a,1}(p) \subset \mathbb{B}(0,\tau_0)$ and $\Omega^-_{a,1}(p) \subset \{\rho_H < -\delta\}$. By a continuity argument, we can also assume there exists $\epsilon_0 > 0$ so that $H \cap \partial\Omega_{a,1+\epsilon_0}(p) \subset \mathbb{B}(0,\tau_0)$ and H intersects $\partial\Omega_{a,b}(p)$ transversally for any $b \in (1-\epsilon_0, 1+\epsilon_0)$. Moreover, we might shrink δ so that the origin is inside $T(p,(z,w)) \le 1-\epsilon_0$. This is because if we shrink δ, we can let p close to the origin, which makes the origin is easy to be contained in $T(p,(z,w)) \le 1-\epsilon_0$. We now modify ρ_H to $\tilde{\rho}_H$ so that $\tilde{\rho}_H = \rho_H$ for $T(p,(z,w)) \le 1-\epsilon_0$ and $\tilde{\rho}_H$ is strictly plurisubharmonic whenever $T(p,(z,w)) \ge 1-\epsilon_0$. This can be achieved because the origin is the only non-strongly pseudoconvex point and around strongly pseudoconvex points, there exist plurisubharmonic defining functions. Hence, one can apply a procedure due to Kohn to get a defining function strictly plurisubharmonic away from weakly pseudoconvex points (see Noell [24]).

We let

$$\rho = K\chi_1(T(p,(z,w))) + \chi_2(\tilde{\rho}_H(z,w)),$$

where $K > 2$ is to be determined later. Here we let $\chi_1(t)$ be a real-valued, C^∞ increasing function on $\mathbb{R}$ with $\chi_1(t) \equiv 0$ if $t < 1-\epsilon_0$ for an $0 < \epsilon_0 < \frac{1}{2}$ and $\chi_1''(t) > 0$ if $t > 1-\epsilon_0$ and there is $t_0 \in (1-\epsilon_0, 1+\epsilon_0)$ so that $\chi_1(t_0) = t_0$. We also let $\chi_2(t) : \mathbb{R} \to \mathbb{R}$ be a C^∞ convex increasing function such that $\chi_2(t) \equiv -\delta$ if $t \le -\delta$ and $\chi_2(t) = t$ if $t > -\frac{1}{2}\delta$.

Let

$$\tilde{\Omega} := \{(z,w) \in \mathbb{C}^2 : \rho(z,w) < 0\},$$

where K is big enough to ensure $\tilde{\Omega}$ being bounded. We divide $\partial\tilde{\Omega}$ into three sets:

$$
\begin{aligned}
B_1 &= \{(z,w) \in \partial\tilde{\Omega} : T(p,(z,w)) < 1-\epsilon_0\},\\
B_2 &= \{(z,w) \in \partial\tilde{\Omega} : \tilde{\rho}_H(z,w) < -\delta\},\\
B_3 &= \{(z,w) \in \partial\tilde{\Omega} : T(p,(z,w)) \ge 1-\epsilon_0 \quad \text{and} \quad \tilde{\rho}_H(z,w) > -\delta\}.
\end{aligned}
$$

For B_1, $\chi_1 = 0$ so the $\partial\tilde{\Omega}$ is defined by $\tilde{\rho}_H$. For B_2, observe first that if $T(p,(z,w)) = 1-\epsilon_0$, then χ_1 vanishes and

$$\rho = -\delta < 0.$$

When $(z,w)\in\mathbb{C}^2$ is such that $T(p,(z,w))=t_0\in(1-\epsilon_0,1+\epsilon_0)$,

$$\rho = KT(p,(z,w)) - \delta > 2-2\epsilon_0-\delta > 0,$$

by shrinking $\delta>0$. Hence $\partial\tilde{\Omega}$ can be defined on B_2 by

$$a|z|^2 + 2e^{-1/|w-w_0|^2} = \chi_1^{-1}(\frac{\delta}{K}),$$

where $\chi_1^{-1}(\frac{\delta}{K})<2$. Moreover Levi-flat sets of B_2 is a real curve transversal to $(1,0)$-tangent vector fields. Since B_3 is away from origin, ρ is a plurisubharmonic function on B_3, because $T(p,(z,w))$ and $\tilde{\rho}_H$ are both plurisubharmonic when (z,w) is away from origin. So $\tilde{\Omega}$ is strongly pseudoconvex on B_3. Since

$$\nabla\rho = K\chi_1'\nabla T(p,(z,w)) + \chi_2'\nabla\tilde{\rho}_H,$$

and H intersects $\partial\Omega_{a,b}(p)$ transversally for any $b\in(1-\epsilon_0,1+\epsilon_0)$, $\nabla\rho$ is not vanishing.

The Levi-flat set contains only a point union with a real curve transversal to $(1,0)$-tangent vector fields. Hence, we obtain the main theorem in the current section.

Theorem 5.2. *$\tilde{\Omega}$ is a bounded pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary which satisfies the following two properties*

1. *$\tilde{\Omega}$ is neither a domain of finite type nor a domain admitting a defining function which is plurisubharmonic on the boundary.*
2. *The Diederich-Fornæss index of $\tilde{\Omega}$ is 1.*

Appendix

Proof of Lemma 3.1

Since $L(\rho)=\overline{L}(\rho)=0$, we have that,

$$\begin{aligned}
&\mathrm{Hess}_{-(-re^{\psi})^{\eta}e^{-\delta\eta|z|^2}}(aL+bN,aL+bN)\\
=\;& -\eta e^{-\delta\eta|z|^2}(-re^{\psi})^{\eta-1}\big(|a|^2\big((\delta^2\eta)(-re^{\psi})L(|z|^2)\overline{L}(|z|^2)\\
&-\delta(-re^{\psi})\,\mathrm{Hess}_{|z|^2}(L,L) - e^{\psi}L(\psi)\overline{L}(r) - e^{\psi}L(r)\overline{L}(\psi) - re^{\psi}L(\psi)\overline{L}(\psi)\\
&-e^{\psi}\,\mathrm{Hess}_r(L,L) - re^{\psi}\,\mathrm{Hess}_{\psi}(L,L)\big) + 2\,\mathrm{Re}\,\big(a\bar{b}(\delta\eta e^{\psi}L(|z|^2)\overline{N}(r)\\
&+\delta\eta re^{\psi}L(|z|^2)\overline{N}(\psi) + (\delta^2\eta)(-re^{\psi})L(|z|^2)\overline{N}(|z|^2) - \delta(-re^{\psi})\,\mathrm{Hess}_{|z|^2}(L,N)\\
&-e^{\psi}L(\psi)\overline{N}(r) - e^{\psi}L(r)\overline{N}(\psi) - re^{\psi}L(\psi)\overline{N}(\psi) - e^{\psi}\,\mathrm{Hess}_r(L,N)\\
&-re^{\psi}\,\mathrm{Hess}_{\psi}(L,N))\big) + |b|^2(\delta\eta N(\rho)\overline{N}(|z|^2) + \delta\eta N(|z|^2)\overline{N}(\rho)\\
&+(\delta^2\eta)(-\rho)N(|z|^2)\overline{N}(|z|^2) - \delta(-\rho)\,\mathrm{Hess}_{|z|^2}(N,N) - \mathrm{Hess}_{\rho}(N,N)\\
&-\frac{1}{\rho}(\eta-1)N(\rho)\overline{N}(\rho))\big)\\
=\;& -\eta e^{-\delta\eta|z|^2}(-re^{\psi})^{\eta-1}\left(|a|^2 I + 2\,\mathrm{Re}(a\bar{b}II) + |b|^2 III\right).
\end{aligned}$$

Recall that $\text{Hess}_{|z|^2}(L,L)=1$. From these, we can simplify the term I a little bit in $\mathbb{C}^2$ with the following two identities,

$$\begin{aligned}&e^{\psi}(L\psi)(\overline{L}r)+re^{\psi}(L\psi)(\overline{L}\psi)=(L\psi)(\overline{L}\rho)=0 \quad \text{and}\\&e^{\psi}(Lr)(\overline{L}\psi)+re^{\psi}(L\psi)(\overline{L}\psi)=(\overline{L}\psi)(L\rho)=0.\end{aligned}$$

We have that

$$\begin{aligned}I&=(\delta^2\eta)(-re^{\psi})L(|z|^2)\overline{L}(|z|^2)-\delta(-re^{\psi})\,\text{Hess}_{|z|^2}(L,L)-e^{\psi}L(\psi)\overline{L}(r)\\&\quad-e^{\psi}L(r)\overline{L}(\psi)-re^{\psi}L(\psi)\overline{L}(\psi)-e^{\psi}\,\text{Hess}_r(L,L)-re^{\psi}\,\text{Hess}_{\psi}(L,L)\\&=(\delta^2\eta)(-re^{\psi})L(|z|^2)\overline{L}(|z|^2)-\delta(-re^{\psi})\,\text{Hess}_{|z|^2}(L,L)+re^{\psi}L(\psi)\overline{L}(\psi)\\&\quad-e^{\psi}\,\text{Hess}_r(L,L)-re^{\psi}\,\text{Hess}_{\psi}(L,L)\\&=e^{\psi}\big((\delta^2\eta)(-r)L(|z|^2)\overline{L}(|z|^2)-\delta(-r)\,\text{Hess}_{|z|^2}(L,L)+rL(\psi)\overline{L}(\psi)\\&\quad-\text{Hess}_r(L,L)-r\,\text{Hess}_{\psi}(L,L)\big)\\&=e^{\psi}\big((\delta^2\eta)(-r)L(|z|^2)\overline{L}(|z|^2)-\delta(-r)+rL(\psi)\overline{L}(\psi)\\&\quad-\text{Hess}_r(L,L)-r\,\text{Hess}_{\psi}(L,L)\big).\end{aligned}$$

Since $\text{Hess}_{|z|^2}(L,N)=0$ we can further simplify II a little:

$$\begin{aligned}II&=\delta\eta e^{\psi}L(|z|^2)\overline{N}(r)+\delta\eta re^{\psi}L(|z|^2)\overline{N}(\psi)+(\delta^2\eta)(-re^{\psi})L(|z|^2)\overline{N}(|z|^2)\\&\quad-\delta(-re^{\psi})\,\text{Hess}_{|z|^2}(L,N)-e^{\psi}L(\psi)\overline{N}(r)-e^{\psi}L(r)\overline{N}(\psi)-re^{\psi}L(\psi)\overline{N}(\psi)\\&\quad-e^{\psi}\,\text{Hess}_r(L,N)-re^{\psi}\,\text{Hess}_{\psi}(L,N)\\&=e^{\psi}\big(\delta\eta L(|z|^2)\overline{N}(r)+\delta\eta rL(|z|^2)\overline{N}(\psi)+(\delta^2\eta)(-r)L(|z|^2)\overline{N}(|z|^2)\\&\quad-\delta(-r)\,\text{Hess}_{|z|^2}(L,N)-L(\psi)\overline{N}(r)-L(r)\overline{N}(\psi)-rL(\psi)\overline{N}(\psi)\\&\quad-\text{Hess}_r(L,N)-r\,\text{Hess}_{\psi}(L,N)\big)\\&=e^{\psi}\big(\delta\eta L(|z|^2)\overline{N}(r)+\delta\eta rL(|z|^2)\overline{N}(\psi)+(\delta^2\eta)(-r)L(|z|^2)\overline{N}(|z|^2)\\&\quad-L(\psi)\overline{N}(r)-L(r)\overline{N}(\psi)-rL(\psi)\overline{N}(\psi)-\text{Hess}_r(L,N)-r\,\text{Hess}_{\psi}(L,N)\big).\end{aligned}$$

More specifically, on $\partial\Omega$, we have an estimate.

$$|II|<e^{\psi}\big(\delta\eta|L(|z|^2)\overline{N}(r)|+|L(\psi)\overline{N}(r)|+|\,\text{Hess}_r(L,N)|\big).$$

Now, for III,

$$\begin{aligned}III&=\delta\eta N(\rho)\overline{N}(|z|^2)+\delta\eta N(|z|^2)\overline{N}(\rho)+(\delta^2\eta)(-\rho)N(|z|^2)\overline{N}(|z|^2)\\&\qquad-\delta(-\rho)\,\text{Hess}_{|z|^2}(N,N)-\text{Hess}_{\rho}(N,N)-\frac{1}{\rho}(\eta-1)N(\rho)\overline{N}(\rho)\\&=\delta\eta N(\rho)\overline{N}(|z|^2)+\delta\eta N(|z|^2)\overline{N}(\rho)+(\delta^2\eta)(-\rho)N(|z|^2)\overline{N}(|z|^2)\\&\qquad-\delta(-\rho)\,\text{Hess}_{|z|^2}(N,N)-\text{Hess}_{\rho}(N,N)-\frac{1}{re^{\psi}}(\eta-1)(e^{2\psi}|N(r)|^2\\&\qquad+e^{2\psi}r^2|N(\psi)|^2+re^{2\psi}N(r)\overline{N}(\psi)+re^{2\psi}N(\psi)\overline{N}(r))\end{aligned}$$

$$= \delta\eta N(\rho)\overline{N}(|z|^2) + \delta\eta N(|z|^2)\overline{N}(\rho) + (\delta^2\eta)(-\rho)N(|z|^2)\overline{N}(|z|^2)$$
$$- \delta(-\rho) - \mathrm{Hess}_\rho(N,N) - \frac{1}{re^\psi}(\eta-1)(e^{2\psi}|N(r)|^2 + e^{2\psi}r^2|N(\psi)|^2$$
$$+ re^{2\psi}N(r)\overline{N}(\psi) + re^{2\psi}N(\psi)\overline{N}(r)).$$

By the previous equality, after shrinking the neighborhood of $\partial\Omega$ in $\overline{\Omega}$ so that r is sufficiently small, we can also obtain an estimate for III:

$$III < \frac{e^\psi}{-2r}(\eta-1)|N(r)|^2.$$

Proof of Lemma 3.2

We rewrite the proof with the language of differential geometry. This proof is essentially due to Fornæss–Herbig in [12]. Let $\xi = \mathrm{Hess}_r(N_r, L_r)$ and then $\psi = -C\xi\bar{\xi}$. We observe that

$$L_r(\psi) = -CL_r(\xi\bar{\xi}) = -C\xi L_r(\bar{\xi}) - C\bar{\xi}L_r(\xi).$$

The preceding equation is 0 because

$$\xi = \mathrm{Hess}_r(N_r, L_r) = 0 = \mathrm{Hess}_r(L_r, N_r) = \bar{\xi}$$

on Σ.

Now we are going to prove (2). Observe that on $\partial\Omega$, $L = L_r$ and

$$\mathrm{Hess}_{\xi\bar{\xi}}(L_r, L_r) = g(\nabla_{L_r}\nabla(\xi\bar{\xi}), L_r) = g(\nabla_{L_r}(\xi\nabla\bar{\xi}), L_r) + g(\nabla_{L_r}(\bar{\xi}\nabla\xi), L_r)$$
$$= 2\,\mathrm{Re}(\xi\,\mathrm{Hess}_{\bar{\xi}}(L_r, L_r)) + |L_r\xi|^2 + |L_r\bar{\xi}|^2$$

which implies

$$\mathrm{Hess}_\psi(L,L) = \mathrm{Hess}_\psi(L_r, L_r) \le -C|L_r\,\mathrm{Hess}_r(N_r, L_r)|^2$$

because $\xi = 0$ on Σ.

Next, we prove

$$L_r\,\mathrm{Hess}_r(N_r, L_r) = N_r\,\mathrm{Hess}_r(L_r, L_r)$$

on Σ. We have

$$L_r\,\mathrm{Hess}_r(N_r, L_r) = L_r g(\nabla_{N_r}\nabla r, L_r)$$
$$= g(\nabla_{L_r}\nabla_{N_r}\nabla r, L_r) + g(\nabla_{N_r}\nabla r, \nabla_{\overline{L}_r}L_r)$$
$$= g(\nabla_{L_r}\nabla_{N_r}\nabla r, L_r) + \mathrm{Hess}_r(N_r, \nabla_{\overline{L}_r}L_r).$$

Since

$$\mathrm{Hess}_r(L_r, N_r) = \mathrm{Hess}_r(N_r, L_r) = 0$$

and

$$0 = \mathrm{Hess}_r(L_r, L_r) = L_r(\overline{L}_r r) - (\nabla_{L_r}\overline{L}_r)r = -(\nabla_{L_r}\overline{L}_r)r,$$

we have that $\nabla_{\overline{L}_r}L_r$ is proportional to L_r and $\mathrm{Hess}_r(N_r, \nabla_{\overline{L}_r}L_r) = 0$. Hence

$$L_r\,\mathrm{Hess}_r(N_r, L_r) = g(\nabla_{L_r}\nabla_{N_r}\nabla r, L_r)$$
$$= g(\nabla_{N_r}\nabla_{L_r}\nabla r, L_r) + g(\nabla_{[L_r,N_r]}\nabla r, L_r),$$

because of the vanishing of the sectional curvature of $\mathbb{C}^2$. Therefore,

$$\begin{aligned} L_r \operatorname{Hess}_r(N_r, L) &= g(\nabla_{N_r}\nabla_{L_r}\nabla r, L_r) + g(\nabla_{[L_r,N_r]}\nabla r, L_r) \\ &= g(\nabla_{N_r}\nabla_{L_r}\nabla r, L_r) + \overline{g(\nabla_{L_r}\nabla r, [L_r, N_r])} \\ &= N_r g(\nabla_{L_r}\nabla r, L_r) - g(\nabla_{L_r}\nabla r, \nabla_{\overline{N}_r} L_r) \\ &\quad + \overline{g(\nabla_{L_r}\nabla r, [L_r, N_r])}, \end{aligned}$$

where the second term vanishes because

$$0 = \operatorname{Hess}_r(N_r, L_r) = N_r(\overline{L}_r r) - (\nabla_{N_r}\overline{L}_r)r = -(\nabla_{N_r}\overline{L}_r)r$$

implies that $\nabla_{\overline{N}_r} L_r$ is proportional to L_r and the third term vanishes because $[L_r, N_r]$ is linearly spanned by L_r and N_r and $g(\nabla_{L_r}\nabla r, L_r) = g(\nabla_{L_r}\nabla r, N_r) = 0$. Thus

$$L_r \operatorname{Hess}_r(N_r, L_r) = N_r g(\nabla_{L_r}\nabla r, L_r) = N \operatorname{Hess}_r(L_r, L_r),$$

which completes the proof.

Acknowledgments. The authors appreciate the kind help from Dr. Anne-Katrin Gallagher. They also thank to the referee for careful reading of the paper and for constructive comments and to Dr. Yuan Yuan for his interests and nice suggestions. Last but not least, they thank to Dr. Bun Wong, Dr. Qi S. Zhang, Xinghong Pan, Dr. Lihan Wang and Dr. Jihun Yum for fruitful conversations.

References

[1] M. Adachi and J. Brinkschulte, *A global estimate for the Diederich-Fornaess index of weakly pseudoconvex domains*, Nagoya Math. J. **220** (2015), 67–80.

[2] D. E. Barrett, *Behavior of the Bergman projection on the Diederich-Fornæss worm*, Acta Math. **168** (1992), no. 1-2, 1–10.

[3] M. Behrens, *Plurisubharmonic defining functions of weakly pseudoconvex domains in* $\mathbf{C}^2$, Math. Ann. **270** (1985), no. 2, 285–296.

[4] B. Berndtsson and P. Charpentier, *A Sobolev mapping property of the Bergman kernel*, Math. Z. **235** (2000), no. 1, 1–10.

[5] H. P. Boas and E. J. Straube, *Sobolev estimates for the* $\overline{\partial}$*-Neumann operator on domains in* $\mathbf{C}^n$ *admitting a defining function that is plurisubharmonic on the boundary*, Math. Z. **206** (1991), no. 1, 81–88.

[6] ———, *de Rham cohomology of manifolds containing the points of infinite type, and Sobolev estimates for the* $\overline{\partial}$*-Neumann problem*, J. Geom. Anal. **3** (1993), no. 3, 225–235.

[7] D. Catlin, *Subelliptic estimates for the* $\overline{\partial}$*-Neumann problem on pseudoconvex domains*, Ann. of Math. (2) **126** (1987), no. 1, 131–191.

[8] J.-P. Demailly, *Mesures de Monge-Ampère et mesures pluriharmoniques*, Math. Z. **194** (1987), no. 4, 519–564.

[9] K. Diederich and J. E. Fornæss, *Pseudoconvex domains: an example with nontrivial Nebenhülle*, Math. Ann. **225** (1977), no. 3, 275–292.

[10] ———, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), no. 2, 129–141.

[11] ———, *Pseudoconvex domains: existence of Stein neighborhoods*, Duke Math. J. **44** (1977), no. 3, 641–662.

[12] J. E. Fornæss and A.-K. Herbig, *A note on plurisubharmonic defining functions in* $\mathbb{C}^2$, Math. Z. **257** (2007), no. 4, 769–781.

[13] ———, *A note on plurisubharmonic defining functions in* $\mathbb{C}^n$, Math. Ann. **342** (2008), no. 4, 749–772.

[14] S. Fu and M.-C. Shaw, *The Diederich-Fornæss exponent and non-existence of Stein domains with Levi-flat boundaries*, J. Geom. Anal. **26** (2016), no. 1, 220–230.

[15] P. S. Harrington, *The order of plurisubharmonicity on pseudoconvex domains with Lipschitz boundaries*, Math. Res. Lett. **15** (2008), no. 3, 485–490.

[16] ———, *Bounded plurisubharmonic exhaustion functions for Lipschitz pseudoconvex domains in* $\mathbb{CP}^n$, J. Geom. Anal. **27** (2017), no. 4, 3404–3440.

[17] A.-K. Herbig and J. D. McNeal, *Convex defining functions for convex domains*, J. Geom. Anal. **22** (2012), no. 2, 433–454.

[18] ———, *Oka's lemma, convexity, and intermediate positivity conditions*, Illinois J. Math. **56** (2012), no. 1, 195–211 (2013).

[19] N. Kerzman and J.-P. Rosay, *Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut*, Math. Ann. **257** (1981), no. 2, 171–184.

[20] J. J. Kohn, *Quantitative estimates for global regularity*, in Analysis and geometry in several complex variables (Katata, 1997), 97–128, Trends Math, Birkhäuser Boston, Boston, MA, 1999.

[21] S. G. Krantz and M. M. Peloso, *Analysis and geometry on worm domains*, J. Geom. Anal. **18** (2008), no. 2, 478–510.

[22] J. M. Lee, *Introduction to Smooth Manifolds*, second edition, Graduate Texts in Mathematics, **218**, Springer, New York, 2013.

[23] J. D. McNeal, *Lower bounds on the Bergman metric near a point of finite type*, Ann. of Math. (2) **136** (1992), no. 2, 339–360.

[24] A. Noell, *Local and global plurisubharmonic defining functions*, Pacific J. Math. **176** (1996), no. 2, 421–426.

[25] T. Ohsawa and N. Sibony, *Bounded p.s.h. functions and pseudoconvexity in Kähler manifold*, Nagoya Math. J. **149** (1998), 1–8.

[26] ———,, *Kähler identity on Levi flat manifolds and application to the embedding*, Nagoya Math. J. **158** (2000), 87–93.

[27] P. Petersen, *Riemannian Geometry*, second edition, Graduate Texts in Mathematics, **171**, Springer, New York, 2006.

[28] S. Pinton and G. Zampieri, *The Diederich-Fornaess index and the global regularity of the* $\bar{\partial}$*-Neumann problem*, Math. Z. **276** (2014), no. 1-2, 93–113.

Steven George Krantz
Mathematics Department
Washington University
St. Louis, USA
Email address: sk@math.wustl.edu

Bingyuan Liu
Mathematics Department
University of California
Riverside, USA
Email address: bingyuan@ucr.edu

Marco Maria Peloso
Dipartimento di Matematica
Università degli Studi di Milano
Italy
Email address: marco.peloso@unimi.it