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REGULARIZATION FOR THE PROBLEM OF FINDING A SOLUTION OF A SYSTEM OF NONLINEAR MONOTONE ILL-POSED EQUATIONS IN BANACH SPACES

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Abstract. The purpose of this paper is to present an operator method of regularization for the problem of finding a solution of a system of nonlinear ill-posed equations with a monotone hemicontinuous mapping and N inverse-strongly monotone mappings in Banach spaces. A regularization parameter choice is given and convergence rate of the regularized solutions is estimated. We also give the convergence and convergence rate for regularized solutions in connection with the finite-dimensional approximation. An iterative regularization method of zero order in a real Hilbert space and two examples of numerical expressions are also given to illustrate the effectiveness of the proposed methods.

1. Introduction

The inverse problem we are interested in consists in determining an unknown physical quantity from a finite set of data in Banach spaces (see [18]). In practical situations, we do not know the data exactly. Instead, we have only approximate measured data satisfying some conditions. The finite set of data mentioned above is obtained by indirect measurements of a parameter, this process being described by a model of system of nonlinear equations (SNEs) in Banach spaces, which is, in general, a typical ill-posed problem.

Standard solution methods for solving SNEs are based on the use of iterative-type regularization methods (see [5, 7, 19, 24, 25]) or Tikhonov-type regularization methods (see [19, 30, 33, 37]) after rewriting SNEs as a single equation. However, these methods become inefficient if the number of equations of SNEs is large. In such a situation, Kaczmarz-type methods (see [22, 23, 29, 31]) which cyclically consider each equation in SNEs separately are much faster and are often the method of choice in practice. Some modifications of this method are studied for solving SNEs in Hilbert spaces, when each mapping is weakly

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sequentially continuous and the corresponding domain of definition is weakly closed (see [9, 15, 16, 20]).

In 2006, in order to solve SNEs, Buong [10] presented a regularization method of Browder–Tikhonov (RMBT) when each mapping is monotone, hemicontinuous and potential, and presented the so-called *generalized discrepancy principle* to choose a value of the regularization parameter. He also gave a convergence rate estimate for the regularized solutions. For a literature concerning RMBT, please refer to [11–13, 26, 27, 34]. In [11], the method RMBT was modified for Hilbert spaces without considering the choice problem of the regularization parameter. Note that the regularization method RMBT and some of its variants can be used for parallel computing (see [2–4]). Some extensions and generalizations of variational inequalities have been studied in [14, 17, 28, 32, 35, 36].

In what follows, we are interested in regularization methods for solving SNEs, where each equation in SNEs is ill-posed. The present work is motivated by interesting ideas on regularization for SNEs involving monotone mappings in [10].

We propose in this paper a new variant of the method RMBT of [10] and [11] with simpler conditions imposed on mappings in the framework of Banach spaces. We also consider the choice of the regularization parameter by the *residual principle*.

The rest of this paper is divided into three sections. In Section 2, we recall some definitions and results that will be used in the proof of our main theorems. In Section 3 we present a method to construct approximate solutions. In Section 4 we present a choice of the regularization parameter, and give an analysis of convergence rate results. In Section 5 we give the convergence and convergence rate for regularized solutions in connection with the finite-dimensional approximation. In the last section, we present an iterative regularization method of zero order in a real Hilbert space and two examples of numerical expressions.

2. Preliminaries

Let E be a real reflexive Banach space and E^* be the its dual space, which both are assumed to be strictly convex. For the sake of simplicity, norms in E and in E^* are denoted by the same symbol $\|\cdot\|$, and $\langle x^*, x\rangle$ denotes the value of the continuous linear functional $x^* \in E^*$ at the point $x \in E$. When $\{x_n\}$ is a sequence in E, $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x, and $x_n \to x$ means the strong convergence.

In what follows, we collect some definitions on monotone operators and their useful properties. We refer the reader [1] for more details.

Definition 2.1. A mapping $A : \mathcal{D}(A) \subset E \to E^*$ is said to be Lipschitz continuous with a constant $L > 0$ (or L-Lipschitz continuous) if

$$\|A(x) - A(y)\| \leq L\|x - y\| \qquad \forall\, x, y \in \mathcal{D}(A).$$

Definition 2.2. A mapping $A : \mathcal{D}(A) \subset E \to E^*$ is called

(i) monotone if

$$\langle A(x) - A(y), x - y\rangle \geq 0 \qquad \forall\, x, y \in \mathcal{D}(A);$$

(ii) maximal monotone if A is monotone and $G(A)$, the graph of A, is not properly contained in the graph of any other monotone mapping;

(iii) λ-inverse strongly monotone (or λ-cocoercive) if there exists a positive constant λ such that

$$\langle A(x) - A(y), x - y\rangle \geq \lambda \|A(x) - A(y)\|^2 \qquad \forall\, x, y \in \mathcal{D}(A).$$

It is well known that if F is a continuously Fréchet differentiable convex functional on E and its gradient ∇F is $\frac{1}{\lambda}$-Lipschitz continuous, then ∇F is λ-inverse strongly monotone (see [6, Corollary 10]).

Definition 2.3. A mapping $A : E \to E^*$ is called

(i) hemicontinuous at a point $x_0 \in \mathcal{D}(A)$ if $A(x_0 + tx) \rightharpoonup Ax_0$ as $t \to 0$ for any x such that $x_0 + tx \in \mathcal{D}(A)$;

(ii) demicontinuous at a point $x_0 \in \mathcal{D}(A)$ if for any sequence $\{x_n\} \subset \mathcal{D}(A)$ such that $x_n \to x_0$, the convergence $Ax_n \rightharpoonup Ax_0$ holds (it is evident that hemicontinuity of A follows from its demicontinuity).

If A is hemicontinuous at every point of $\mathcal{D}(A)$, then A is said to be hemicontinuous.

Obviously, any λ-inverse strongly monotone mapping A is monotone and L-Lipschitz continuous with constant $L = \frac{1}{\lambda}$. And any monotone and hemicontinuous operator $A : E \to E^*$ with $\mathcal{D}(A) = E$ is maximal monotone.

Definition 2.4. A reflexive Banach space E is said to be a **E**-space if it is strictly convex and has the Kadec-Klee property: for any sequence $\{x_n\}$, the weak convergence $x_n \rightharpoonup x$ and convergence of norms $\|x_n\| \to \|x\|$ imply strong convergence $x_n \to x$.

Note that Hilbert spaces as well as reflexive locally uniformly convex spaces are **E**-spaces. Therefore, L^p, l^p, W^p_m $(1 < p < \infty)$ are also **E**-spaces.

Throughout this paper, we assume that the normalized duality mapping $J : E \to E^*$ satisfying the relation

$$\langle x, J(x)\rangle = \|x\|^2 = \|J(x)\|^2 \qquad \forall\, x \in E$$

is single-valued. Note that this assumption is satisfied if E^* is strictly convex. Furthermore, we have the following properties for the mapping J (see [1]): $J(-x) = -J(x)$ for all $x \in \mathcal{D}(J)$, $J(tx) = tJ(x)$ for all $x \in \mathcal{D}(J)$ and $t \in [0, \infty)$, and

$$\langle J(x) - J(y), x - y\rangle \geq (\|x\| - \|y\|)^2 \qquad \forall\, x, y \in E. \tag{2.1}$$

If E is a strictly convex space, then J is a strictly monotone mapping. If E^* is strictly convex, then J is demicontinuous (hence, hemicontinuous).

3. Problem and method

In this paper, we consider the problem of finding a solution of a system of nonlinear ill-posed operator equations:

$$A_i(x) = f_i, \quad i = 0, 1, \dots, N, \tag{3.1}$$

where $N \geq 1$ is an integer, A_0 is monotone and hemicontinuous, the other mappings A_i, $i = 1, \dots, N$, are λ_i-inverse strongly monotone with domain $\mathcal{D}(A_i) = E$, and $f_i \in E^*$ for all $i = 0, 1, \dots, N$.

We are interested in the situation that the solution of (3.1) does not depend continuously on the data f_i (see, for instance, [1]). In addition, we assume that we are only given 'noisy data' $f_i^\delta \in E^*$ with known noise level $\delta > 0$, that is,

$$\|f_i - f_i^\delta\| \leq \delta, \quad i = 0, 1, \dots, N. \tag{3.2}$$

Denote by S_i the solution set of the i-th equation in (3.1), that is,

$$S_i = \{x \in E : A_i(x) = f_i\}.$$

Throughout this paper, we assume that

$$S := \bigcap_{i=0}^{N} S_i \neq \emptyset.$$

In what follows we introduce and analyze a new variant of equation (1.5) in [10] and variational inequality (7) in [11] in Banach spaces, where the *regularized solution* is defined by a solution of the following aggregated operator equation:

$$A_0(x) + \alpha^\mu \sum_{i=1}^{N} (A_i(x) - f_i^\delta) + \alpha J(x - x^+) = f_0^\delta, \tag{3.3}$$

with a regularization parameter $\alpha > 0$, a fixed number $\mu \in (0, 1)$, and an initial point $x^+ \notin S$.

Lemma 3.1. *Let E be a reflexive Banach space, E^* be a strictly convex Banach space, $J : \mathcal{D}(J) = E \to E^*$ be a normalized duality mapping. Suppose that $A_0 : \mathcal{D}(A_0) = E \to E^*$ is monotone and hemicontinuous, the other mappings $A_i : \mathcal{D}(A_i) = E \to E^*$, $i = 1, \dots, N$, are λ_i-inverse strongly monotone. Let $f_i^\delta \in E^*$ for all $\delta > 0$ and all $i = 0, 1, \dots, N$. Assume that condition* (3.2) *holds. Then, equation* (3.3) *has a unique solution x_α^δ for each $\alpha > 0$.*

Proof. Clearly, for each fixed $\alpha > 0$, the mapping

$$A^N(\cdot) := \alpha^\mu \sum_{i=1}^{N} (A_i(\cdot) - f_i^\delta),$$

is monotone and $(\alpha^\mu \sum_{i=1}^{N} L_i)$-Lipschitz continuous with $\mathcal{D}(A^N) = E$, where $L_i = \frac{1}{\lambda_i}$. So, A^N is hemicontinuous. Consequently, the mapping $A = A_0 + A^N$ is monotone and hemicontinuous with $\mathcal{D}(A) = E$. Hence, A is maximally monotone (see [1]). Furthermore, under our assumptions, we have that J is demicontinuous and single-valued, and $\mathcal{D}(J) = E$. Therefore, Theorem 1.7.4 (see [1]) ensures the solvability of equation (3.3). On the other hand, because J is strictly monotone, the mapping $A + \alpha J$ is also strictly monotone. Thus, equation (3.3) has a unique solution x_α^δ for each $\alpha > 0$. $\square$

Note that we can use an explicit method which is similar to (27) and (28) in [2] to formulate a numerical procedure to implement equation (3.3).

4. Convergence rate result and choice of the regularization parameter α

It is our purpose in this section to present our convergence results:

(1) The strong convergence of $\{x_\alpha^\delta\}$ to a solution of the system of equations (3.1) in the sense of regularization methods, that is, when $\alpha \to 0$ and $\frac{\delta}{\alpha} \to 0$ as $\delta \to 0$.

(2) The regularization parameter choice for $\alpha = \alpha(\delta)$ by using the principle $\rho(\alpha) = K\delta^p$, where $\rho(\alpha) = \alpha\|x_\alpha^\delta - x^+\|$ and $K > 3$.

(3) The convergence rate estimate for $\left\{x_{\alpha(\delta)}^\delta\right\}$ under the conditions:

(i)

$$\|A_0(y) - f_0 - [A_0^{'}(x^0)]^*(y - x^0)\| \leq \tau\|A_0(y) - f_0\|, \tag{4.1}$$

for y in some neighborhood of $x^0 \in S$, where $A_0^{'}(x^0)$ denotes the derivative of A_0 at x^0, $[A_0^{'}(x^0)]^*$ is the adjoint of $A_0^{'}(x^0)$, and τ is some positive constant;

(ii)

$$\langle J(x) - J(y), x - y\rangle \geq m_J\,\|x - y\|^s, \quad \forall\, x, y \in E,\ m_J > 0,\ s \geq 2. \tag{4.2}$$

The conditions (4.1) and (4.2) are standard in the literature. The condition (4.1) is called *tangential cone condition* and is widely used in the analysis of regularization methods for solving nonlinear ill-posed inverse problems (see [21]).

Note that when $A^N \equiv 0$, i.e., $A_i(x) = f_i$ for all $x \in E$ and all $i = 1, \dots, N$, we have that $\rho(\alpha) = \|A_0(x_\alpha^\delta) - f_0^\delta\|$ and the principle mentioned above is the residual one, investigated in [1] and references therein.

Theorem 4.1. *Let E be a* $\mathbf{E}$*-space. Assume that the conditions in Lemma 3.1 are satisfied. Suppose that*

$$S := \bigcap_{i=0}^{N} S_i \neq \emptyset.$$

If a regularization parameter choice $\alpha(\delta)$ satisfies the limit conditions

$$\alpha(\delta) \to 0 \quad \textit{and} \quad \frac{\delta}{\alpha(\delta)} \to 0 \quad \textit{as} \quad \delta \to 0, \tag{4.3}$$

then the sequence $\{x_\alpha^\delta\}$ *of associated regularized solutions of the operator equation* (3.3) *converges strongly in* E *to an* x^+*-minimum norm solution* x^0*, that is,*

$$x^0 \in S \quad \text{and} \quad \|x^0 - x^+\| = \min_{z\in S} \|z - x^+\|, \tag{4.4}$$

where $x^+ \notin S$ *is an initial guess.*

Proof. It follows from (3.1) and (3.3) that

$$\begin{aligned} &A_0(x_\alpha^\delta) - A_0(z) + \alpha^\mu \sum_{i=1}^N [A_i(x_\alpha^\delta) - A_i(z)] + \alpha J(x_\alpha^\delta - x^+) \\ &= f_0^\delta - f_0 + \alpha^\mu \sum_{i=1}^N (f_i^\delta - f_i) \end{aligned} \tag{4.5}$$

for any $z \in S$, where x_α^δ is a solution of (3.3). By (4.5) we have that

$$\begin{aligned} &\langle A_0(x_\alpha^\delta) - A_0(z) + \alpha^\mu \sum_{i=1}^N [A_i(x_\alpha^\delta) - A_i(z)] + \alpha J(x_\alpha^\delta - x^+), x_\alpha^\delta - z\rangle \\ &= \langle f_0^\delta - f_0 + \alpha^\mu \sum_{i=1}^N (f_i^\delta - f_i), x_\alpha^\delta - z\rangle. \end{aligned} \tag{4.6}$$

Using (3.2) with the monotonicity of A_i, it follows from (4.6) that

$$\langle J(x_\alpha^\delta - x^+), x_\alpha^\delta - z\rangle \le \frac{\delta}{\alpha}(1 + N\alpha^\mu)\|x_\alpha^\delta - z\|. \tag{4.7}$$

Consequently,

$$\begin{aligned} \|x_\alpha^\delta - x^+\| &\le \frac{1}{2}\Bigg\{\|x^+ - z\| + \frac{c(\delta,\alpha)}{\alpha} \\ &\qquad + \sqrt{\left(\|x^+ - z\| + \frac{c(\delta,\alpha)}{\alpha}\right)^2 + \frac{4c(\delta,\alpha)}{\alpha}\|x^+ - z\|}\Bigg\} \\ &\le \|x^+ - z\| + \frac{c(\delta,\alpha)}{\alpha} + \sqrt{\frac{c(\delta,\alpha)}{\alpha}\|x^+ - z\|}, \end{aligned} \tag{4.8}$$

where $c(\delta,\alpha) = \delta(1 + N\alpha^\mu)$. Hence, $\{x_\alpha^\delta\}$ is bounded, because $\delta/\alpha,\ \alpha \to 0$. Since E is reflexive, there exists a subsequence of $\{x_\alpha^\delta\}$, that converges weakly to some element $\overline{x} \in E$. For the sake of simplicity, assume that $x_\alpha^\delta \rightharpoonup \overline{x}$, as $\delta/\alpha, \alpha \to 0$.

First, we prove that $\overline{x} \in S_0$. Indeed, by virtue of the monotonicity of A_i and J, it follows from (3.3) that for all $x \in E$,

$$\begin{aligned} &\langle A_0(x) - f_0^\delta, x - x_\alpha^\delta\rangle \ge \langle A_0(x_\alpha^\delta) - f_0^\delta, x - x_\alpha^\delta\rangle \\ &\qquad = \alpha^\mu \sum_{i=1}^N \langle A_i(x_\alpha^\delta) - f_i^\delta, x_\alpha^\delta - x\rangle + \alpha\langle J(x_\alpha^\delta - x^+), x_\alpha^\delta - x\rangle \end{aligned} \tag{4.9}$$

$$\geq \alpha^{\mu} \sum_{i=1}^{N} \langle A_i(x) - f_i^{\delta}, x_{\alpha}^{\delta} - x\rangle + \alpha\langle J(x - x^{+}), x_{\alpha}^{\delta} - x\rangle.$$

Letting δ, $\alpha \to 0$ with $\delta/\alpha \to 0$ in (4.9), we obtain that

$$\langle A_0(x) - f_0, x - \overline{x}\rangle \geq 0 \qquad \forall x \in E.$$

Thus, $\overline{x} \in S_0$ (see [38]). Now, we shall prove that $\overline{x} \in S_i$, $i = 1, 2, \dots, N$. Again, from (3.3), the monotonicity of A_0 and (3.2), it follows that, for any $z \in S$,

$$\begin{aligned}\sum_{i=1}^{N}\langle A_i(x_{\alpha}^{\delta}) - f_i, x_{\alpha}^{\delta} - z\rangle = &\sum_{i=1}^{N}\langle f_i^{\delta} - f_i, x_{\alpha}^{\delta} - z\rangle + \alpha^{1-\mu}\langle J(x_{\alpha}^{\delta} - x^{+}), z - x_{\alpha}^{\delta}\rangle \\ &+ \frac{1}{\alpha^{\mu}}\langle f_0^{\delta} - A_0(x_{\alpha}^{\delta}) + A_0(z) - f_0, x_{\alpha}^{\delta} - z\rangle \\ \leq &\frac{\delta}{\alpha^{\mu}}(1+N\alpha^{\mu})\|x_{\alpha}^{\delta}-z\| + \alpha^{1-\mu}\langle J(x_{\alpha}^{\delta} - x^{+}), z - x_{\alpha}^{\delta}\rangle,\end{aligned}$$

which together with the λ_i-inverse-strongly monotonicity of A_i and the monotonicity of J, implies

$$\begin{aligned}&\sum_{i=1}^{N}\lambda_i\|A_i(x_{\alpha}^{\delta}) - A_i(z)\|^2 \\ &\leq \frac{\delta}{\alpha}\alpha^{1-\mu}(1 + N\alpha^{\mu})\|x_{\alpha}^{\delta} - z\| + \alpha^{1-\mu}\|x_{\alpha}^{\delta} - x^{+}\|\|z - x_{\alpha}^{\delta}\|.\end{aligned}$$

Thus, $\|A_i(x_{\alpha}^{\delta}) - f_i\| \to 0$ as δ, $\alpha \to 0$ with $\delta/\alpha \to 0$. Each mapping A_i is maximal monotone (see, [8, Theorem 1.3]). As we know (see [1, Lemma 1.4.5]), the graph $G(A)$ of any maximal monotone mapping A from a reflexive Banach space E to E^* is demiclosed, that is, $x_n \to x$, $y_n \rightharpoonup f$ or $x_n \rightharpoonup x$, $y_n \to f$, where $(x_n, y_n) \in G(A)$, imply that $(x, f) \in G(A)$. Thus, $A_i(\overline{x}) = f_i$, $i = 1, 2, \dots, N$, that is, $\overline{x} \in S_i$.

Next, since all of S_i are closed convex, S is also closed convex. Therefore, the element x^0 in S with x^+-minimal norm in the strictly convex Banach space E is unique. And now, from (4.8) with z replaced by $\overline{x}$, it implies that

$$\|x_{\alpha}^{\delta} - x^{+}\| \to \|\overline{x} - x^{+}\| \quad \text{and} \quad \|\overline{x} - x^{+}\| \leq \|z - x^{+}\|$$

for all $z \in S$. Hence, $x_{\alpha}^{\delta} \to \overline{x}$ (because E is an **E**-space), which is the element x^0, that we have to find. □

Now, we consider the choice of $\bar{\alpha} = \alpha(\delta)$ by using the principle $\rho(\alpha) = K\delta^p$, where $\rho(\alpha) = \alpha\|x_{\alpha}^{\delta} - x^{+}\|$ and $K > 3$.

Lemma 4.2. *Let E, E^*, J, S, A_i, and f_i^{δ} $(i = 0, 1, \dots, N)$ be as in Lemma 3.1. Let α_0 be some positive number satisfying $\frac{\alpha_0}{\delta} \geq c_0$, where $c_0 > 0$ is independent of δ. Then we have the following statements*:

(1) *The function $\rho(\alpha)$ is continuous on $[\alpha_0, +\infty)$ for each $\alpha_0 > 0$.*

(2) *We have that*

$$\lim_{\alpha\to+\infty}\rho(\alpha)=+\infty,$$

provided that A_i, for each $i=1,\dots,N$, is continuous at x^+ and

$$\left\|\sum_{i=1}^{N}\left[A_i(x^+)-f_i^\delta\right]\right\|>0 \quad \textit{for all} \quad \delta\geq 0, \tag{4.10}$$

and $f_0^\delta=f_0$.

Proof. (1) Let α and β be any two numbers in $[\alpha_0,+\infty)$, $\alpha_0>0$. It follows from (3.3) that

$$\begin{aligned}A_0(x_\alpha^\delta)-A_0(x_\beta^\delta)+\alpha^\mu\sum_{i=1}^{N}[A_i(x_\alpha^\delta)-f_i^\delta]-\beta^\mu\sum_{i=1}^{N}[A_i(x_\beta^\delta)-f_i^\delta]\\+\alpha J(x_\alpha^\delta-x^+)-\beta J(x_\beta^\delta-x^+)=0,\end{aligned}$$

which is equivalent to the following equality

$$\begin{aligned}\alpha\langle J(x_\alpha^\delta-x^+)-J(x_\beta^\delta-x^+),x_\alpha^\delta-x_\beta^\delta\rangle+(\alpha-\beta)\langle J(x_\beta^\delta-x^+),x_\alpha^\delta-x_\beta^\delta\rangle\\+\alpha^\mu\sum_{i=1}^{N}\langle A_i(x_\alpha^\delta)-A_i(x_\beta^\delta),x_\alpha^\delta-x_\beta^\delta\rangle\\+(\alpha^\mu-\beta^\mu)\sum_{i=1}^{N}\langle A_i(x_\beta^\delta)-f_i^\delta,x_\alpha^\delta-x_\beta^\delta\rangle\leq 0.\end{aligned}$$

This equality, together with the well-known inequality (2.1) and the monotonicity of A_i, implies that

$$\begin{aligned}(\|x_\alpha^\delta-x^+\|-\|x_\beta^\delta-x^+\|)^2\leq\left[\frac{|\alpha-\beta|}{\alpha_0}\|x_\beta^\delta-x^+\|+\frac{|\alpha^\mu-\beta^\mu|}{\alpha_0}\sum_{i=1}^{N}\|A_i(x_\beta^\delta)-f_i^\delta\|\right]\\\times(\|x_\alpha^\delta\|+\|x_\beta^\delta\|).\end{aligned}$$

So, from the last inequality and (4.8) with $c(\delta,\alpha)/\alpha$ replaced by $\delta[1/\alpha_0+N/\alpha_0^{1-\mu}]$, we get the continuity of $\|x_\alpha^\delta-x^+\|$ at any $\beta\in[\alpha_0,+\infty)$. Thus, $\rho(\alpha)$ is continuous on $[\alpha_0,+\infty)$.

(2) Now, it follows from (3.3) that

$$\begin{aligned}A_0(x_\alpha^\delta)-A_0(x^+)+\alpha^\mu\sum_{i=1}^{N}[A_i(x_\alpha^\delta)-A_i(x^+)]+\alpha J(x_\alpha^\delta-x^+)\\=f_0^\delta-A_0(x^+)+\alpha^\mu\sum_{i=1}^{N}[f_i^\delta-A_i(x^+)].\end{aligned}$$

Acting on the last equality by $x_\alpha^\delta - x^+$ and using the monotonicity of A_i and the definition of J, we obtain that

$$\|x_\alpha^\delta - x^+\| \leq \frac{\|f_0^\delta - A_0(x^+)\|}{\alpha} + \frac{1}{\alpha^{1-\mu}} \sum_{i=1}^N \|f_i^\delta - A_i(x^+)\|.$$

Thus,

$$\lim_{\alpha \to +\infty} \|x_\alpha^\delta - x^+\| = 0.$$

Clearly, the conclusion of Lemma 4.2 is proved by using (4.10), the last equality,

$$\rho(\alpha) \geq \alpha^\mu \left\| \sum_{i=1}^N \left[A_i(x_\alpha^\delta) - f_i^\delta\right] \right\| - \|A_0(x_\alpha^\delta) - f_0^\delta\|,$$

the continuity of A_i at x^+, for $i = 1, 2, \ldots, N$, and the local boundedness of A_0 (see [1, Theorem 1.3.16]). $\square$

Now, we show that the regularization parameter α can be chosen by the residual principle.

Theorem 4.3. *Let E, E^*, J, S, A_i, and f_i^δ $(i = 0, 1, \ldots, N)$ be as in Lemma 3.1. Let E be an* **E***-space and x^+ be a point in E satisfying* (4.10) *and $x^+ \notin S$. Then, there exists at least a value $\bar{\alpha} = \alpha(\delta)$ such that*

$$\bar{\alpha} \geq \frac{(K-3)\delta^p}{\|z - x^+\|}, \quad z \in S, \tag{4.11}$$

and

$$\rho(\bar{\alpha}) = K\delta^p, \quad K > 3, \quad 0 < p \leq 1. \tag{4.12}$$

In addition, letting $\delta \to 0$, we have that

(1) $\alpha(\delta) \to 0$;
(2) *if $p \in (0,1)$ then $\frac{\delta}{\alpha(\delta)} \to 0$, and $x_{\alpha(\delta)}^\delta \to x^0 \in S$, where x^0 is an x^+-minimum norm solution*;
(3) *if $p = 1$ and $S = \{x^0\}$, then $x_{\alpha(\delta)}^\delta$ converges weakly to x^0, and the inequality $\frac{\delta}{\alpha(\delta)} \leq C$ holds, where C is some positive constant that does not depend on δ.*

Proof. It follows from (4.8) that

$$\alpha\|x_\alpha^\delta - x^+\| \leq \alpha\|z - x^+\| + c(\delta, \alpha) + \sqrt{\alpha c(\delta, \alpha)\|x^+ - z\|} \tag{4.13}$$

for any $z \in S$. It is clear that, for every fixed $\delta > 0$,

$$\alpha\|z - x^+\| < (K-3)\delta^p, \quad K > 3, \ p \in (0, 1] \tag{4.14}$$

for sufficiently small α. Then, for $\alpha \leq \min\{1/N, \delta/(2\|x^+ - z\|)\}$ and $0 < \delta \leq 1$, we have that

$$\rho(\alpha) < (K-3)\delta^p + 3\delta < (K-3)\delta^p + 3\delta^p = K\delta^p. \tag{4.15}$$

Now, consider the function

$$d(\alpha) = \rho(\alpha) - K\delta^p \tag{4.16}$$

for $\alpha \geq \alpha_0 > 0$. By Lemma 4.2, we have

$$\lim_{\alpha\to+\infty} d(\alpha) = +\infty.$$

Obviously, by (4.15), (4.16), there is a value of $\alpha > 0$ such that $d(\alpha) < 0$. Since $d(\alpha)$ is continuous on $[\alpha_0, +\infty)$, there exists a value $\overline{\alpha}$ such that $d(\overline{\alpha}) = 0$, i.e., $\overline{\alpha} = \alpha(\delta)$ satisfies (4.12) and the symbol "$<$" in (4.14) is replaced by "$\geq$" for $\alpha = \overline{\alpha}$. It means that $\overline{\alpha} = \alpha(\delta)$ satisfies (4.11).

Next, we prove that:

(1) $\alpha(\delta) \to 0$ as $\delta \to 0$. If it is not true, then there exists a sequence $\overline{\alpha}_k = \alpha(\delta_k)$ with $\delta_k \to 0$ such that $\overline{\alpha}_k \to C_0$ (some positive constant) or $\overline{\alpha}_k \to \infty$ as $k \to \infty$.

Consider the first case, that is, $\overline{\alpha}_k \to C_0$ (some positive constant), as $k \to \infty$. From (4.12), we obtain that $C_0 \lim_{k\to\infty} \|x^{\delta_k}_{\overline{\alpha}_k} - x^+\| = 0$. Now, replacing δ, α and x in (3.3), respectively, by δ_k, $\overline{\alpha}_k$ and $x^{\delta_k}_{\overline{\alpha}_k}$, and taking the limit as $k \to +\infty$, we obtain that

$$A_0(x^+) - A_0(z) + C_0^\mu \sum_{i=1}^N [A_i(x^+) - A_i(z)] = 0 \tag{4.17}$$

for some $z \in S$. Acting on (4.17) by $x^+ - z$ and using the monotonicity of A_0 and as A_i is λ_i-inverse strongly monotone for $i = 1, 2, \ldots, N$, we have

$$\sum_{i=1}^N \lambda_i \|A_i(x^+) - A_i(z)\|^2 \leq 0.$$

Consequently, $\|A_i(x^+) - A_i(z)\| = 0$ for $i = 1, \ldots, N$. It means that $x^+ \in \cap_{i=1}^N S_i$. Therefore, it follows from (4.17) that $x^+ \in S_0$. Hence $x^+ \in S$, this is a contradiction to the assumption that $x^+ \notin S$.

In the second case, that is, $\overline{\alpha}_k \to \infty$ as $k \to \infty$. From (4.12), we have

$$\lim_{k\to+\infty} \|x^{\delta_k}_{\overline{\alpha}_k} - x^+\| = \lim_{k\to+\infty} \frac{\rho(\overline{\alpha}_k)}{\overline{\alpha}_k} = K \lim_{k\to+\infty} \frac{\delta_k^p}{\overline{\alpha}_k} = 0. \tag{4.18}$$

Again, replacing δ, α and x in (3.3), respectively, by δ_k, $\overline{\alpha}_k$ and $x^{\delta_k}_{\overline{\alpha}_k}$, we obtain that

$$\overline{\alpha}_k^\mu \left\| \sum_{i=1}^N A_i(x^{\delta_k}_{\overline{\alpha}_k}) - f_i^{\delta_k} \right\| - \|A_0(x^{\delta_k}_{\overline{\alpha}_k}) - f_0^{\delta_k}\| \leq \overline{\alpha}_k \|x^{\delta_k}_{\overline{\alpha}_k} - x^+\| = \rho(\overline{\alpha}_k) = K\delta_k^p.$$

Taking limits as $k \to +\infty$ in the last inequality and using (4.10), (4.18), the local boundedness of A_0 and the fact that $\overline{\alpha}_k \to \infty$ and $\delta_k \to 0$, we obtain the inequality $+\infty \leq 0$, that is impossible. So, there exists only the case that $\overline{\alpha} = \alpha(\delta) \to 0$ as $\delta \to 0$.

(2) Further, from (4.11), we obtain that

$$\frac{\delta}{\alpha(\delta)} \le \frac{\delta^{1-p}\|z - x^+\|}{(K-3)}.$$

So, in the case, when $0 < p < 1$, $\delta/\overline{\alpha} \to 0$ as $\delta \to 0$. By Theorem 4.1, $x^\delta_{\alpha(\delta)} \to x^0 \in S$ which solves (4.4).

(3) Obviously, when $p = 1$, we have $\delta/\alpha(\delta) \le C = \|z - x^+\|/(K-3)$. Then, from (4.13) it implies the boundedness of $\{x^\delta_{\alpha(\delta)}\}$. Since E is reflexive, there exists a subsequence $\{x_k := x^{\delta_k}_{\alpha(\delta_k)}\}$ which converges weakly to some element $x_\infty \in E$ as $k \to \infty$. By (3.3) with α, δ and x replaced by α_k, δ_k and x_k, respectively, we obtain that

$$\|A_0(x_k) - f_0^{\delta_k}\| \le \alpha_k^\mu \sum_{i=1}^N \|A_i(x_k) - f_i^{\delta_k}\| + \alpha_k\|x_k - x^+\|.$$

Thus, $\|A_0(x_k) - f_0^{\delta_k}\| \to 0$ as $k \to \infty$. By Lemma 1.4.5 ([1]), $A_0(x_\infty) = f_0$. Now, from (3.3) and the properties of A_i, it follows that

$$\begin{aligned}\sum_{i=1}^N \lambda_i\|A_i(x_k) - f_i\|^2 &\le \left\langle \sum_{i=1}^N (f_i^{\delta_k} - f_i) - \alpha_k^{1-\mu} J(x_k - x^+), x_k - x_\infty \right\rangle \\ &\quad + \frac{\delta_k}{\alpha_k^\mu}\|x_k - x_\infty\| \\ &\le \Big(N\delta_k + \alpha_k^{1-\mu}\|x_k - x^+\| + \frac{\alpha_k^{1-\mu}\delta_k}{\alpha_k}\Big)\|x_k - x_\infty\|.\end{aligned}$$

Therefore, $\|A_i(x_k) - f_i\| \to 0$ as $k \to \infty$, for $i = 1, \dots, N$. Again, by the above reason, $A_i(x_\infty) = f_i$. It means that $x_\infty \in S$, i.e., $x_\infty = x^0$ and all nets $\{x^\delta_{\alpha(\delta)}\}$ converge weakly to x^0, because S contains only one element x^0. This completes the proof. □

Theorem 4.4. *Assume that the following conditions hold*:

(i) A_0 *is continuously Fréchet differentiable and the tangential cone condition* (4.1) *is satisfied*;

(ii) *there exists an element* $\omega \in E$ *such that*

$$[A_0'(x_0)]^*\omega = J(x^0 - x^+),$$

where J *verifies condition* (4.2); *and*

(iii) *the regularization parameter* $\alpha(\delta)$ *is chosen by the residual principle* (4.12).

Then, for $0 < p < 1$, *we obtain the convergence rate result:*

$$\|x^\delta_{\alpha(\delta)} - x^0\| = O(\delta^\gamma) \ \text{ as } \delta \to 0, \quad \gamma = \min\left\{\frac{1-p}{s-1}, \frac{\mu p}{s}\right\}, \quad s \ge 2.$$

Proof. From (3.2), (3.3), (4.2) and the properties of A_i, it follows that

$$
\begin{aligned}
m_J\|x_\alpha^\delta - x^0\|^s &\leq \left\langle J(x^0 - x^+) - J(x_\alpha^\delta - x^+), x^0 - x_\alpha^\delta \right\rangle \\
&\leq \langle J(x^0 - x^+), x^0 - x_\alpha^\delta \rangle \\
&\quad + \frac{1}{\alpha}\left\langle A_0(x_\alpha^\delta) - A_0(x^0) + \alpha^\mu \sum_{i=1}^{N}[A_i(x_\alpha^\delta) - A_i(x^0)], x^0 - x_\alpha^\delta \right\rangle \qquad (4.19)\\
&\quad + \frac{1}{\alpha}\left\langle f_0 - f_0^\delta + \alpha^\mu \sum_{i=1}^{N}(f_i - f_i^\delta), x^0 - x_\alpha^\delta \right\rangle \\
&\leq \langle J(x^0 - x^+), x^0 - x_\alpha^\delta \rangle + \frac{\delta}{\alpha}(1 + N\alpha^\mu)\|x_\alpha^\delta - x^0\|.
\end{aligned}
$$

On the other hand, using (4.1), condition (ii) in the theorem and (3.3), we can write

$$
\begin{aligned}
&\left\langle J(x^0 - x^+), x^0 - x_\alpha^\delta \right\rangle \\
&= \langle \omega, A_0'(x^0)(x^0 - x_\alpha^\delta) \rangle \\
&\leq \|\omega\|(\tau + 1)\|A_0(x_\alpha^\delta) - A_0(x^0)\| \\
&\leq \|\omega\|(\tau + 1)\left[\|f_0^\delta - f_0\| + \alpha^\mu \sum_{i=1}^{N} \|f_i^\delta - A_i(x_\alpha^\delta)\| + \alpha\|x_\alpha^\delta - x^+\|\right],
\end{aligned}
$$

which together with (4.19) implies that

$$
\begin{aligned}
m_J\|x_\alpha^\delta - x^0\|^s &\leq \frac{\delta}{\alpha}(1 + N\alpha^\mu)\|x_\alpha^\delta - x^0\| \\
&\quad + \|\omega\|(\tau + 1)\left[\delta + \alpha^\mu \sum_{i=1}^{N} \|f_i^\delta - A_i(x_\alpha^\delta)\| + \alpha\|x_\alpha^\delta - x^+\|\right].
\end{aligned}
$$

Note that when α is chosen by (4.12), for sufficiently small δ, we have that $N\alpha^\mu(\delta) \leq 1$ and $\|x_{\alpha(\delta)}^\delta - x^0\| \leq \|x_{\alpha(\delta)}^\delta - x^+\|$. Therefore,

$$
\begin{aligned}
m_J\|x_{\alpha(\delta)}^\delta - x^0\|^s &\leq 2\frac{\delta}{\alpha(\delta)}\|x_{\alpha(\delta)}^\delta - x^0\| \\
&\quad + \|\omega\|(\tau + 1)\left[\delta + \alpha^\mu\left(N\delta + \sum_{i=1}^{N}\frac{1}{\lambda_i}\|x_{\alpha(\delta)}^\delta - x^+\|\right) + K\delta^p\right] \\
&\leq 2\frac{\delta}{\alpha(\delta)}\|x_{\alpha(\delta)}^\delta - x^0\| \\
&\quad + \|\omega\|(\tau + 1)\left[2\delta + (K\delta^p)^\mu c^{1-\mu}\sum_{i=1}^{N}\frac{1}{\lambda_i} + K\delta^p\right] \\
&\leq C_1\delta^{1-p}\|x_{\alpha(\delta)}^\delta - x^0\| + C_2\delta^{p\mu},
\end{aligned}
$$

where c is the right hand side of (4.8) with $c(\alpha,\delta)$ replaced by 2δ and C_1, C_2 are two positive constants. Using the implication:

$$a,\ b,\ c\ \geq\ 0,\ s\ >\ t,\ a^s \leq ba^t + c \Longrightarrow a^s = O(b^{s/(s-t)} + c),$$

we obtain that

$$\|x^\delta_{\alpha(\delta)} - x^0\| = O\big(\delta^\gamma\big).$$

This completes the proof. □

5. Finite-dimensional approximation

In computation, the finite-dimensional approximation for (3.3) is the important problem. As usually, it can be approximated by the following equation:

$$(5.1)\qquad A^n_0(x) + \alpha^\mu \sum_{i=1}^N (A^n_i(x) - f^{n\delta}_i) + \alpha J^n(x) = f^{n\delta}_0, \quad \alpha > 0,\ x \in E_n,$$

where $A^n_i = P^*_n A_i P_n$, $J^n = P^*_n J P_n$, $f^{n\delta}_i = P^*_n f^\delta_i$, $P_n : E \to E_n$ is a linear projection operator from E onto the finite dimensional subspace E_n of E, $P^*_n : E^* \to E^*_n$ is conjugate operator to P_n, and

$$E_n \subset E_{n+1}, \quad \forall n, \quad P_n x \to x, \quad \forall x \in E.$$

Without loss of generality, we suppose that $\|P_n\| = 1$. As also for (3.3), this equation has a unique solution $x^\delta_{\alpha,n}$ for all $\delta, \alpha > 0$ and n.

Theorem 5.1. *The sequence* $\{x^\delta_{\alpha,n}\}$ *of solutions of the equation* (5.1) *converges to a solution* x^δ_α *of* (3.3) *as* $n \to \infty$.

Proof. It follows from (5.1) that

$$(5.2)\quad \langle A^n_0(x^\delta_{\alpha,n}) - f^{n\delta}_0, x^\delta_{\alpha,n} - P_n x^\delta_\alpha\rangle + \alpha^\mu \sum_{i=1}^N \langle A^n_i(x^\delta_{\alpha,n}) - f^{n\delta}_i, x^\delta_{\alpha,n} - P_n x^\delta_\alpha\rangle + \alpha\langle J^n(x^\delta_{\alpha,n}), x^\delta_{\alpha,n} - P_n x^\delta_\alpha\rangle = 0.$$

By using (4.2), we have

$$\alpha m_J \|x^\delta_{\alpha,n} - P_n x^\delta_\alpha\|^s \leq \alpha\langle J(x^\delta_{\alpha,n}) - J(P_n x^\delta_\alpha), x^\delta_{\alpha,n} - P_n x^\delta_\alpha\rangle = \alpha\langle J^n(x^\delta_{\alpha,n}) - J^n(P_n x^\delta_\alpha), x^\delta_{\alpha,n} - P_n x^\delta_\alpha\rangle.$$

From (5.2), we have

$$(5.3)\quad \alpha m_J \|x^\delta_{\alpha,n} - P_n x^\delta_\alpha\|^s \leq \langle A^n_0(x^\delta_{\alpha,n}) - f^{n\delta}_0, P_n x^\delta_\alpha - x^\delta_{\alpha,n}\rangle + \alpha^\mu \sum_{i=1}^N \langle A^n_i(x^\delta_{\alpha,n}) - f^{n\delta}_i, P_n x^\delta_\alpha - x^\delta_{\alpha,n}\rangle + \alpha\langle J^n(P_n x^\delta_\alpha), P_n x^\delta_\alpha - x^\delta_{\alpha,n}\rangle.$$

Since $A^n_i = P^*_n A_i P_n$, $J^n = P^*_n J P_n$, $f^{n\delta}_i = P^*_n f^\delta_i$, it follows from (5.3) that

$$(5.4)\qquad \alpha m_J \|x^\delta_{\alpha,n} - P_n x^\delta_\alpha\|^s$$

$$\begin{aligned}&\leq \langle A_0(x^\delta_{\alpha,n}) - A_0(P_nx^\delta_\alpha) + A_0(P_nx^\delta_\alpha) - f^\delta_0, P_nx^\delta_\alpha - x^\delta_{\alpha,n}\rangle \\ &\quad + \alpha^\mu \sum_{i=1}^N \langle A_i(x^\delta_{\alpha,n}) - A_i(P_nx^\delta_\alpha) + A_i(P_nx^\delta_\alpha) - f^\delta_i, P_nx^\delta_\alpha - x^\delta_{\alpha,n}\rangle \\ &\quad + \alpha\langle J(P_nx^\delta_\alpha), P_nx^\delta_\alpha - x^\delta_{\alpha,n}\rangle.\end{aligned}$$

Using the monotonicity of A_i, it follows from (5.4) that

$$\begin{aligned}\alpha m_J\|x^\delta_{\alpha,n} - P_nx^\delta_\alpha\|^s &\leq \langle A_0(P_nx^\delta_\alpha) - f^\delta_0, P_nx^\delta_\alpha - x^\delta_{\alpha,n}\rangle \\ &\quad + \alpha^\mu \sum_{i=1}^N \langle A_i(P_nx^\delta_\alpha) - f^\delta_i, P_nx^\delta_\alpha - x^\delta_{\alpha,n}\rangle \\ &\quad + \alpha\langle J(P_nx^\delta_\alpha), P_nx^\delta_\alpha - x^\delta_{\alpha,n}\rangle.\end{aligned} \tag{5.5}$$

Which leads to the following inequality

$$\begin{aligned}&\alpha m_J\|x^\delta_{\alpha,n} - P_nx^\delta_\alpha\|^s \\ &\leq \left[\|A_0(P_nx^\delta_\alpha)\| + \|f^\delta_0\| + \alpha^\mu \sum_{i=1}^N \left(\|A_i(P_nx^\delta_\alpha)\| + \|f^\delta_i\|\right)\right]\|P_nx^\delta_\alpha - x^\delta_{\alpha,n}\| \\ &\quad + \alpha\|P_nx^\delta_\alpha\|\|P_nx^\delta_\alpha - x^\delta_{\alpha,n}\|.\end{aligned} \tag{5.6}$$

This implies that the sequence $\{x^\delta_{\alpha,n}\}$ is bounded. Without loss of generality, we suppose that $\{x^\delta_{\alpha,n}\}$ is convergent weakly to $\overline{x}^\delta_\alpha$. Since $A^n_i = P^*_nA_iP_n$, $J^n = P^*_nJP_n$, $f^{n\delta}_i = P^*_nf^\delta_i$, the monotonicity of A_i and J, it follows from (5.1) that

$$\langle A_0(x^n) + \alpha^\mu \sum_{i=1}^N (A_i(x^n) - f^\delta_i) + \alpha J(x^n) - f^\delta_0, x^n - x^\delta_{\alpha,n}\rangle \geq 0,$$

where $\alpha > 0$, $x^n = P_nx \in E_n$.

By letting $n \to \infty$ in this inequality, using the property of A_i, P_n and $x^\delta_{\alpha,n} \rightharpoonup \overline{x}^\delta_\alpha$, we have

$$\langle A_0(x) + \alpha^\mu \sum_{i=1}^N (A_i(x) - f^\delta_i) + \alpha J(x) - f^\delta_0, x - \bar{x}^\delta_\alpha\rangle \geq 0 \quad \forall x \in E.$$

Since, (3.3) has a unique solution, it follows that $\overline{x}^\delta_\alpha = x^\delta_\alpha$ and sequence $\{x^\delta_{\alpha,n}\}$ converges weakly to x^δ_α. From (5.5) deduce the sequence $\{x^\delta_{\alpha,n}\}$ converges strongly to x^δ_α as $n \to \infty$. □

Let

$$\gamma_n(z) = \|(I - P_n)(z)\|, \quad z \in S,$$

where I denotes the identity operator in E.

Theorem 5.2. *Let E, E^*, J, S, A_i, and f^δ_i $(i = 0, \dots, N)$ be as in Lemma 3.1. Support that E is an* **E***-space. If δ/α and $\gamma_n(z)/\alpha \to 0$ as $\alpha \to 0$ and $n \to \infty$, then the sequence $\{x^\delta_{\alpha,n}\}$ converges to $x^0 \in S$.*

Proof. For $z \in S$, $z^n = P_n z$, it follows from (5.1) that

$$\langle A_0^n(x_{\alpha,n}^\delta) - f_0^{n\delta}, x_{\alpha,n}^\delta - z^n\rangle + \alpha^\mu \Big\langle \sum_{i=1}^N (A_i^n(x_{\alpha,n}^\delta) - f_i^{n\delta}), x_{\alpha,n}^\delta - z^n \Big\rangle + \alpha\langle J^n(x_{\alpha,n}^\delta), x_{\alpha,n}^\delta - z^n\rangle = 0, \tag{5.7}$$

where $x_{\alpha,n}^\delta$ is a solution of (5.1). It follows from (5.7), $P_nP_n = P_n$, $A_i^n = P_n^* A_i P_n$, $f_i^{n\delta} = P_n^* f_i^\delta$, $J^n = P_n^* J P_n$, and the monotonicity of A_i that

$$\begin{aligned}
&\alpha\langle J(x_{\alpha,n}^\delta), x_{\alpha,n}^\delta - z^n\rangle \\
&= \alpha\langle J^n(x_{\alpha,n}^\delta), x_{\alpha,n}^\delta - z^n\rangle = \langle A_0^n(x_{\alpha,n}^\delta) - f_0^{n\delta}, z^n - x_{\alpha,n}^\delta\rangle \\
&\quad + \alpha^\mu \Big\langle \sum_{i=1}^N (A_i^n(x_{\alpha,n}^\delta) - f_i^{n\delta}), z^n - x_{\alpha,n}^\delta \Big\rangle \\
&\le \langle A_0(z^n) - f_0^\delta, z^n - x_{\alpha,n}^\delta\rangle + \alpha^\mu \Big\langle \sum_{i=1}^N (A_i(z^n) - f_i^\delta), z^n - x_{\alpha,n}^\delta \Big\rangle.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\alpha\langle J(x_{\alpha,n}^\delta), x_{\alpha,n}^\delta - z^n\rangle &\le \langle A_0(z^n) - A_0(z) + f_0 - f_0^\delta, z^n - x_{\alpha,n}^\delta\rangle \\
&\quad + \alpha^\mu \Big\langle \sum_{i=1}^N (A_i(z^n) - A_i(z) + f_i - f_i^\delta), z^n - x_{\alpha,n}^\delta \Big\rangle \\
&\le \Big[\|A_0(z^n) - A_0(z)\| + \|f_0 - f_0^\delta\|\Big]\|z^n - x_{\alpha,n}^\delta\| \qquad (5.8)\\
&\quad + \alpha^\mu \sum_{i=1}^N \Big[\|A_i(z^n) - A_i(z)\| + \|f_i - f_i^\delta\|\Big]\|z^n - x_{\alpha,n}^\delta\|.
\end{aligned}$$

On the other hand, by using (3.2) and

$$\|A_i(z^n) - A_i(z)\| \le \bar{K}\gamma_n(z), \tag{5.9}$$

where $\bar{K}$ is some positive constant depending only on z, it follows from (5.1) that

$$\langle J(x_{\alpha,n}^\delta), x_{\alpha,n}^\delta - z^n\rangle \le \frac{\delta + \bar{K}\gamma_n(z)}{\alpha}(1 + N\alpha^\mu)\|z^n - x_{\alpha,n}^\delta\|.$$

Hence, we have

$$\langle J(x_{\alpha,n}^\delta), x_{\alpha,n}^\delta\rangle - \langle J(x_{\alpha,n}^\delta), z^n\rangle \le \frac{\delta + \bar{K}\gamma_n(z)}{\alpha}(1 + N\alpha^\mu)(\|x_{\alpha,n}^\delta\| + \|z^n\|). \tag{5.10}$$

Thus, we have

$$\|x_{\alpha,n}^\delta\|^2 - \|x_{\alpha,n}^\delta\|\Big[\|z\| + \frac{\bar{c}(\delta,\alpha)}{\alpha}\Big] - \frac{\bar{c}(\delta,\alpha)}{\alpha}\|z\| \le 0,$$

where $\bar{c}(\delta,\alpha) = (\delta + K\gamma_n(z))(1 + N\alpha^\mu)$. Consequently, we have

$$\|x^\delta_{\alpha,n}\| \le \frac{1}{2}\left\{\|z\| + \frac{\bar{c}(\delta,\alpha)}{\alpha} + \sqrt{\left(\|z\| + \frac{\bar{c}(\delta,\alpha)}{\alpha}\right)^2 + \frac{4\bar{c}(\delta,\alpha)}{\alpha}\|z\|}\right\} \tag{5.11}$$
$$\le \|z\| + \frac{\bar{c}(\delta,\alpha)}{\alpha} + \sqrt{\frac{\bar{c}(\delta,\alpha)}{\alpha}\|z\|}.$$

Since δ/α, $\gamma_n(z)/\alpha \to 0$ as $\alpha \to 0$ and $n \to \infty$, it means that $\{x^\delta_{\alpha,n}\}$ is bounded. Since E is reflexive, there exists a subsequence of $\{x^\delta_{\alpha,n}\}$, that converges weakly to some element $\overline{x} \in E$. For the sake of simplicity, assume that $x^\delta_{\alpha,n} \rightharpoonup \overline{x}$ as $\alpha \to 0$ and $n \to \infty$. First, we prove that $\overline{x} \in S_0$. Indeed, by virtue of $A^n_i = P^*_n A_i P_n$, $J^n = P^*_n J P_n$, $f^{n\delta}_i = P^*_n f^\delta_i$, the monotonicity of A_i and J, it follows from (5.1) that

$$\begin{aligned}
&\langle A^n_0(P_n x) - f^{n\delta}_0, P_n x - x^\delta_{\alpha,n}\rangle \\
&= \langle A^n_0(P_n x) - A^n_0(x^\delta_{\alpha,n}) + A^n_0(x^\delta_{\alpha,n}) - f^{n\delta}_0, P_n x - x^\delta_{\alpha,n}\rangle \\
&\ge \langle A_0(x^\delta_{\alpha,n}) - f^\delta_0, P_n x - x^\delta_{\alpha,n}\rangle \\
&= \alpha^\mu \sum_{i=1}^N \langle A_i(x^\delta_{\alpha,n}) - f^\delta_i, x^\delta_{\alpha,n} - P_n x\rangle + \alpha\langle J(x^\delta_{\alpha,n}), x^\delta_{\alpha,n} - P_n x\rangle \\
&\ge \alpha^\mu \sum_{i=1}^N \langle A_i(P_n x) - f^\delta_i, x^\delta_{\alpha,n} - P_n x\rangle + \alpha\langle J(P_n x), x^\delta_{\alpha,n} - P_n x\rangle, \quad \forall x \in E.
\end{aligned} \tag{5.12}$$

Since $P_n P_n = P_n$, the last inequality has form

$$\begin{aligned}
\langle A_0(P_n x) - f^\delta_0, P_n x - x^\delta_{\alpha,n}\rangle \ge \alpha^\mu \sum_{i=1}^N &\langle A_i(P_n x) - f^\delta_i, x^\delta_{\alpha,n} - P_n x\rangle \\
&+ \alpha\langle J(P_n x), x^\delta_{\alpha,n} - P_n x\rangle, \quad \forall x \in E.
\end{aligned} \tag{5.13}$$

After tending δ, $\alpha \to 0$, and $n \to \infty$ in this inequality, we obtain

$$\langle A_0(x) - f_0, x - \overline{x}\rangle \ge 0, \quad \forall x \in E.$$

Thus, $\overline{x} \in S_0$ (see [38]). Now, we shall prove that $\overline{x} \in S_i$, $i = 1, 2, \dots, N$. Again, from (5.1), the monotonicity of A_i, J, (3.2), and (5.9), we have

$$\begin{aligned}
&\sum_{i=1}^N \langle A_i(x^\delta_{\alpha,n}) - A_i(P_n z), x^\delta_{\alpha,n} - P_n z\rangle \\
&= \sum_{i=1}^N \langle A^n_i(x^\delta_{\alpha,n}) - A^n_i(P_n z), x^\delta_{\alpha,n} - P_n z\rangle \\
&= \sum_{i=1}^N \langle A^n_i(x^\delta_{\alpha,n}) - f^{n\delta}_i + f^{n\delta}_i - A^n_i(P_n z), x^\delta_{\alpha,n} - P_n z\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \langle f_i^{n\delta} - A_i^n(P_n z), x_{\alpha,n}^{\delta} - P_n z\rangle + \alpha^{1-\mu}\langle J^n(x_{\alpha,n}^{\delta}), P_n z - x_{\alpha,n}^{\delta}\rangle \\
&\quad + \frac{1}{\alpha^{\mu}}\langle f_0^{n\delta} - A_0^n(x_{\alpha,n}^{\delta}), x_{\alpha,n}^{\delta} - P_n z\rangle \\
&\le \sum_{i=1}^{N} \langle f_i^{\delta} - A_i(P_n z) + A_i(z) - f_i, x_{\alpha,n}^{\delta} - P_n z\rangle + \alpha^{1-\mu}\langle J(P_n z), P_n z - x_{\alpha,n}^{\delta}\rangle \\
&\quad + \frac{1}{\alpha^{\mu}}\langle f_0^{\delta} - A_0(P_n z) + A_0(z) - f_0, x_{\alpha,n}^{\delta} - P_n z\rangle \\
&\le \left[\sum_{i=1}^{N} \|f_i^{\delta} - f_i\| + \sum_{i=1}^{N} \|A_i(z) - A_i(P_n z)\|\right]\|x_{\alpha,n}^{\delta} - P_n z\| \\
&\quad + \alpha^{1-\mu}\langle J(P_n z), P_n z - x_{\alpha,n}^{\delta}\rangle \\
&\quad + \frac{1}{\alpha^{\mu}}\left[\|f_0^{\delta} - f_0\| + \|A_0(z) - A_0(P_n z)\|\right]\|x_{\alpha,n}^{\delta} - P_n z\| \\
&\le \frac{1}{\alpha^{\mu}}\left[\delta + \delta N\alpha^{\mu} + \bar{K}\gamma_n(z) + \bar{K}\gamma_n(z)N\alpha^{\mu}\right]\|x_{\alpha,n}^{\delta} - P_n z\| \\
&\quad + \alpha^{1-\mu}\langle J(P_n z), P_n z - x_{\alpha,n}^{\delta}\rangle, \quad \forall z \in S.
\end{aligned}
$$

Which together with the λ_i-inverse strongly monotone property of A_i implies

$$
\begin{aligned}
&\sum_{i=1}^{N} \lambda_i \|A_i(x_{\alpha,n}^{\delta}) - A_i(P_n z)\|^2 \\
&\le \sum_{i=1}^{N} \langle A_i(x_{\alpha,n}^{\delta}) - A_i(P_n z), x_{\alpha,n}^{\delta} - P_n z\rangle \\
&\le \left[\frac{\delta}{\alpha}\alpha^{1-\mu}(1 + N\alpha^{\mu}) + \frac{\gamma_n(z)}{\alpha}\alpha^{1-\mu}(\bar{K} + N\bar{K}\alpha^{\mu})\right]\|x_{\alpha,n}^{\delta} - P_n z\| \\
&\quad + \alpha^{1-\mu}\|P_n(z)\|\|P_n z - x_{\alpha,n}^{\delta}\|, \quad \forall z \in S.
\end{aligned}
$$

Thus, $\|A_i(x_{\alpha,n}^{\delta}) - A_i(z)\| \to 0$ as δ, $\alpha \to 0$, $n \to \infty$ with $\delta/\alpha \to 0$, and $\gamma_n(z)/\alpha \to 0$. Note that, each mapping A_i is maximal monotone (see [8], Theorem 1.3, p. 40). As we know that (see [1], Lemma 1.4.5, p. 39), the graph $G(A)$ of any maximal monotone mapping A from a reflexive Banach space E to E^* is demiclosed, that is, $x_n \to x$, $y_n \rightharpoonup f$ or $x_n \rightharpoonup x$, $y_n \to f$, where $(x_n, y_n) \in G(A)$, imply that $(x, f) \in G(A)$. Thus, $A_i(\bar{x}) = f_i$, $i = 1, 2, \dots, N$, that is, $\bar{x} \in S_i$. Next, since each S_i is closed convex, S is also closed convex. Therefore, the element x^0 in S with minimal norm in the strictly convex Banach space E is unique. And now, from (5.11) with z replaced by $\bar{x}$, it implies that $\|x_{\alpha,n}^{\delta}\| \to \|\bar{x}\|$ and $\|\bar{x}\| \le \|z\|$, for all $z \in S$. Hence, $x_{\alpha,n}^{\delta} \to \overline{x}$ (because E is an **E**-space), which is the element x^0, that we have to find. □

Theorem 5.3. *Assume that the following conditions hold*:

(i) A_0 *is continuously Fréchet differentiable with* (4.1) *for* $x = x^0$*, and the other each* A_i *is* L_i*-Lipschitz continuous in some neighbourhood of* x^0;
(ii) *there exists an element* $\omega \in E$ *such that*

$$[A_0'(x^0)]^*\omega = J(x^0),$$

where J *satisfies condition* (4.2);
(iii) *the parameter* $\alpha = \alpha(\delta)$ *is chosen by* $\alpha \sim (\delta + \gamma_n)^\nu$, $0 < \nu < 1$, *where* $\gamma_n = \max_{x\in S} \gamma_n(x)$.

Then

$$\|x^\delta_{\alpha,n} - x^0\| = O((\delta + \gamma_n)^h + \gamma_n^l),$$

where

$$h = \min\left\{\frac{1-\nu}{s-1}, \frac{\mu\nu}{s}\right\}, \quad l = \min\left\{\frac{1}{s}, \frac{\nu}{s-1}\right\}, \ s \geq 2.$$

Proof. Replacing $P_n x^\delta_\alpha$ by $x^0_n = P_n x^0$ in (5.5), we obtain

$$\begin{aligned}(5.14)\qquad & \alpha m_J \|x^\delta_{\alpha,n} - x^0_n\|^s \\ & \leq \langle A_0(x^0_n) - f^\delta_0, x^0_n - x^\delta_{\alpha,n}\rangle + \alpha^\mu \sum_{i=1}^N \langle A_i(x^0_n) - f^\delta_i, x^0_n - x^\delta_{\alpha,n}\rangle \\ & \quad + \alpha\langle J(x^0_n), x^0_n - x^\delta_{\alpha,n}\rangle.\end{aligned}$$

We have

$$\begin{aligned}(5.15)\qquad \langle A_0(x^0_n) - f^\delta_0, x^0_n - x^\delta_{\alpha,n}\rangle & \leq \Big(\|A_0(x^0_n) - A_0(x^0)\| + \delta\Big)\|x^0_n - x^\delta_{\alpha,n}\| \\ & \leq (\widetilde{C_0}\gamma_n + \delta)\|x^0_n - x^\delta_{\alpha,n}\|,\end{aligned}$$

where $\widetilde{C_0}$ is a positive constant depending only on x^0. And also, we have

$$\begin{aligned}(5.16)\qquad & \sum_{i=1}^N \langle A_i(x^0_n) - f^\delta_i, x^0_n - x^\delta_{\alpha,n}\rangle \\ & \leq \Big[\sum_{i=1}^N \Big(\|A_i(x^0_n) - A_i(x^0)\| + \delta\Big)\Big]\|x^0_n - x^\delta_{\alpha,n}\| \\ & \leq \Big(\sum_{i=1}^N \widetilde{C_i}\gamma_n + N\delta\Big)\|x^0_n - x^\delta_{\alpha,n}\| \\ & \leq \Big(\widetilde{C}\gamma_n + N\delta\Big)\|x^0_n - x^\delta_{\alpha,n}\|,\end{aligned}$$

where $\widetilde{C_i}$ is a positive constant depending only on x^0 and $\widetilde{C} = \sum_{i=1}^N \widetilde{C_i}$, and

$$(5.17)\qquad \langle J(x^0_n) - J(x^0), x^0_n - x^\delta_{\alpha,n}\rangle \leq C(\widetilde{R})\gamma_n^\nu \|x^0_n - x^\delta_{\alpha,n}\|, \ 0 < \nu < 1,$$

where $\widetilde{R} > \|x^0\|$ and

$$(5.18)\qquad \langle J(x^0), x^0_n - x^\delta_{\alpha,n}\rangle = \langle J(x^0), x^0_n - x^0\rangle + \langle J(x^0), x^0 - x^\delta_{\alpha,n}\rangle$$

$$
\begin{aligned}
&= \langle J(x^0), x_n^0 - x^0\rangle + \langle \omega, A_0'(x^0)(x^0 - x_{\alpha,n}^\delta)\rangle \\
&\le \|J(x^0)\|\|x_n^0 - x^0\| + \|\omega\|\|A_0'(x^0)(x^0 - x_{\alpha,n}^\delta)\| \\
&\le \|x^0\|\|(I-P_n)x^0\| + \|\omega\|\|A_0'(x^0)(x^0 - x_{\alpha,n}^\delta)\| \\
&\le \widetilde{R}\gamma_n + \|\omega\|(\tau+1)\|A_0(x_{\alpha,n}^\delta) - f_0\|,
\end{aligned}
$$

and

$$
\begin{aligned}
(5.19)\qquad &\|A_0(x_{\alpha,n}^\delta) - f_0\| \\
&\le \delta + \|A_0(x_{\alpha,n}^\delta) - f_0^\delta\| \\
&\le \delta + \alpha^\mu \sum_{i=1}^N \|A_i(x_{\alpha,n}^\delta) - f_i^\delta\| + \alpha\|x_{\alpha,n}^\delta\| \\
&\le \delta + \alpha^\mu \sum_{i=1}^N \Big(\|A_i(x_{\alpha,n}^\delta) - A_i(x^0)\| + \delta\Big) + \alpha\|x_{\alpha,n}^\delta\| \\
&\le \delta + \alpha^\mu \sum_{i=1}^N \Big(\|A_i(x_{\alpha,n}^\delta) - A_i(x_n^0)\| + \|A_i(x_n^0) - A_i(x^0)\| + \delta\Big) \\
&\qquad + \alpha\|x_{\alpha,n}^\delta\| \\
&\le \delta + \alpha^\mu \sum_{i=1}^N L_i\|x_{\alpha,n}^\delta - x_n^0\| + \alpha^\mu\widetilde{C}\gamma_n + \alpha^\mu N\delta + \alpha\|x_{\alpha,n}^\delta\|.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
(5.20)\qquad &\alpha m_J\|x_{\alpha,n}^\delta - x_n^0\|^s \\
&\le (\widetilde{C_0}\gamma_n + \delta)\|x_n^0 - x_{\alpha,n}^\delta\| + \alpha^\mu\Big(\widetilde{C}\gamma_n + N\delta\Big)\|x_n^0 - x_{\alpha,n}^\delta\| \\
&\quad + \alpha C(\widetilde{R})\gamma_n^\nu\|x_n^0 - x_{\alpha,n}^\delta\| + \alpha\widetilde{R}\gamma_n \\
&\quad + \alpha\|\omega\|(\tau+1)\Big[(1+\alpha^\mu N)\delta + \alpha^\mu\sum_{i=1}^N L_i\|x_{\alpha,n}^\delta - x_n^0\| + \alpha^\mu\widetilde{C}\gamma_n + \alpha\|x_{\alpha,n}^\delta\|\Big] \\
&\le \Big[\widetilde{C_0}\gamma_n + \delta + \alpha^\mu(\widetilde{C}\gamma_n + N\delta) + \alpha C(\widetilde{R})\gamma_n^\nu + \alpha^{\mu+1}\|\omega\|(\tau+1)\sum_{i=1}^N L_i\Big] \\
&\quad \times \|x_n^0 - x_{\alpha,n}^\delta\| + \alpha\widetilde{R}\gamma_n + \alpha\|\omega\|(\tau+1)\delta(1+N\alpha^\mu) \\
&\quad + \alpha^{\mu+1}\|\omega\|(\tau+1)\widetilde{C}\gamma_n + \alpha^2\|\omega\|(\tau+1)\|x_{\alpha,n}^\delta\|.
\end{aligned}
$$

The last inequality implies that

$$
\begin{aligned}
(5.21)\qquad &m_J\|x_{\alpha,n}^\delta - x_n^0\|^s \\
&\le \Big[\frac{\widetilde{C_0}\gamma_n + \delta}{\alpha} + \alpha^\mu\frac{\widetilde{C}\gamma_n + N\delta}{\alpha} + C(\widetilde{R})\gamma_n^\nu + \alpha^\mu\|\omega\|(\tau+1)\sum_{i=1}^N L_i\Big]\|x_n^0 - x_{\alpha,n}^\delta\|
\end{aligned}
$$

$$+ \widetilde{R}\gamma_n + \|\omega\|(\tau+1)\delta(1+N\alpha^\mu)$$
$$+ \alpha^\mu\|\omega\|(\tau+1)\widetilde{C}\gamma_n + \alpha\|\omega\|(\tau+1)\|x^\delta_{\alpha,n}\|.$$

If α is chosen by condition (iii), then $\alpha \le 1$, from (5.21) we have

$$\|x^\delta_{\alpha,n} - x^0_n\|^s \le \Big[C_1(\gamma_n+\delta)^{1-\nu} + C_2(\gamma_n+\delta)^{\mu\nu} + C_3\gamma_n^\nu\Big]\|x^0_n - x^\delta_{\alpha,n}\| + C_4\gamma_n + C_5(\gamma_n+\delta)^{\mu\nu}, \tag{5.22}$$

where C_i, $i = 1,\dots,5$ are some positive constants. Using the implication

$$a,b,c \ge 0, \quad s > t, \quad a^s \le ba^t + c \Rightarrow a^s = \mathrm{O}(b^{s/(s-t)} + c),$$

we obtain

$$\|x^\delta_{\alpha,n} - x^0_n\| = \mathrm{O}((\delta+\gamma_n)^h + \gamma_n^l).$$

Thus

$$\|x^\delta_{\alpha,n} - x^0\| = \mathrm{O}((\delta+\gamma_n)^h + \gamma_n^l). \qquad \square$$

6. Iterative regularization method of zero order

Now we consider the following iterative regularization method of zero order, where z_{m+1} is defined by [10]

(6.1)
$$z_{m+1} = z_m - \beta_m\Big[(A_0(z_m) - f_0) + \alpha_m^\mu\sum_{i=1}^N(A_i(z_m) - f_i) + \alpha_m^{\mu+1}(z_m - x^+)\Big], \ z_0 \in H,$$

where H is a real Hilbert space, $\{\alpha_m\}$ and $\{\beta_m\}$ are sequences of positive numbers.

We have the following results and prove them by similar methods to Theorems 2.4 and 2.5 in [34].

Theorem 6.1. *Assume that the conditions in Lemma 3.1 are satisfied. Then we have the following statements.*

(1) *For each $\alpha_m > 0$, problem*

$$A_0(x) + \alpha_m^\mu\sum_{i=1}^N(A_i(x) - f_i) + \alpha_m^{\mu+1}(x - x^+) = f_0 \tag{6.2}$$

has a unique solution x_m.

(2) *If $0 < \alpha_m \le 1$, $\alpha_m \to 0$ as $m \to \infty$, then $\lim_{m\to\infty} x_m = x^0 \in S$ with the minimum norm x^+.*

Theorem 6.2. *Assume that $\{\alpha_m\}$ and $\{\beta_m\}$ in the problem* (6.1) *satisfy the following conditions*:

(i) $1 \ge \alpha_m \searrow 0$, $\beta_m \to 0$ *as* $m \to +\infty$;

(ii) $\lim_{m\to+\infty} \dfrac{|\alpha_{m+1} - \alpha_m|}{\beta_m\alpha_m^{2(\mu+1)}} = 0$, $\lim_{m\to+\infty}\dfrac{\beta_m}{\alpha_m^{\mu+1}} = 0$;

(iii) $\sum_{m=0}^\infty \beta_m\alpha_m^{\mu+1} = +\infty$.

Then, $\lim_{m\to+\infty} z_m = x^0 \in S$ *with the minimum norm* x^+.

Remark 6.1. There are 2 advantages of methods (6.1) in comparison with the other regularization algorithms listed below:

(i) Weaker assumption on operators A_i, $i = 0, 1, \ldots, N$.

To be more specific, iterative regularization method of zero order in [10] was considered for solving system of equations with $A_0, A_1, \ldots, A_N$ were monotone potential operators. Iterative regularization method of zero order in [34] was considered for solving system of equations with $A_0, A_1, \ldots, A_N$ were inverse-strongly monotone operators. Iterative regularization method of zero order in (6.1) was considered for solving system of equations with A_0 is hemicontinuous monotone operator and $A_1, \ldots, A_N$ were inverse-strongly monotone operators.

(ii) Faster rate of convergence (see Examples 6.3 and 6.4 for more details).

Now we give two examples of numerical expressions to illustrate the effectiveness of the proposed methods. We consider the problem: find an element $x^0 \in H$ such that

$$\varphi_j(x^0) = \min_{x\in H} \varphi_j(x), \quad i = 0, 1, \ldots, N, \tag{6.3}$$

where φ_j is weakly lower semi-continuous proper convex function in a real Hilbert space H.

Example 6.3. We consider the case, when the function $\varphi_j(x)$ is defined by

$$\varphi_j(x) = \frac{1}{2}\langle A_j x, x\rangle.$$

Then x^0 is a solution to the problem (6.3) if and only if $x^0 \in S$ with $A_j x = \varphi_j'(x)$ where $A_j = B_j^T B_j$ is an $M\times M$ matrix, $B_j = (b^j_{lk})^M_{l,k=1}$ is determined as follows

$$\begin{cases} b^j_{1k} = \sin(j+1),\ j = 0,1,2,\ k = 1,2,\ldots,M, \\ b^j_{2k} = 2\sin(j+1),\ j = 0,1,2,\ k = 1,2,\ldots,M, \\ b^j_{lk} = \cos((j+1)l)\sin((j+1)k),\ j = 0,1,2,\ l = 3,\ldots,M,\ k = 1,2,\ldots,M. \end{cases} \tag{6.4}$$

We have $x^0 = (0,\ldots,0)^T \in \mathbb{R}^M$ is a solution of (6.3). Since $\det(A_j) = 0$, $j = 0,1,2$, each equation in $A_j(x) = 0$ is ill-posed. Consequently, the problem (6.3) in this case is ill-posed too. We apply method (2.11) in [34] with $\alpha_m = (1+m)^{-1/12}$, $\beta_m = (1+m)^{-1/2}$, $z_0 = (1,\ldots,1)^T \in \mathbb{R}^M$ and our method (6.1) with the same starting point z_0 and the same value of parameters α_m, β_m with choosing $\mu = 1/2$, $M = 50$, we obtain the following table and figure of numerical results:

From Table 6.1 and Table 6.2, we can see that the iterative regularization method (6.1) converges faster than the iterative regularization method (2.11) in [34].

TABLE 6.1. Numerical of method (6.1)

m	err_1	$\|x_0 - z_m^1\|$
10	1.3639×10^{-3}	1.8174×10^{-4}
20	6.0425×10^{-5}	8.0625×10^{-6}
50	2.2479×10^{-7}	3.0007×10^{-8}
100	7.0664×10^{-10}	9.4203×10^{-11}
122	9.9058×10^{-11}	1.3198×10^{-11}

TABLE 6.2. Numerical of method (2.11) in [34]

m	err_2	$\|x_0 - z_m^2\|$
10	7.6449×10^{-3}	1.0171×10^{-3}
20	7.1677×10^{-4}	9.4292×10^{-5}
50	1.2291×10^{-5}	1.5681×10^{-6}
100	2.2234×10^{-7}	2.6792×10^{-8}
122	5.8411×10^{-8}	6.8293×10^{-9}

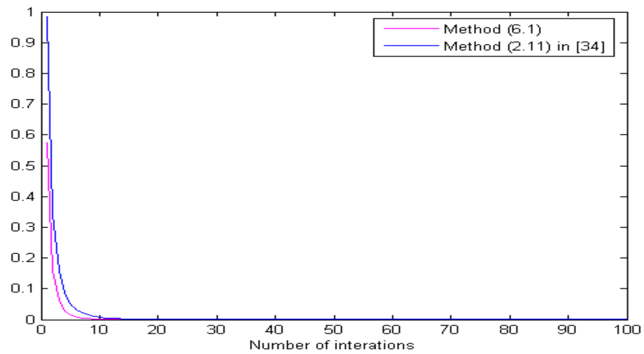

FIGURE 6.1. Numerical of methods (6.1) and (2.11) in [34]

Example 6.4. We consider the case, when the function $\varphi_j : L_2[0,1] \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi_j(x) = F\left(\frac{1}{2}\langle B_j(x), x\rangle\right), \quad j = 0, 1, 2,$$

with $F : \mathbb{R} \to \mathbb{R}$ is chosen as follows

$$F(t) = \begin{cases} 0, & t \le a_0, \\ c(t - a_0), & t > a_0, \end{cases}$$

where c, a_0 are the positive constants, and

$$B_j : L^2[0,1] \to L^2[0,1]$$

are defined by

$$B_j x(t) = \int_0^1 k_j(t,s)x(s)ds, \quad j = 0,1,2,$$

with

$$k_0(t,s) = \begin{cases} s(1-t), & s \le t, \\ t(1-s), & s > t, \end{cases}$$

$$k_1(t,s) = \begin{cases} \dfrac{(1-s)^2 st^2}{2} - \dfrac{(1-s)^2 t^3(1+2s)}{6} + \dfrac{(t-s)^3}{6}, & t \ge s, \\ \dfrac{s^2(1-s)(1-t)^2}{2} + \dfrac{s^2(1-t^3)(2s-3)}{6} + \dfrac{(s-t)^3}{6}, & t < s, \end{cases}$$

$$k_2(t,s) = ts$$

are kernel functionals defined on the square $\{0 \le t, s \le 1\}$. We can approximate $\varphi_j(x)$ by the following function

$$\varphi_{j\varepsilon}(x) = F_\varepsilon\left(\frac{1}{2}\langle B_j(x), x\rangle\right),\ j = 0,1,2,$$

with $F_\varepsilon : \mathbb{R} \to \mathbb{R}$ is chosen as follows

$$F_\varepsilon(t) = \begin{cases} 0, & t \le a_0, \\ \dfrac{c(t-a_0)^2}{2\varepsilon}, & a_0 < t \le a_0 + \varepsilon, \\ c(t - a_0 - \dfrac{\varepsilon}{2}), & t > a_0 + \varepsilon. \end{cases}$$

Then, $\overline{x}$ of (6.3) is a solution of operator equation

$$\varphi'_{j\varepsilon}(x) = f_j,$$

where $f_j = \theta \in L^2[0,1]$, and

$$\varphi'_{j\varepsilon}(x) = F'_\varepsilon\left(\frac{1}{2}\langle B_j(x), x\rangle\right) B_j(x)$$

are monotone operators from $L^2[0,1]$ to $L^2[0,1]$, with

$$F'_\varepsilon(t) = \begin{cases} 0, & t \le a_0, \\ \dfrac{c(t-a_0)}{\varepsilon}, & a_0 < t \le a_0 + \varepsilon, \\ c & t > a_0 + \varepsilon. \end{cases}$$

We compute the regularized solutions $x^\delta_{\alpha,n}$ by approximating $H = L^2[0,1]$ by the sequence of the linear subspaces H_n, which is a set of all linear combinations of $\{\phi_1, \phi_1, \dots, \phi_n\}$ defined on uniform grid of $n+1$ points in $[0,1]$

$$\phi_k(t) = \begin{cases} 0, & t \notin (t_{k-1}, t_k], \\ 1, & t \in (t_{k-1}, t_k], \end{cases}$$

and

$$P_n(x) = \sum_{k=1}^{n} x(t_k)\phi_k(t).$$

We have $\|P_n\| = 1$ and $\|(I - P_n)x\| = O(n^{-1})$, for all $x \in L^2[0,1]$. Then, the finite-dimensional regularized equation (5.1) is of the form

$$\widetilde{A}_0^n(\widetilde{x}) + \alpha^\mu \left(\widetilde{A}_1^n(\widetilde{x}) + \widetilde{A}_2^n(\widetilde{x}) - f_1^{n\delta} - f_2^{n\delta} \right) + \alpha\widetilde{x} = f_0^{n\delta}, \quad \alpha > 0, \quad \widetilde{x} \in H_n,$$

with

$$\widetilde{A}_j^n(\widetilde{x}) = \varphi'_{j\varepsilon}(\widetilde{x}) = F'_\varepsilon\left(\frac{1}{2}\langle \widetilde{B}_j(\widetilde{x}), \widetilde{x}\rangle\right)\widetilde{B}_j(\widetilde{x}),$$

where

$$\widetilde{A}_j^n(\widetilde{x}) = \left(\varphi'_{j\varepsilon}(\widetilde{x}_1), \varphi'_{j\varepsilon}(\widetilde{x}_2), \ldots, \varphi'_{j\varepsilon}(\widetilde{x}_n)\right)^T,$$

$$\widetilde{B}_j = \left(lk_j(t_u, t_v)\right)_{u,v=1}^{n}, \quad l = \frac{1}{n},$$

$$f_j^{n\delta} = (\delta, \delta, \ldots, \delta)^T,$$

$$\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T, \quad \widetilde{x}_k = x(t_k), \quad k = 1, 2, \ldots, n.$$

Taking account of the iterative method (2.11) in [34] with $z_0 = (1.5, \ldots, 1.5)^T \in \mathbb{R}^n$, $\alpha_m = (1+m)^{-1/16}$, $\beta_m = (1+m)^{-1/2}$ and our method (6.1) with the same starting point z_0 and the same value of parameters α_m, β_m with choosing $\mu = 1/2$, $n = 50$, $\delta = 10^{-10}$, we obtain the following table of numerical results:

Table 6.3. Numerical of method (6.1)

m	err_1	$\|x_0 - z_m^1\|$
10	3.3993×10^{-2}	1.0652×10^{-1}
20	2.4607×10^{-3}	1.254×10^{-2}
50	2.737×10^{-6}	2.5522×10^{-5}
100	3.1152×10^{-8}	4.5182×10^{-7}
200	1.0017×10^{-10}	2.6778×10^{-9}

Table 6.4. Numerical of method (2.11) in [34]

m	err_2	$\|x_0 - z_m^2\|$
10	5.759×10^{-2}	2.4046×10^{-1}
20	6.7532×10^{-3}	4.8016×10^{-2}
50	2.2141×10^{-5}	3.0834×10^{-3}
100	9.4185×10^{-7}	2.1547×10^{-5}
200	2.1407×10^{-8}	7.9956×10^{-7}

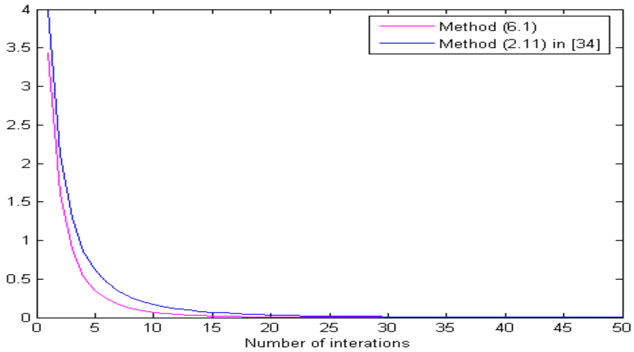

FIGURE 6.2. Numerical of methods (6.1) and (2.11) in [34]

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