J. Korean Math. Soc. **55** (2018), No. 4, pp. 809–832
https://doi.org/10.4134/JKMS.j170464
pISSN: 0304-9914 / eISSN: 2234-3008

THE KÜNNETH SPECTRAL SEQUENCE FOR COMPLEXES OF BANACH SPACES

HeeSook Park

Abstract. In this paper, we form the basis of the abstract theory for constructing the Künneth spectral sequence for a complex of Banach spaces. As the category of Banach spaces is not abelian, several difficulties occur and hinder us from applying the usual method of homological algebra directly. The most notable facts are the image of a morphism of Banach spaces is not necessarily a Banach space, and also the closed summand of a Banach space need not be a topological direct summand. So, we consider some conditions and categorical terms that fit the category of Banach spaces to modify the familiar method of homological algebra.

1. Introduction

Motivated by Noskov's construction [10] of the Hochschild-Serre spectral sequence for bounded cohomology of a discrete group G with real coefficients $\mathbb{R}$, the first attempt for this paper is to construct the Künneth spectral sequence for bounded cohomology.

Before proceeding, we define the following terms in advance for a simple description.

Definition 1.1. Let $f : U \to V$ be a bounded linear map of Banach spaces. We define the terms Ker f and Im f as

$$\text{Ker } f = \{u \in U \mid f(u) = 0\} \text{ and } \text{Im } f = f(U) = \{f(u) \mid u \in U\}.$$

As Ker $f = f^{-1}(0)$, Ker f is a closed subspace of U and so it is a Banach space. However, Im f may not be a closed subspace of V and so it may not be a Banach space.

Now, we recall briefly the definition of bounded cohomology $\widehat{H}^*(G)$ of a discrete group G with coefficients in $\mathbb{R}$. For each $n > 0$, we consider the space $B^n(G)$ and a boundary operator $d^n : B^n(G) \to B^{n+1}(G)$, where

Received July 10, 2017; Revised November 2, 2017; Accepted January 30, 2018.
2010 *Mathematics Subject Classification.* Primary 55T99; Secondary 18G40.
Key words and phrases. Künneth spectral sequence, Banach complex.
This paper was supported by Sunchon National University Research Fund in 2016.

(1) G^n is the n-product of G so that $G^n = \underbrace{G \times \cdots \times G}_{n}$;

(2) $B^n(G)$ is the space of all bounded functions $f : G^n \to \mathbb{R}$ with the norm $\|f\| = \sup\{|f(x)| \mid x \in G^n\}$;

(3) $d^n : B^n(G) \to B^{n+1}(G)$ is defined by the formula

$$\begin{aligned} & d^n(f)(g_1, g_2, \ldots, g_{n+1}) \\ &= f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}g_i, \ldots, g_{n+1}) + f(g_1, \ldots, g_n). \end{aligned}$$

It is easy to check that every d_n is a bounded linear map with $d^{n+1} \circ d^n = 0$. Notice that $B^n(G)$ has a Banach space structure with respect to this norm $\|\cdot\|$. So

$$0 \xrightarrow{d^{-1}=0} \mathbb{R} \xrightarrow{d^0=0} B^1(G) \xrightarrow{d^1} B^2(G) \xrightarrow{d^2} \cdots$$

is a complex of Banach spaces and its nth cohomology is defined as $\widehat{H}^n(G) = \text{Ker } d^n / \text{Im } d^{n-1}$. As Ker d^n is a closed subspace of the normed space $B^n(G)$, $\widehat{H}^n(G)$ is the quotient space of a normed space. However, as the space of coboundaries Im d^{n-1} may not be closed; $\widehat{H}^n(G)$ has the natural seminorm induced by the norm on $B^n(G)$ and it may not have a Banach space structure.

From this point of view, Mitsumatsu [9] defined its reduced cohomology as

$$\overline{\widehat{H^n}(G)} = \text{Ker } d^n / \overline{\text{Im } d^{n-1}},$$

where $\overline{\text{Im } d^{n-1}}$ denotes the norm closure of Im d^{n-1}. Then this reduced cohomology $\overline{\widehat{H^n}(G)}$ has a norm and so a Banach space structure.

In general, let $\mathbf{V} = \{\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots\}$ be a complex of Banach spaces, where V^* are Banach spaces and boundary operators d^* are bounded linear maps.

As usual, its nth cohomology $H^n(\mathbf{V})$ is defined as

$$H^n(\mathbf{V}) = \text{Ker } d^n / \text{Im } d^{n-1}.$$

In this case, $H^n(\mathbf{V})$ has the natural seminorm induced by a norm on V^n and so has a topological vector space structure rather than a Banach space structure. On the other hand, its reduced cohomology defined as

$$\overline{H^n(\mathbf{V})} = \text{Ker } d^n / \overline{\text{Im } d^{n-1}}$$

is a normed space and so a Banach space. Notice that, in both $H^n(\mathbf{V})$ and $\overline{H^n(\mathbf{V})}$ cases, a boundary operator $d^n : V^n \to V^{n+1}$ does not provide a familiar short exact sequence of Banach spaces. For example, neither the sequence from $H^n(\mathbf{V})$

$$0 \to \text{Ker } d^n \to V^n \to \text{Im } d^n \to 0$$

nor the sequence from $\overline{H^n(\mathbf{V})}$

$$0 \to \text{Ker } d^n \to V^n \to \overline{\text{Im } d^n} \to 0$$

can be exact in the category of Banach spaces, unless Im d^n is closed. Furthermore, let

$$0 \to \mathbf{A} \xrightarrow{\varphi} \mathbf{B} \xrightarrow{\psi} \mathbf{C} \to 0 \tag{1.1}$$

be a short exact sequence of complexes of Banach spaces. As shown in [5], there is an exact sequence

$$\cdots \to H^{n-1}(\mathbf{C}) \xrightarrow{\delta^{n-1}} H^n(\mathbf{A}) \xrightarrow{H^n\varphi} H^n(\mathbf{B}) \xrightarrow{H^n\psi} H^n(\mathbf{C}) \to \cdots \tag{1.2}$$

of cohomology as topological vector spaces, and a semiexact sequence

$$\cdots \to \overline{H^{n-1}}(\mathbf{C}) \xrightarrow{\overline{\delta^{n-1}}} \overline{H^n}(\mathbf{A}) \xrightarrow{\overline{H^n}\varphi} \overline{H^n}(\mathbf{B}) \xrightarrow{\overline{H^n}\psi} \overline{H^n}(\mathbf{C}) \to \cdots \tag{1.3}$$

of reduced cohomology as Banach spaces. Let d_A^*, d_B^*, and d_C^* denote the boundary operators of the complexes $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, respectively. The conditions for the images of $d_\bullet^*$ to be closed affect the exactness of the sequences (1.2) and (1.3). If the images of all these boundary operators $d_\bullet^*$ are closed, then the sequences (1.2) and (1.3) are the same. In fact, from Theorem 2 in [5] we have the followings:

Theorem 1.2. *From the exact sequence* (1.1) *of complexes of Banach spaces, the followings hold*;

(1) *If the image of d_A^n is closed, then the image of d_C^{n-1} is closed if and only if the subspace* $\mathrm{Im} H^n\varphi$ *is closed in* $H^n(\boldsymbol{B})$;
(2) *If the image of d_B^{n-1} is closed, then the image of d_A^{n-1} is closed if and only if the subspace* $\mathrm{Im} H^n\psi$ *is closed in* $H^n(\boldsymbol{C})$;
(3) *If the image of d_C^{n-1} is closed, then the image of d_B^{n-1} is closed if and only if the subspace* $\mathrm{Im}\delta^{n-1}$ *is closed in* $H^n(\boldsymbol{A})$.

Now we return to our main subject: the Künneth spectral sequence. We recall the following theorem in [7] as the ordinary case of abelian category which we will modify.

Theorem 1.3. *Let (K^*, d_K) and (L^*, d_L) be differential graded modules over a ring R with K^* flat. Then there is a spectral sequence with*

$$E_2^{p,q} = \bigoplus_{s+t=q} Tor_R^p \left(H^s(K^*), H^t(L^*)\right).$$

If K^* and L^* have differentials of degree $+1$, then this is a second quadrant spectral sequence. When $E_r^{*,*}$ converges, it does so to $H(K^* \otimes_R L^*, d_\otimes)$.

If we consider only the vector space structure of a complex of Banach spaces, then we have the same result as Theorem 1.3. But, as explained above, considering the Banach space structure of it, the ordinary method of homological algebra will not work properly on a complex of Banach spaces and so on bounded cohomology. Hence, we must consider some properties of the category of Banach spaces. For example, we need to check several categorical terms such as exact sequences of Banach spaces, tensor product, projective resolutions for Tor functor, flat Banach spaces, and so forth. In the following section, we briefly study these properties. Based on them, in Section 3, we construct the Künneth type spectral sequence for a complex of Banach spaces first in a general form and then, as an example, apply it to bounded cohomology.

Throughout this paper, we consider only Banach spaces over the field of real numbers $\mathbb{R}$.

2. Basic properties in the category of Banach spaces

In this section, we study some terms and properties of Banach spaces that are needed for our construction of the Künneth spectral sequence. Although we must explain these terms in the category of Banach spaces, to avoid some categorical difficulties, we only collect the necessary properties for our purposes here and refer all details to [1], [2], and [3].

Definition 2.1. Let U and V be Banach spaces. A morphism $f : U \to V$ is a linear map of the underlying vector spaces which is continuous with respect to the topologies defined by the norms.

Notice that a morphism of Banach spaces is a bounded linear map. It is easy to check that the class of Banach spaces with morphisms forms a category. We denote this category by $\mathcal{B}an$.

Remark 2.2. $\mathcal{B}an$ is an additive category that has *finite* (co)products. However, infinite (co)products do not exist in $\mathcal{B}an$ as explained in [2].

In the following, we review the categorical definitions of the objects relating to a morphism in $\mathcal{B}an$.

Remark 2.3. Let $f : U \to V$ be a morphism in $\mathcal{B}an$.

(1) The kernel of f is identified with $\mathrm{Ker}\ f$.
(2) The cokernel of f is identified with $V/\overline{f(U)}$ and is denoted by $\mathrm{Cok}\ f$.
(3) The image of f is defined as the kernel of its cokernel. It is identified with $\overline{f(U)}$.
(4) The coimage of f is defined as the cokernel of its kernel. It is identified with $U/\mathrm{Ker}\ f$.

Notice that our definition of $\mathrm{Im}\ f$ is not an object in $\mathcal{B}an$. In fact, the categorical definition of the image of $f : U \to V$ in $\mathcal{B}an$ is $\overline{f(U)}$. Since the coimage and image of f are not isomorphic as Banach spaces, $\mathcal{B}an$ is not an abelian category.

Remark 2.4. Let $f : U \to V$ be a surjective morphism in $\mathcal{B}an$.

(1) The quotient space $U/\mathrm{Ker}\ f$ and V are isomorphic as Banach spaces by the Open Mapping Theorem. Thus V and the cokernel of the inclusion morphism $i : \mathrm{Ker}\ f \to U$ are isomorphic as Banach spaces.

(2) f is a cokernel, that is, there is a morphism $g : W \to U$ such that $U/\overline{g(W)}$ and V are isomorphic as Banach spaces. Conversely, if a morphism in $\mathcal{B}an$ is a cokernel, then it is surjective. Thus, in $\mathcal{B}an$, a morphism is a cokernel if and only if it is surjective.

For Banach spaces U with a norm $\|\cdot\|_U$ and V with a norm $\|\cdot\|_V$, we consider their algebraic tensor product $U \otimes V$ and define the *projective tensor norm* $\|\cdot\|_\pi$ on $U \otimes V$ as follows: for $\omega \in U \otimes V$,

$$\|\omega\|_\pi = \inf \left\{ \sum_{i=1}^{n} \|u_i\|_U \|v_i\|_V, \text{ where } \omega = \sum_{i=1}^{n} u_i \otimes v_i \right\},$$

where the infimum is taken over all representations of $\omega \in U \otimes V$.

Definition 2.5. The projective tensor product of Banach spaces U and V is defined as the completion of $U \otimes V$ with respect to the projective tensor norm $\|\cdot\|_\pi$. It is denoted by $U \widehat{\otimes} V$.

Notice that $w \in U \widehat{\otimes} V$ if and only if, for every $\epsilon > 0$, there are $u_i \in U$ and $v_i \in V$ such that

$$\omega = \sum_{i=1}^{\infty} u_i \otimes v_i \text{ and } \|\omega\|_\pi \leq \sum_{i=1}^{\infty} \|u_i\|_U \|v_i\|_V \leq \|\omega\|_\pi + \epsilon$$

and also $\sum_{i=1}^{\infty} \|u_i\|_U \|v_i\|_V < \infty$.

As we know it, the tensor product $\otimes$ in the category of vector spaces is a right exact functor and so it preserves epimorphisms. Recall that, in general, a morphism $f : A \to B$ in a category is called an epimorphism if $g \circ f = 0$ for every morphism $g : B \to C$ implies $g = 0$. Also, in the category of vector spaces, a morphism is an epimorphism if and only if it is surjective.

Proposition 2.6. *A morphism $f : U \to V$ of Banach spaces is an epimorphism if and only if Im f is dense in V, that is $\overline{f(U)} = V$.*

Proof. Suppose $\overline{\text{Im } f} = \overline{f(U)} = Y$ and $g \circ f = 0$ for a morphism $g : V \to W$. Then $g(f(U)) = 0$ and so, by continuity of g, $g\left(\overline{f(U)}\right) = g(V) = 0$. Hence $g = 0$ and so f is an epimorphism.

Conversely, let f be an epimorphism and consider a projection $p : V \to V/\overline{f(U)}$. Then $p \circ f = 0$ and so $p = 0$. Hence $V = \overline{f(U)}$. □

Remark 2.7. We state some elementary properties of the projective tensor product on Banach spaces. We refer to [2] for proofs and in details.

Let U, V, and W be Banach spaces and $\text{Hom}_{\mathcal{B}an}(U, V)$ denote the space of all morphisms $f : U \to V$.

(1) $U \widehat{\otimes} V$ is a Banach space.
(2) $\widehat{\otimes}$ is symmetric, associative, and additive.
(3) $U \widehat{\otimes} \mathbb{R} = \mathbb{R} \widehat{\otimes} U = U$.

(4) Let $f \in \mathrm{Hom}_{\mathcal{B}an}(U, V)$ and $g \in \mathrm{Hom}_{\mathcal{B}an}(U_1, V_1)$. Then the linear map $f \otimes g : U \otimes U_1 \to V \otimes V_1$ extends to a morphism $f \widehat{\otimes} g : U \widehat{\otimes} U_1 \to V \widehat{\otimes} V_1$. Furthermore, suppose $f_1 \in \mathrm{Hom}_{\mathcal{B}an}(V, W)$ and $g_1 \in \mathrm{Hom}_{\mathcal{B}an}(V_1, W_1)$. Then
$$\left(f_1 \widehat{\otimes} g_1\right) \circ \left(f \widehat{\otimes} g\right) = (f_1 \circ f) \widehat{\otimes} (g_1 \circ g).$$
(5) $\mathrm{Hom}_{\mathcal{B}an}(U \widehat{\otimes} V, W) = \mathrm{Hom}_{\mathcal{B}an}(U, \mathrm{Hom}_{\mathcal{B}an}(V, W))$.
(6) $\widehat{\otimes}$ preserves colimits.
(7) For an epimorphism $f : U \to V$, $f \widehat{\otimes} 1_W : U \widehat{\otimes} W \to V \widehat{\otimes} W$ is an epimorphism.
(8) Let A be a closed subspace of U. If $f : U \to U/A$ is the quotient map, then $1_V \widehat{\otimes} f : V \widehat{\otimes} U \to V \widehat{\otimes} (U/A)$ is also a quotient map. The kernel of $1_V \widehat{\otimes} f$ is the norm closure of $V \otimes A$ in $V \widehat{\otimes} U$.

Proposition 2.8. *Let $f : U \to V$ be a surjective morphism of Banach spaces and X be a Banach space. Then $1_X \widehat{\otimes} f : X \widehat{\otimes} U \to X \widehat{\otimes} V$ is surjective. In particular, the projective tensor product preserves cokernels.*

Proof. Since f is surjective, V and $U/\mathrm{Ker}\ f$ are topologically isomorphic. So we can write $f : U \to U/\mathrm{Ker}\ f$ as the quotient map. Then
$$1_X \widehat{\otimes} f : X \widehat{\otimes} U \to X \widehat{\otimes} (U/\mathrm{Ker}\ f)$$
is also a quotient map by Remark 2.7 and so surjective. Since $X \widehat{\otimes} (U/\mathrm{Ker}\ f)$ and $X \widehat{\otimes} V$ are isomorphic as Banach spaces, $1_X \widehat{\otimes} f$ is also surjective.

Recall that a morphism is surjective if and only if it is a cokernel by Remark 2.4. Hence the second statement follows. □

Definition 2.9. A sequence of Banach spaces and morphisms
$$\cdots \to U_{n+1} \xrightarrow{\partial_{n+1}} U_n \xrightarrow{\partial_n} U_{n-1} \to \cdots \tag{2.9.1}$$
is said to be exact if $\mathrm{Ker}\ f = \mathrm{Im}\ f$.

Proposition 2.10. *Let*
$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$
be a short exact sequence of Banach spaces. Then there are isomorphisms
$$\mathrm{Ker}\ g \cong U \quad \text{and} \quad \mathrm{Cok}\ f \cong W$$
as Banach spaces. In particular, $\mathrm{Im}\ f$ *is a closed subspace of* V.

Proof. Notice that f is injective. Since $\mathrm{Im}\ f = \mathrm{Ker}\ g$, $\mathrm{Im}\ f$ is a closed subspace of V. Hence U and $\mathrm{Im}\ f$ are topologically isomorphic. So U and $\mathrm{Ker}\ g$ are topologically isomorphic. Also, notice that g is surjective. Hence $\mathrm{Cok}\ f$ and W are topologically isomorphic from Remark 2.4. □

Recall that the dual space X^* of a Banach space X is the normed space of all bounded linear functional on X with the operator norm and is a Banach space. Notice that $X^* = \mathrm{Hom}_{\mathcal{B}an}(X, \mathbb{R})$.

Corollary 2.11. *A short sequence of Banach spaces*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is exact if and only if the sequence of Banach spaces

$$0 \to W^* \xrightarrow{g^*} V^* \xrightarrow{f^*} U^* \to 0$$

is exact.

Proof. By the duality principle in [1], the dual space functor preserves and reflects kernel-cokernel exact sequence. By Proposition 2.10, our exact sequence of Banach spaces are kernel-cokernel exact sequence. □

Definition 2.12. A Banach space X is said to be flat if, for every short exact sequence of Banach spaces

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0, \tag{2.12.1}$$

the sequence of Banach spaces

$$0 \to X\widehat{\otimes}U \xrightarrow{1_X\widehat{\otimes}f} X\widehat{\otimes}V \xrightarrow{1_X\widehat{\otimes}g} X\widehat{\otimes}W \to 0 \tag{2.12.2}$$

is exact.

Corollary 2.13. *Consider the exact sequences of Banach spaces* (2.12.1) *and* (2.12.2) *in Definition* 2.12. *Then there are isomorphisms of Banach spaces*:

(1) $\mathrm{Im}\,(1_X\widehat{\otimes}f) \cong X\widehat{\otimes}\mathrm{Im}\, f$;
(2) $\mathrm{Ker}\,(1_X\widehat{\otimes}g) \cong X\widehat{\otimes}\mathrm{Ker}\, g$;
(3) $\mathrm{Cok}\,(1_X\widehat{\otimes}f) \cong X\widehat{\otimes}\mathrm{Cok}\, f$.

Proof. Since the sequences (2.12.1) and (2.12.2) are exact, Im f and Im $(1_X\widehat{\otimes}f)$ are closed by Proposition 2.10. So f and $1_X\widehat{\otimes}f$ are topologically injective. This shows the first two isomorphisms. For the third isomorphism, recall that $W \cong \mathrm{Cok}\, f$ and $X\widehat{\otimes}W \cong \mathrm{Cok}\,(1_X\widehat{\otimes}f)$ by Proposition 2.10. □

Corollary 2.14. *Let*

$$\boldsymbol{U}: \cdots \to U^{n-1} \xrightarrow{d^{n-1}} U^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots$$

be a complex of Banach spaces such that Im d^n is closed for every n. Suppose X is a flat Banach space. Then there are isomorphisms of Banach spaces:

(1) $\mathrm{Im}\,(1_X\widehat{\otimes}d^n) \cong X\widehat{\otimes}\mathrm{Im}\, d^n$;
(2) $\mathrm{Ker}\,(1_X\widehat{\otimes}d^n) \cong X\widehat{\otimes}\mathrm{Ker}\, d^n$;
(3) $\mathrm{Ker}\,(1_X\widehat{\otimes}d^n)/\mathrm{Im}\,(1_X\widehat{\otimes}d^{n-1}) \cong X\widehat{\otimes}H^n(\boldsymbol{U})$.

Proof. Since every image of d^n is closed, we have the following exact sequences of Banach spaces

$$0 \to \mathrm{Ker}\, d^n \xrightarrow{i^n} U^n \xrightarrow{d^n} \mathrm{Im}\, d^n \to 0; \tag{2.14.1}$$

$$0 \to \mathrm{Im}\, d^{n-1} \xrightarrow{j^n} \mathrm{Ker}\, d^n \xrightarrow{p^n} H^n(\mathbf{U}) \to 0. \tag{2.14.2}$$

Since X is flat, there are induced exact sequences of Banach spaces

$$0 \to X\widehat{\otimes}\text{Ker } d^n \xrightarrow{1_X\widehat{\otimes}i^n} X\widehat{\otimes}U^n \xrightarrow{1_X\widehat{\otimes}d^n} X\widehat{\otimes}\text{Im } d^n \to 0; \tag{2.14.3}$$

$$0 \to X\widehat{\otimes}\text{Im } d^{n-1} \xrightarrow{1_X\widehat{\otimes}j^n} X\widehat{\otimes}\text{Ker } d^n \xrightarrow{1_X\widehat{\otimes}p^n} X\widehat{\otimes}H^n(\mathbf{U}) \to 0. \tag{2.14.4}$$

From Corollary 2.13, we get the first two topological isomorphisms directly. For the third one, recall that $X\widehat{\otimes}\text{Im } d^{n-1}$ and $\text{Im }(1_X\widehat{\otimes}j^n)$ are topologically isomorphic. Then

$$\begin{aligned} X\widehat{\otimes}H^n(\mathbf{U}) &\cong \text{Cok }(1_X\widehat{\otimes}j^n) \\ &\cong X\widehat{\otimes}\text{Ker } d^n/X\widehat{\otimes}\text{Im } d^{n-1} \cong \text{Ker }(1_X\widehat{\otimes}d^n)/\text{Im }(1_X\widehat{\otimes}d^{n-1}). \end{aligned}$$ □

Definition 2.15. A short exact sequence of Banach spaces

$$0 \to U \to V \to W \to 0$$

is called pure if its dual exact sequence

$$0 \to W^* \to V^* \to U^* \to 0$$

is split.

Proposition 2.16. *Let*

$$0 \to X \to Y \to Z \to 0 \tag{2.16.1}$$

be a short exact sequence of Banach spaces.

(1) *Assume Z is flat. Then X is flat if and only if Y is flat.*
(2) *Assume Y is flat. Then the sequence* (2.16.1) *is pure if and only if Z is flat.*
(3) *Assume the sequence* (2.16.1) *is pure. Then, for every Banach space V, the sequence*

$$0 \to X\widehat{\otimes}V \to Y\widehat{\otimes}V \to Z\widehat{\otimes}V \to 0$$

is exact.

Proof. This is Corollary 2.5.3 and Proposition 2.5.4 in [1]. □

Notice that if the sequences (2.14.1) and (2.14.2) in Corollary 2.14 are pure, then for every Banach space X, we have the same exact sequences and results as the sequences (2.14.3) and (2.14.4).

As the Künneth type spectral sequence involves the Tor functor, we need to construct a projective resolution and a Tor functor in $\mathcal{B}an$.

Remark 2.17. By definition, an object P in a category $\mathcal{C}$ is projective if for any epimorphism $f : X \to Y$ and any morphism $g : P \to Y$, there is a morphism $\bar{f} : P \to X$ such that $f \circ \bar{f} = g$. In $\mathcal{B}an$, consider a strict inclusion morphism $\beta\colon X \hookrightarrow Y$ such that β is an epimorphism. Then a morphism $\gamma : \mathbb{R} \to Y$ such that $\gamma(1) = y \in Y \setminus X$ can not be lifted to X. So the ground field $\mathbb{R}$ is not categorically projective. Recall that a ground field $\mathbb{R}$ is a direct summand of every nonzero Banach space. Hence only the zero object is projective in $\mathcal{B}an$.

Definition 2.18. A Banach space U is called projective if, for a short exact sequence of Banach spaces $0 \to X \to Y \to Z \to 0$, the induced sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}an}(U,X) \to \mathrm{Hom}_{\mathcal{B}an}(U,Y) \to \mathrm{Hom}_{\mathcal{B}an}(U,Z) \to 0$$

is exact in **Ab**: the category of abelian groups.

Notice that a projective Banach space is not a projective object in $\mathcal{B}an$. Especially, the ground field $\mathbb{R}$ is projective in the sense of Definition 2.18.

Remark 2.19. We state some basic properties of projective Banach spaces.

(1) Let S be an index set. Then $\ell^1_S(\mathbb{R}) = \coprod_{s\in S} \mathbb{R}_s$, where $\mathbb{R}_s \equiv \mathbb{R}$.
(2) Every Banach space U can be written as a quotient of an ℓ^1-space $\ell^1_S(\mathbb{R})$ for some index set S.
(3) Every $\ell^1_S(\mathbb{R})$ for some index set S is a projective Banach space.
(4) There are enough projectives in the category of Banach spaces.
(5) A Banach space P is projective if and only if P is isomorphic to ℓ^1_S for some index set S.

For the proof for Remark 2.19, we refer to [1] and [2].

Proposition 2.20. *Every projective Banach space is flat.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of Banach spaces and U be a projective Banach space. Recall that by the duality principle, the dual sequence of Banach spaces

$$0 \to Z^* \to Y^* \to X^* \to 0$$

is also exact. Since U is projective, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}an}(U,Z^*) \to \mathrm{Hom}_{\mathcal{B}an}(U,Y^*) \to \mathrm{Hom}_{\mathcal{B}an}(U,X^*) \to 0 \tag{2.20.1}$$

is exact. Recall that, by Corollary 2.11, the sequence

$$0 \to U\widehat{\otimes}X \to U\widehat{\otimes}Y \to U\widehat{\otimes}Z \to 0 \tag{2.20.2}$$

is exact if and only if the sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}an}(U\widehat{\otimes}Z,\mathbb{R}) \to \mathrm{Hom}_{\mathcal{B}an}(U\widehat{\otimes}Y,\mathbb{R}) \to \mathrm{Hom}_{\mathcal{B}an}(U\widehat{\otimes}X,\mathbb{R}) \to 0 \tag{2.20.3}$$

is exact. Notice that the sequences (2.20.1) and (2.20.3) are equivalent by Remark 2.7. Thus the sequence (2.20.3) is exact. Hence the sequence (2.20.2) is exact and U is flat. ☐

Now we construct a projective resolution of a Banach space.

Definition 2.21. A resolution of a Banach space U is a sequence of Banach spaces and morphisms

$$\cdots \xrightarrow{\partial_4} P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} U \xrightarrow{\partial_{-1}=0} 0 \tag{2.21.1}$$

which is exact.

This resolution (2.21.1) is called projective if every P_n is projective for $n \geq 0$. Also we say this resolution is of finite length if there is an integer n such that $P_k = 0$ for $k > n$.

As a resolution (2.21.1) of a Banach space U is exact, every image of ∂_n is closed for $n \geq 0$. Hence, for each $n \geq 0$, there is an induced short exact sequence of Banach spaces of the form

$$0 \to \text{Ker } \partial_n \to P_n \to \text{Im } \partial_n \to 0.$$

Proposition 2.22. *Let U be a Banach space. Then there is a projective resolution of U.*

Proof. Since a Banach space U is a quotient of a projective Banach space by Remark 2.19, there is a projective Banach space P_0 and a quotient map $\epsilon_0 : P_0 \to U$. Since ϵ_0 is surjective, $P_0/\text{Ker } \epsilon_0$ and U are isomorphic as Banach spaces. So we have a short exact sequence of Banach spaces

$$0 \to \text{Ker } \epsilon_0 \xrightarrow{i_0} P_0 \xrightarrow{\epsilon_0} \text{Im } \epsilon_0 \to 0,$$

where i_0 is an inclusion map and so a morphism in $\mathcal{B}an$. Similarly, since Ker ϵ_0 is a Banach space, there is a projective Banach space P_1 and a quotient map $\epsilon_1 : P_1 \to \text{Ker } \epsilon_0$. So there is a short exact sequence of Banach spaces

$$0 \to \text{Ker } \epsilon_1 \xrightarrow{i_1} P_1 \xrightarrow{\epsilon_1} \text{Ker } \epsilon_0 \to 0.$$

By induction, for every $n \geq 1$, there is an exact sequence of Banach spaces

$$0 \to \text{Ker } \epsilon_n \xrightarrow{i_n} P_n \xrightarrow{\epsilon_n} \text{Ker } \epsilon_{n-1} \to 0,$$

where P_n is projective. We define $\partial_n : P_n \to P_{n-1}$ by $\partial_n = i_{n-1} \circ \epsilon_n$ for every $n \geq 1$. It is clear that each ∂_n is a bounded linear map and so a morphism of Banach spaces. Then there is a sequence of morphisms

$$\cdots \to P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon_0} U \to 0$$

in which each P_n is projective. It is clear ϵ_0 is surjective and

$$\text{Im } \partial_1 = \partial_1(P_1) = i_0(\epsilon_1(P_1)) = i_0(\text{Ker } \epsilon_0) = \text{Ker } \epsilon_0.$$

Also, since i_n is injective and ϵ_n is surjective, for each $n \geq 1$, we have

$$\text{Im } \partial_n = \partial_n(P_n) = i_{n-1}(\epsilon_n(P_n)) = \epsilon_n(P_n) = \text{Ker } \partial_{n-1}.$$

Hence this sequence is exact. □

In the category of Banach spaces, a projective resolution of a Banach space U is not unique. As similar to vector spaces, the Comparison Lemma holds in $\mathcal{B}an$ and so any two of them are chain homotopically equivalent as shown in [3].

Let U and V be Banach spaces. For a projective resolution of a Banach space V

$$\cdots \to P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} V \to 0, \tag{2.1}$$

we have a complex of Banach spaces

$$\cdots \to P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0=0} 0.$$

Then the induced sequence

$$\cdots \xrightarrow{1_U\widehat{\otimes}\partial_3} U\widehat{\otimes}P_2 \xrightarrow{1_U\widehat{\otimes}\partial_2} U\widehat{\otimes}P_1 \xrightarrow{1_U\widehat{\otimes}\partial_1} U\widehat{\otimes}P_0 \xrightarrow{1_U\widehat{\otimes}\partial_0=0} 0 \tag{2.2}$$

is a complex of Banach spaces by Remark 2.7.

Definition 2.23. Let U and V be Banach spaces. From the complex of Banach spaces (2.2), we define

$$\mathrm{Tor}_n(U,V) = \mathrm{Ker}\ (1_U\widehat{\otimes}\partial_n)/\mathrm{Im}\ (1_U\widehat{\otimes}\partial_{n+1}).$$

Notice that, since projective resolutions of U are chain homotopically equivalent, Tor is independent of a choice of a projective resolution. Also, $\mathrm{Tor}_n(U,V)$ may not be a Banach space.

Theorem 2.24. *Let U and V be Banach space and U be flat. Then*

(1) *$Tor_0(U,V) = U\widehat{\otimes}V$,*

(2) *$Tor_n(U,V) = 0$ for $n > 0$.*

Proof. Construct the projective resolution (2.1) of V and the induced complex (2.2) obtained by applying the projective tensor product $U\widehat{\otimes}\cdot$.

Let $n \geq 1$. Recall that every image of boundary operator ∂_n in the sequence (2.1) is closed and so a Banach space. So we have an exact sequence of Banach spaces

$$0 \to \mathrm{Ker}\ \partial_n \xrightarrow{i_n} P_n \xrightarrow{\partial_n} \mathrm{Im}\ \partial_n \to 0. \tag{2.24.1}$$

Since U is flat, for $n \geq 1$, the induced sequence of Banach spaces

$$0 \to U\widehat{\otimes}\mathrm{Ker}\ \partial_n \xrightarrow{1_U\widehat{\otimes}i_n} U\widehat{\otimes}P_n \xrightarrow{1_U\widehat{\otimes}\partial_n} U\widehat{\otimes}\mathrm{Im}\ \partial_n \to 0 \tag{2.24.2}$$

is also exact. Hence, by Corollary 2.13, we have

$$\mathrm{Im}\ (1_U\widehat{\otimes}\partial_{n+1}) \cong U\widehat{\otimes}\mathrm{Im}\ (\partial_{n+1}) \quad \text{and}$$
$$\mathrm{Ker}\ (1_U\widehat{\otimes}\partial_n) \cong U\widehat{\otimes}\mathrm{Ker}\ (\partial_n) = U\widehat{\otimes}\mathrm{Im}\ (\partial_{n+1}).$$

This shows that, for $n \geq 1$,

$$\begin{aligned}\mathrm{Tor}_n(U,V) &= \mathrm{Ker}\ (1_U\widehat{\otimes}\partial_n)/\mathrm{Im}\ (1_U\widehat{\otimes}\partial_{n+1})\\ &\cong \big(U\widehat{\otimes}\mathrm{Im}\ \partial_{n+1}\big) \,/\, \big(U\widehat{\otimes}\mathrm{Im}\ \partial_{n+1}\big) = 0.\end{aligned}$$

Now we compute $\mathrm{Tor}_0(U,V)$. Notice that $\mathrm{Ker}\ (1_U\widehat{\otimes}\partial_0) = U\widehat{\otimes}P_0$ from the sequence (2.2). Also, from the sequence (2.1), the morphism ϵ is surjective and $\mathrm{Ker}\ \epsilon = \mathrm{Im}\ \partial_1$. So, from the sequence (2.24.2), we have

$$\mathrm{Im}\ (1_U\widehat{\otimes}\partial_1) \cong U\widehat{\otimes}\mathrm{Im}\ \partial_1 = U\widehat{\otimes}\mathrm{Ker}\ \epsilon.$$

Since U is flat and the sequence of Banach spaces

$$0 \to \mathrm{Ker}\ \epsilon \xrightarrow{i_n} P_0 \xrightarrow{\epsilon} V \to 0$$

is exact, the induced sequence of Banach spaces

$$0 \to U\widehat{\otimes}\mathrm{Ker}\ \epsilon \xrightarrow{1_U\widehat{\otimes}i_n} U\widehat{\otimes}P_0 \xrightarrow{1_U\widehat{\otimes}\epsilon} U\widehat{\otimes}V \to 0$$

is also exact. Hence

$$\begin{aligned}
\mathrm{Tor}_0(U,V) &= \mathrm{Ker}\ (1_U\widehat{\otimes}\partial_0)/\mathrm{Im}\ (1_U\widehat{\otimes}\partial_1)\\
&\cong \left(U\widehat{\otimes}P_0\right)/U\widehat{\otimes}\mathrm{Ker}\ \epsilon\\
&\cong \left(U\widehat{\otimes}P_0\right)/\mathrm{Ker}\ (1_U\widehat{\otimes}\epsilon)\\
&\cong U\widehat{\otimes}V.
\end{aligned}$$

□

3. The Künneth spectral sequence for a complex of Banach spaces

In this section, we frame the Künneth spectral sequence for a complex of Banach spaces by modifying the Theorem 1.3 in [7] which we stated in the Introduction.

Since there exist only finite (co)products in $\mathcal{B}an$, we need some finiteness conditions on complexes. In addition, as our complexes of Banach spaces have differentials of degree $+1$, we deal with a second quadrant spectral sequence as cohomology and so we have to be careful about the convergence.

Definition 3.1. Let $\mathbf{V} = \{\cdots \to V^n \xrightarrow{\partial^n} V^{n+1} \to \cdots\}$ be a complex of Banach spaces. A projective resolution of $\mathbf{V}$ is a sequence

$$\cdots \to \mathbf{P}^{-2} \to \mathbf{P}^{-1} \to \mathbf{P}^{0} \to \mathbf{V} \to 0$$

such that the following commutative diagram (3.1) of projective Banach spaces $P^{*,*}$ whose columns $P^{p,*}$ are complexes and whose rows are exact:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & P^{-2,2} \arrow[u] \arrow[r] & P^{-1,2} \arrow[u] \arrow[r] & P^{0,2} \arrow[u] \arrow[r] & V^{2} \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & P^{-2,1} \arrow[u] \arrow[r] & P^{-1,1} \arrow[u] \arrow[r] & P^{0,1} \arrow[u] \arrow[r] & V^{1} \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & P^{-2,0} \arrow[u, "\lambda_{-2}"] \arrow[r, "\delta_P"] & P^{-1,0} \arrow[u, "\lambda_{-1}"] \arrow[r, "\delta_P"] & P^{0,0} \arrow[u, "\lambda_0"] \arrow[r, "\epsilon"] & V^{0} \arrow[u, "\partial_0"] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

This projective resolution (3.1) of $\mathbf{V}$ is called proper if the following sequences are projective resolutions: for every n,

$$\text{(P1):}\qquad \cdots \to P^{-2,n} \to P^{-1,n} \to P^{0,n} \to V^n \to 0,$$

$$\text{(P2):}\qquad \cdots \to Z^n(\mathbf{P}^{-2}) \to Z^n(\mathbf{P}^{-1}) \to Z^n(\mathbf{P}^0) \to Z^n(\mathbf{V}) \to 0,$$

$$\text{where}\quad Z^n(\mathbf{P}^k) = \text{kernel of } \lambda_k : P^{k,n} \to P^{k,n+1},$$

$$\text{(P3):}\qquad \cdots \to H^n(\mathbf{P}^{-2}) \to H^n(\mathbf{P}^{-1}) \to H^n(\mathbf{P}^0) \to H^n(\mathbf{V}) \to 0.$$

Notice that, in a proper projective resolution (3.1) of $\mathbf{V}$, the image of every morphism in the rows is closed.

Remark 3.2. Let $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ and $0 \to U' \xrightarrow{f'} V' \xrightarrow{g'} W' \to 0$ be exact sequences of Banach spaces. Consider the sequence of direct sums

$$0 \to U \oplus U' \xrightarrow{f \oplus f'} V \oplus V' \xrightarrow{g \oplus g'} W \oplus W' \to 0. \tag{3.2.1}$$

It is clear that this sequence (3.2.1) is algebraically exact. Since a finite direct sum of Banach spaces is also a Banach space, the sequence (3.2.1) is an exact sequence of Banach spaces.

In [3], it is shown that the 3×3 Lemma and the Horseshoe Lemma hold in an *exact category.* As an example, a collection of short exact sequences consisting of kernel-cokernel pairs, such as

$$0 \to X \xrightarrow{f} V \xrightarrow{g} W \to 0$$

for which f = kernel of g and g = cokernel of f, forms an exact category. In our case, the short exact sequences of Banach spaces form such a category.

Proposition 3.3. *Let* $\boldsymbol{V} = \{\cdots \to V^n \xrightarrow{\partial^n} V^{n+1} \to \cdots\}$ *be a complex of Banach spaces such that every* Im ∂^n *is closed. Then* $\boldsymbol{V}$ *admits a proper projective resolution like the diagram* (3.1). *In particular, the image of every vertical arrow in the diagram* (3.1) *is closed.*

Proof. By the assumption that every image of ∂^n is closed, the proof is similar to the case of the complex of vector spaces. So we give a short description and refer details to [7] and [11]. All we have to check is that every object appearing in the diagram is a Banach space. First, notice that Ker ∂^n, Im ∂^n, and $H^n(\mathbf{V})$ are all Banach spaces for every n. Thus there are short exact sequences of Banach spaces

$$0 \to \mathrm{Im}\ \partial^{n-1} \to \mathrm{Ker}\ \partial^n \to H^n(\mathbf{V}) \to 0 \quad \text{and}$$
$$0 \to \mathrm{Ker}\ \partial^n \to V^n \to \mathrm{Im}\ \partial^n \to 0.$$

Recall that every Banach space admits a projective resolution by Proposition 2.22. Let

$$\cdots \to PB^{-1,n} \to PB^{0,n} \to \mathrm{Im}\ \partial^n \to 0 \text{ and } \cdots \to PH^{-1,n} \to PH^{0,n} \to H^n(\mathbf{V}) \to 0$$

be the projective resolutions of Im ∂^n and $H^n(\mathbf{V})$ for each n, respectively.

For every k and n, we set $PZ^{k,n} = PB^{k,n} \bigoplus PH^{k,n}$. It is easy to check that every $PZ^{k,n}$ is a projective Banach space and there is an exact sequence of Banach spaces

$$0 \to PB^{k,n} \to PZ^{k,n} \to PH^{k,n} \to 0$$

given by inclusion and projection. Then, using the 3×3 Lemma and the Horseshoe Lemma as shown in [3], we can show

$$\cdots \to PZ^{-2,n} \to PZ^{-1,n} \to PZ^{0,n} \to \text{Ker } \partial^n \to 0$$

is a projective resolution of Ker ∂^n. Again, by the same argument as above, from the projective resolutions of Ker ∂^* and Im ∂^n, we can construct a projective resolution of V^n as

$$\cdots \to P^{1,n} \to P^{0,n} \to V^n \to 0,$$

where $P^{k,n} = PB^{k,n} \bigoplus PH^{k,n} \bigoplus PB^{k,n+1}$. Notice it fits into an exact sequence

$$0 \to PZ^{k,n} \to P^{k,n} \to PB^{k,n+1} \to 0.$$

Now, for each n and k, we compute $H^n(\mathbf{P}^k)$. First, we have to check the boundary operator $\lambda^k : P^{k,n} \to P^{k,n+1}$. Consider the following commutative diagram of exact sequences of Banach spaces:

$$\begin{tikzcd}
0 \arrow[r] & PZ^{k,n+1} \arrow[r] & P^{k,n+1} \arrow[r] & PB^{k,n+2} \arrow[r] & 0 \\
0 \arrow[r] & PZ^{k,n} \arrow[r] \arrow[u] & P^{k,n} \arrow[r] \arrow[u, "\lambda^k"] & PB^{k,n+1} \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Notice that λ^k is the composition

$$\lambda^k : P^{k,n} \to PB^{k,n+1} \to PZ^{k,n+1} \to P^{k,n+1}.$$

It is clear that Im $\lambda^{k-1} = PB^{k,n}$ and Ker $\lambda^k = PZ^{k,n+1}$. Thus $H^n(\mathbf{P}^k) = PH^{k,n}$. Hence Im λ^{k-1} is closed and $H^n(\mathbf{P}^k)$ is a Banach space. $\square$

The Künneth spectral sequence is a generalization of the Künneth Theorem that allows us to express $H^*\left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right)$ in terms of $H^*(\mathbf{U})$ and $H^*(\mathbf{V})$ for complexes $\mathbf{U}$ and $\mathbf{V}$. We check the Künneth Theorem first.

Definition 3.4. We say a complex of Banach spaces $\mathbf{V} = \{V^n, \partial^n\}$ is positive in the upper indices (in short, positive) if $V^m = 0$ for $m < 0$. Also, we say a complex of Banach spaces $\mathbf{V} = \{V^n, \partial^n\}$ is bounded above if $V^n = 0$ for sufficiently large n.

Let $\mathbf{U} = \{U^n, d^n\}$ and $\mathbf{V} = \{V^n, \partial^n\}$ be the positive complexes of Banach spaces. It is easy to see that their projective tensor product $\mathbf{U}\widehat{\otimes}\mathbf{V}$ forms a positive complex of Banach spaces. In fact, for each nonnegative integer n,

$$\left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right)^n = \bigoplus_{p+q=n} U^p\widehat{\otimes}V^q = \bigoplus_{p=0}^{n} U^p\widehat{\otimes}V^{n-p}$$

and boundary operator $D^n : \left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right)^n \to \left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right)^{n+1}$ is defined as

$$D^n = \sum_{p=0}^{n} \left(d^p\widehat{\otimes}1_{\mathbf{V}} + (-1)^p 1_{\mathbf{U}}\widehat{\otimes}\partial^{n-p}\right).$$

Proposition 3.5. *Let $\boldsymbol{A} = \{A^n, \partial^n\}$ be a positive complex of flat Banach spaces such that every boundary operator ∂^n is zero. Also, let $\boldsymbol{V} = \{V^n, d^n\}$ be a positive complex of Banach spaces such that every Im d^n is closed. Then, for each $n \geq 0$, there is an isomorphism of Banach spaces*

$$H^n(\boldsymbol{A}\widehat{\otimes}\boldsymbol{V}) \cong \bigoplus_{p+q=n} \boldsymbol{A}^p\widehat{\otimes}H^q(\boldsymbol{V}),$$

where $H^*(\boldsymbol{V})$ is construed as a complex with zero differentiation. In particular, the image of every boundary operator of the complex $\boldsymbol{A}\widehat{\otimes}\boldsymbol{V}$ is closed.

Proof. Notice that $\mathbf{A}\widehat{\otimes}\mathbf{V}$ forms a positive complex of Banach spaces with boundary operators $D^n = \sum_{p=0}^{n}(\pm)1_{\mathbf{A}}\widehat{\otimes}d^{n-p}$.

Since every A^p is a flat Banach space and every Im d^q is closed, from Corollary 2.14 there are exact sequences of Banach spaces:

$$0 \to A^p\widehat{\otimes}\mathrm{Ker}\ d^q \xrightarrow{1_{A^p}\widehat{\otimes}i^q} A^p\widehat{\otimes}V^q \xrightarrow{1_{A^p}\widehat{\otimes}d^q} A^p\widehat{\otimes}\mathrm{Im}\ d^q \to 0, \tag{3.5.1}$$

$$0 \to A^p\widehat{\otimes}\mathrm{Im}\ d^{q-1} \xrightarrow{1_{A^p}\widehat{\otimes}j^q} A^p\widehat{\otimes}\mathrm{Ker}\ d^q \xrightarrow{1_{A^p}\widehat{\otimes}\lambda^q} A^p\widehat{\otimes}H^q(\mathbf{V}) \to 0. \tag{3.5.2}$$

By Remark 3.2, the sequence of finite direct sums from (3.5.1)

$$0 \to \bigoplus_{p=0}^{n} A^p\widehat{\otimes}\mathrm{Ker}\ d^q \to$$
$$\xrightarrow{\sum_{p=0}^{n}(-1)^p 1_{A^p}\widehat{\otimes}i^{n-p}} \bigoplus_{p=0}^{n} A^p\widehat{\otimes}V^{n-p} \xrightarrow{\sum_{p=0}^{n}(-1)^p 1_{A^p}\widehat{\otimes}d^{n-p}} \bigoplus_{p=0}^{n} A^p\widehat{\otimes}\mathrm{Im}\ d^{n-p} \to 0$$

is an exact sequence of Banach spaces. Hence

$$\mathrm{Ker}\ D^n = \mathrm{Ker}\ \left(\sum_{p=0}^{n}(-1)^p 1_{A^p}\widehat{\otimes}d^{n-p}\right) \cong \bigoplus_{p=0}^{n} A^p\widehat{\otimes}\mathrm{Ker}\ d^{n-p},$$
$$\mathrm{Im}\ D^n = \mathrm{Im}\ \left(\sum_{p=0}^{n}(-1)^p 1_{A^p}\widehat{\otimes}d^{n-p}\right) \cong \bigoplus_{p=0}^{n} A^p\widehat{\otimes}\mathrm{Im}\ d^{n-p}.$$

Notice that the image of every D^n is closed. Also, from the exact sequence (3.5.2), there is an exact sequence of Banach spaces
(3.5.3)

$$0 \to \bigoplus_{p=0}^{n} A^p \widehat{\otimes} \mathrm{Im}\ d^{n-p-1} \xrightarrow{J^n} \bigoplus_{p=0}^{n} A^p \widehat{\otimes} \mathrm{Ker}\ d^{n-p} \xrightarrow{\Lambda^n} \bigoplus_{p=0}^{n} A^p \widehat{\otimes} H^{n-p}(\mathbf{V}) \to 0,$$

where

$$J^n = \sum_{p=0}^{n} (-1)^p 1_{A^p} \widehat{\otimes} j^{n-p}, \qquad \Lambda^n = \sum_{p=0}^{n} (-1)^p 1_{A^p} \widehat{\otimes} \lambda^{n-p}.$$

Notice that, from the exactness of the sequence (3.5.3), J^n is an injective morphism and its image is closed. Also, Λ^n is surjective. Hence there are isomorphisms

$$\begin{aligned} \bigoplus_{p=0}^{n} A^p \widehat{\otimes} H^{n-p}(\mathbf{V}) &\cong \left(\bigoplus_{p=0}^{n} A^p \widehat{\otimes} \mathrm{Ker}\ d^{n-p} \right) \Big/ \left(\bigoplus_{p=0}^{n} A^p \widehat{\otimes} \mathrm{Im}\ d^{n-p-1} \right) \\ &\cong \mathrm{Ker}\ D^n / \mathrm{Im}\ D^{n-1} \\ &\cong H^n(\mathbf{A} \widehat{\otimes} \mathbf{V}) \end{aligned}$$

as Banach spaces. □

Now, we review the Künneth Theorem for complexes of Banach spaces.

Theorem 3.6. *Let $\{\boldsymbol{U}, d^*\}$ and $\{\boldsymbol{V}, \partial^*\}$ be the positive and bounded above complexes of Banach spaces and satisfy the following conditions*:

(1) *all images of boundary operators d^* and ∂^* are closed*;
(2) *all Im d^* and $H^*(U)$ are flat.*

Then, for each $n \geq 0$, there is an isomorphism

$$\bigoplus_{p+q=n} H^p(\boldsymbol{U}) \widehat{\otimes} H^q(\boldsymbol{V}) \cong H^n(\boldsymbol{U} \widehat{\otimes} \boldsymbol{V})$$

of Banach spaces.

Proof. For $d^p : U^p \to U^{p+1}$, we set Ker $d^p = Z^p$, Im $d^{p-1} = B^p$, and Im $d^p = B_+^p$. Notice that the kernels and images of boundary operators d^* and ∂_* are closed and so Banach spaces. Then $H^*(\mathbf{U})$ and $H^*(\mathbf{V})$ are also Banach spaces. Hence the sequences of Banach spaces

(3.6.1) $\quad 0 \to Z^p \xrightarrow{j^p} U^p \xrightarrow{d^p} B_+^p \to 0 \quad$ and $\quad 0 \to B^p \xrightarrow{i^p} Z^p \xrightarrow{\lambda_p} H^p(\mathbf{U}) \to 0$

are exact.

Since B^p and $H^p(\mathbf{U})$ are flat Banach spaces, the Banach space Z^p in the second sequence (3.6.1) is also flat by Proposition 2.16. Similarly, the Banach space U^p is flat from the first sequence (3.6.1). Hence, both exact sequences (3.6.1) are pure.

First, by applying $\cdot\widehat{\otimes}H^q(\mathbf{V})$ to the second sequence (3.6.1), we have an exact sequence of Banach spaces

(3.6.2)

$$0 \to \left(B^*\widehat{\otimes}H^*(\mathbf{V})\right)^n \xrightarrow{i^*\widehat{\otimes}id_{H^*(\mathbf{V})}} \left(Z^*\widehat{\otimes}H^*(\mathbf{V})\right)^n \xrightarrow{\lambda^*\widehat{\otimes}id_{H^*(\mathbf{V})}} \left(H^*(\mathbf{U})\widehat{\otimes}H^*(\mathbf{V})\right)^n \to 0.$$

Since the first one in the sequences (3.6.1) is pure, we have an exact sequence

$$0 \to Z^p\widehat{\otimes}V^q \to U^p\widehat{\otimes}V^q \to B^p_+\widehat{\otimes}V^q \to 0$$

of Banach spaces. So there is an exact sequence of complexes of Banach spaces

$$0 \to Z^*\widehat{\otimes}\mathbf{V} \xrightarrow{\varphi} \mathbf{U}\widehat{\otimes}\mathbf{V} \xrightarrow{\psi} B^*_+\widehat{\otimes}\mathbf{V} \to 0,$$

where $\varphi = j^*\widehat{\otimes}id_V$ and $\psi = d^*\widehat{\otimes}id_V$. Then, as we saw in the sequence (1.2), there is an induced exact sequence

$$\begin{aligned} \cdots \to H^{n-1}\left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right) &\xrightarrow{H^{n-1}\psi} H^{n-1}\left(B^*_+\widehat{\otimes}\mathbf{V}\right) \xrightarrow{\Lambda^{n-1}} H^n\left(Z^*\widehat{\otimes}\mathbf{V}\right) \\ &\xrightarrow{H^n\varphi} H^n\left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right) \xrightarrow{H^n\psi} H^n\left(B^*_+\widehat{\otimes}\mathbf{V}\right) \xrightarrow{\Lambda^n} H^{n+1}\left(Z^*\widehat{\otimes}\mathbf{V}\right) \to \cdots \end{aligned} \tag{3.6.3}$$

of topological vector spaces. Since Z^* and B^* are considered as complexes of flat Banach spaces having zero boundary operators, by Proposition 3.5 there are isomorphisms

$$\begin{aligned} &H^n\left(Z^*\widehat{\otimes}\mathbf{V}\right) \cong Z^*\widehat{\otimes}H^n(\mathbf{V}) \quad \text{and} \\ &H^{n-1}\left(B^*_+\widehat{\otimes}\mathbf{V}\right) = H^n\left(B^*\widehat{\otimes}\mathbf{V}\right) \cong B^*\widehat{\otimes}H^n(\mathbf{V}) \end{aligned}$$

of Banach spaces. Hence the exact sequence (3.6.3) can be written as

$$\begin{aligned} \cdots \to H^{n-1}\left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right) &\xrightarrow{H^{n-1}\psi} B^*\widehat{\otimes}H^n(\mathbf{V}) \xrightarrow{\Lambda^{n-1}} Z^*\widehat{\otimes}H^n(\mathbf{V}) \\ &\xrightarrow{H^n\varphi} H^n\left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right) \xrightarrow{H^n\psi} B^*\widehat{\otimes}H^{n+1}(\mathbf{V}) \xrightarrow{\Lambda^n} Z^*\widehat{\otimes}H^{n+1}(\mathbf{V}) \to \cdots. \end{aligned} \tag{3.6.4}$$

As in the ordinary case, we can check that every connecting homomorphism Λ^n in (3.6.4) has degree 0 and is equal to $i^*\widehat{\otimes}id_{H^*(\mathbf{V})}$ in (3.6.2). Thus Λ^n is injective. From Proposition 3.5, all images of boundary operators of the complexes $Z^*\widehat{\otimes}H^n(\mathbf{V})$ and $B^*\widehat{\otimes}H^n(\mathbf{V})$ are closed. Also, from the exactness of the sequence (3.6.2),

$$\mathrm{Im}\ \Lambda^* = \mathrm{Im}\ \left(i^*\widehat{\otimes}id_{H^*(\mathbf{V})}\right) = \mathrm{Ker}\ \left(\lambda^*\widehat{\otimes}id_{H^*(\mathbf{V})}\right)$$

and so $\mathrm{Im}\ \Lambda^*$ are closed in $Z^*\widehat{\otimes}H^n(\mathbf{V})$. Hence, from Theorem 1.2(3), all images of boundary operators of $\mathbf{U}\widehat{\otimes}\mathbf{V}$ are closed, so that $H^*(\mathbf{U}\widehat{\otimes}\mathbf{V})$ are also Banach spaces. Thus (3.6.4) is an exact sequence of Banach spaces. Finally, from the exact sequences of Banach spaces (3.6.2) and (3.6.4), we get the following isomorphisms of Banach spaces:

$$\begin{aligned} \left(H^*(\mathbf{U})\widehat{\otimes}H^*(\mathbf{V})\right)^n &\cong \left(\mathbf{Z}^*\widehat{\otimes}H^*(\mathbf{V})\right)/\mathrm{Ker}\ \left(\lambda^*\widehat{\otimes}id_{H^*(\mathbf{V})}\right) && \text{from (3.6.2)} \\ &\cong \left(\mathbf{Z}^*\widehat{\otimes}H^*(\mathbf{V})\right)/\left(\mathbf{B}^*\widehat{\otimes}H^*(\mathbf{V})\right) && \text{from (3.6.2)} \\ &\cong H^*\left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right) && \text{from (3.6.4).} \end{aligned}$$

□

Now, we carry on the Künneth spectral sequence for a complex of Banach spaces.

Theorem 3.7. *Let* $\{\boldsymbol{U}, d^*\}$ *and* $\{\boldsymbol{V}, \partial^*\}$ *be the positive and bounded above complexes of Banach spaces satisfying the following conditions*:

(1) *all images of boundary operators* d^* *and* ∂^* *are closed*;
(2) U^s *is flat for every* $s \geq 0$.

Then there is a spectral sequence with

$$E_2^{p,q} = \bigoplus_{s+t=q} Tor^p \left(H^s(\boldsymbol{U}), H^t(\boldsymbol{V})\right)$$

and it converges to $H^n(\boldsymbol{U}\widehat{\otimes}\boldsymbol{V})$ *as topological vector spaces.*

Proof. Consider a proper projective resolution of $\boldsymbol{V}$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & P^{-2,2} \arrow[r] \arrow[u] & P^{-1,2} \arrow[r] \arrow[u] & P^{0,2} \arrow[r] \arrow[u] & V^2 \arrow[r] \arrow[u] & 0 \\
\cdots \arrow[r] & P^{-2,1} \arrow[r] \arrow[u] & P^{-1,1} \arrow[r] \arrow[u] & P^{0,1} \arrow[r] \arrow[u] & V^1 \arrow[r] \arrow[u] & 0 \\
\cdots \arrow[r] & P^{-2,0} \arrow[r, "\delta_P"] \arrow[u, "\lambda_{-2}"] & P^{-1,0} \arrow[r, "\delta_P"] \arrow[u, "\lambda_{-1}"] & P^{0,0} \arrow[r, "\epsilon"] \arrow[u, "\lambda_0"] & V^0 \arrow[r] \arrow[u, "\partial_0"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

as shown in the diagram (3.1). Recall that each column is a complex and each tth row is a projective resolution of V^t and so the image of each right arrow morphism in this diagram is closed. We consider the double complex

$$M^{p,\,q} = \bigoplus_{s+t=q} U^s\widehat{\otimes}P^{p,\,t}$$

with morphisms

$$D' : M^{p,\,q} \to M^{p+1,\,q}, \quad \text{given by } D' = \sum(-1)^q 1 \otimes \delta_P$$
$$D'' : M^{p,\,q} \to M^{p,\,q+1}, \quad \text{given by } D'' = \sum d^* \otimes 1 + \sum(-1)^s 1 \otimes \lambda_p\ .$$

Notice q is nonnegative and denotes a dimension and p is nonpositive and gives the homological degree. Since s and t are nonnegative, the double complex $M^{p,q}$

for given p and q is a finite coproduct and so is well defined in the category of Banach spaces. For each $n \geq 0$, its total complex is defined as

$$TM_n = \bigoplus_{p+q=n} M^{p,q} = \bigoplus_{p+q=n} \left(\bigoplus_{s+t=q} U^s \widehat{\otimes} P^{p,t} \right).$$

As the complexes $\mathbf{U}$ and $\mathbf{V}$ are positive and bounded above, the total complex TM_n for given n is a finite coproduct. So TM_n is a Banach space.

Now we check the first and the second filtration. From the second filtration

$$F_{II}^q (TM)_n = \bigoplus_{j \geq q} \left(\bigoplus_{s+t=j} U^s \widehat{\otimes} P^{n-j,t} \right),$$

we have

$$\left[F_{II}^q / F_{II}^{q+1} \right]_n = \bigoplus_{s+t=q} U^s \widehat{\otimes} P^{n-q,t} = M^{n-q,q}$$

and so it is the qth row with boundary operators $D' = \sum (-1)^q 1 \widehat{\otimes} \delta_P$. Thus

$${}_{II}E_1^{p,q} = H \left(\bigoplus_{s+t=q} U^s \widehat{\otimes} P^{p,t}, \sum (-1)^q 1 \widehat{\otimes} \delta_P \right)$$

which is the homology of qth row in the diagram (3.1). For, fixed s, t with $s + t = q$, the qth row is

$$\cdots \to U^s \widehat{\otimes} P^{-3,t} \to U^s \widehat{\otimes} P^{-2,t} \to U^s \widehat{\otimes} P^{-1,t} \to U^s \widehat{\otimes} P^{0,t} \to 0.$$

This is the complex obtained by applying $U^s \widehat{\otimes} \cdot$ to a projective resolution $P^{p,t}$ of V^t. Hence

$${}_{II}E_1^{p,q} = \bigoplus_{s+t=q} \mathrm{Tor}^p \left(U^s, V^t \right).$$

Since U^s is flat, $\mathrm{Tor}^p (U^s, V^t) = 0$ for $p \neq 0$. Hence

$${}_{II}E_1^{p,q} = {}_{II}E_1^{0,q} = \bigoplus_{s+t=q} U^s \widehat{\otimes} V^t.$$

Then

$${}_{II}E_2^{0,q} = H^q \left({}_{II}E_1^{0,*} \right) = H^q \left(\mathbf{U} \widehat{\otimes} \mathbf{V} \right).$$

Since this is only a column on the q-axis, it collapses at $E_2 = E_\infty$. This establishes

$${}_{II}E_2^{0,n} = H^n \left(\mathbf{U} \widehat{\otimes} \mathbf{V} \right).$$

Next, we consider the first filtration. For fixed p,

$$F_I^p (TM)_n = \bigoplus_{j \geq p} \left[\bigoplus_{s+t=n-j} U^s \widehat{\otimes} P^{j,t} \right].$$

Then

$$\left[F_I^p / F_I^{p+1} \right]_n = \bigoplus_{s+t=n-p} U^s \widehat{\otimes} P^{p,t} = M^{p,n-p}.$$

Hence

$$
\begin{aligned}
{}_I E_1^{p,q} &= H^q\,(p\text{th column of } M^{*,*}) \\
&= H^q\left(U^*\widehat{\otimes}P^{p,*},\ D'' = \sum d_U \otimes 1 + \sum (-1)^s 1 \otimes \lambda_p\right).
\end{aligned}
$$

As shown in Proposition 3.3, for each p, the image of every vertical arrow $\lambda_p : P^{p,t} \to P^{p,t+1}$ in the diagram (3.1) is closed. So its cohomology $H^t(P^{p,*})$ is a Banach space. Also, by Definition 3.1, the kernel $Z^t(\mathbf{P}^p)$ of $\lambda_p\colon P^{p,*} \to P^{p,*}$ and its cohomology $H^t(P^{p,*})$ are projective Banach spaces. So they are flat by Proposition 2.20. Hence, by Proposition 2.16 and the exact sequence of Banach spaces

$$
0 \to \mathrm{Im}\ \lambda_{p,t-1} \to \mathrm{Ker}\ \lambda_{p,t} \to H^t(P^{p,*}) \to 0,
$$

$\mathrm{Im}\ \lambda_{p,t-1}$ is also flat. So, from Theorem 3.6, we have a topological isomorphism

$$
{}_I E_1^{p,q} = H^q\left(U^*\widehat{\otimes}P^{p,*}\right) \cong \bigoplus_{s+t=q} H^s(\mathbf{U})\,\widehat{\otimes} H^t(P^{p,*}).
$$

Then

$$
{}_I E_2^{p,q} = H^p\left({}_I E_1^{*,q}\right) = H^p\left(\bigoplus_{s+t=q} H^s(\mathbf{U})\,\widehat{\otimes} H^t(P^{*,*})\right).
$$

So we need to compute $H^p\left(\bigoplus_{s+t=q} H^s(\mathbf{U})\,\widehat{\otimes} H^t(P^{*,*})\right)$. Since the sequence

$$
\cdots \to H^t(\mathbf{P}^p) \to \cdots \to H^t\left(\mathbf{P}^{-1}\right) \to H^t(\mathbf{P}^p) \to H^t(\mathbf{V}) \to 0
$$

is a projective resolution of $H^t(\mathbf{V})$, the cohomology of its induced complex obtained by applying $H^s(\mathbf{U})\widehat{\otimes}\cdot$ is defined as $\mathrm{Tor}^p\left(H^s(\mathbf{U}), H^t(\mathbf{V})\right)$. Hence

$$
{}_I E_2^{p,q} = H^p\left(\bigoplus_{s+t=q} H^s(\mathbf{U})\,\widehat{\otimes} H^t(P^{*,*})\right) = \bigoplus_{s+t=q} \mathrm{Tor}^p\left(H^s(\mathbf{U}), H^t(\mathbf{V})\right).
$$

Lastly, to show the convergence of this spectral sequence, we check the filtration is bounded. Since the complexes $\mathbf{U}$ and $\mathbf{V}$ are positive, $U^s = 0$ and $V^t = 0$ for every negative integers s and t. Also, since they are bounded above, there exist positive integers s_0 and t_0 such that $U^s = 0$ for every $s > s_0$ and $V^t = 0$ for every $t > t_0$. Let $m = s_0 + t_0 - 2$. Then, since p is nonpositive, $(TM)_n = 0$ for every $n > m$. It is clear that, for both the first and the second filtrations, $\{0\} \subset F^0(TM)_0 = TM_0$.

From the first filtration, it is easy to check that for each dimension $n > 0$,

$$
\{0\} \subset F^0(TM)_n \subset F^{-1}(TM)_n \subset \cdots \subset F^{n-m}(TM)_n = (TM)_n.
$$

Also, for each dimension $n < 0$,

$$
\{0\} \subset F^n(TM)_n \subset F^{n-1}(TM)_n \subset \cdots \subset F^{n-m}(TM)_n = (TM)_n.
$$

Similarly, from the second filtration, we can check for each dimension n with $0 < n \le m$,

$$
\{0\} \subset F^m(TM)_n \subset F^{m-1}(TM)_n \subset \cdots \subset F^n(TM)_n = (TM)_n.
$$

Also, for each dimension $n < 0$,

$$\{0\} \subset F^m(TM)_n \subset F^{m-1}(TM)_n \subset \cdots \subset F^0(TM)_n = (TM)_n.$$

Hence the filtrations are bounded and so the spectral sequence converges to $H^n\left(\mathbf{U}\widehat{\otimes}\mathbf{V}\right)$. □

We consider the case of bounded cohomology. First, notice that we need the *bounded above* condition on the complex in Theorem 3.7 to define the total complex as a finite product in the category of Banach spaces and also to get a bounded filtration.

However, for a discrete group G, the space of any n-dimensional bounded cochains $B^n(G) = \{f : G^n \to \mathbb{R} \mid \|f\| < \infty\}$ is never 0. For example, even for the trivial group $\{e\}$, we have $B^n(\{e\}) = \mathbb{R}$ for every $n \geq 0$. Thus, the *bounded above* condition on the complex in Theorem 3.7 is not applicable to bounded cochain complex $\{B^*(G)\}$ in general, and so we need to modify it.

Remark 3.8. (1) In Theorem 3.6, the condition for Im d_* and $H^*(\mathbf{U})$ to be flat is replaced by the condition for the exact sequences of Banach spaces

(3.6.1) $0 \to Z^p \xrightarrow{j_p} U^p \xrightarrow{d_p} B^p_+ \to 0$ and $0 \to B^p \xrightarrow{i_p} Z^p \xrightarrow{\lambda_p} H^p(\mathbf{U}) \to 0$

to be pure.

(2) In Theorem 3.7, the condition for U^* to be flat is needed to compute the first spectral sequence from the second filtration

$$_{II}E_1^{p,q} = \bigoplus_{s+t=q} \mathrm{Tor}^p\left(U^s, V^t\right).$$

Proposition 3.9. *Let V be a Banach space. Suppose U is either a finite dimensional subspace or a closed and finite codimensional subspace of V. Then U is complemented. In particular, the exact sequence of Banach spaces $0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$ is split.*

Proof. We assume $U \neq \{0\}$.

Suppose U is a finite dimensional subspace of V. Notice that U is a closed subspace and so a Banach subspace of V. Let $\{u_1, u_2, \ldots, u_n\}$ be a basis of U and $\{u^1, u^2, \ldots, u^n\}$ be its dual basis. Recall that every u^i has a continuous extension v^i to V and $v^i(u_j) = \delta_{ij}$, where δ_{ij} is a Kronecker delta. Consider $P : V \to V$ defined by $P(v) = \sum_{i=1}^n v^i(v)u_i$. Then it is easy to check that P is an idempotent morphism, that is, $(P \circ P)(v) = P(v)$ for every $v \in V$, and also $1_V = P + (1_V - P)$. Hence $U = \mathrm{Im}\ P$ and U is complemented.

Similarly, suppose U is a closed and finite codimensional subspace of V. Then V/U is a finite dimensional Banach space. Recall that V/U is topologically isomorphic with $\mathbb{R}^m$ for some positive integer $m < \infty$. Notice that V/U is projective by Remark 2.19. Hence, for the surjective morphism $\pi : V \to V/U$, there is a morphism $\lambda : V/U \to V$ such that $\pi \circ \lambda = 1_{V/U}$. Then $V \cong \mathrm{Im}\ \lambda \oplus \mathrm{Ker}\ \pi$ and $\mathrm{Ker}\ \pi \cong U$. This shows U is complemented.

Now, suppose $0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$ is an exact sequence of Banach spaces. Since U is complemented, there is a Banach space X such that $V \cong U \oplus X$. Since V/U and X are topologically isomorphic, X and W are also topologically isomorphic. In [8], it is shown that algebraic direct sum of closed subspaces of a Banach space is also a topological direct sum. So $V \cong U \oplus W$ as Banach spaces. Hence the sequence is split. $\square$

Proposition 3.10. *Let* $\boldsymbol{V} = \{0 \to V^0 \to V^1 \xrightarrow{\partial^1} V^2 \xrightarrow{\partial^2} \cdots\}$ *be a complex of Banach spaces. Suppose every* Im ∂^n *is a finite dimensional subspace of* V^{n+1}*. Then, for each* $n \geq 0$*, the sequences of Banach spaces*

$$0 \to \mathrm{Im}\ \partial^{n-1} \xrightarrow{i_n} \mathrm{Ker}\ \partial^n \xrightarrow{\lambda^n} H^n(\boldsymbol{V}) \to 0, \tag{3.10.1}$$

$$0 \to \mathrm{Ker}\ \partial^n \xrightarrow{j^n} V^n \xrightarrow{\partial^n} \mathrm{Im}\ \partial^n \to 0 \tag{3.10.2}$$

are exact and pure.

Proof. Since every Im ∂^* is a Banach space, the sequences (3.10.1) and (3.10.2) are exact. Notice that every Im ∂^n is a finite dimensional subspace of V^{n+1} and also of Ker d^{n+1}. Also every Ker d^n is a closed and finite codimensional subspace of V^n. So the exact sequences of Banach spaces (3.10.1) and (3.10.2) are split. Then the dual of these sequences are also split and so pure. The given sequences are pure by definition. $\square$

Corollary 3.11. *For the discrete groups G and K, let the bounded cochains* $\{B^*(G), d^*\}$ *and* $\{B^*(K), \partial^*\}$ *satisfy the following conditions*: *for* $n \geq 0$,

(1) *every image of boundary operator* ∂^n *is closed*;
(2) *every Im* d^n *is a finite dimensional subspace of* $B^{n+1}(G)$.

Then there is an isomorphism

$$\bigoplus_{p+q=n} H^p(B^*(G))\widehat{\otimes} H^q(B^*(K)) \cong H^n(B^*(G)\widehat{\otimes} B^*(K))$$

of Banach spaces.

Proof. From Theorem 3.6, we set $\mathbf{U} = B^*(G)$ and $\mathbf{V} = B^*(K)$. Since every Im d^n is a finite dimensional subspace of $B^{n+1}(G)$, the exact sequences of the form in the sequence (3.6.1) are pure by Proposition 3.10. By Remark 3.8, the condition (2) for flatness in Theorem 3.6 is replaced by the given condition (2) here. Then the rest of the proof is the same as in Theorem 3.6. $\square$

Corollary 3.12. *Let the bounded cochains* $\{B^*(G), d^*\}$ *and* $\{B^*(K), \partial^*\}$ *satisfy the following conditions*:

(1) *all images of boundary operators* d^* *and* ∂^* *are closed*;
(2) $B^s(G)$ *is flat for every* $s \geq 0$;
(3) $\widehat{H}^s(G) = 0$ *and* $\widehat{H}^t(K) = 0$ *for every* $s > s_0$ *and* $t > t_0$.

Then there is a spectral sequence with

$$E_2^{p,q} = \bigoplus_{s+t=q} Tor^p \left(\widehat{H}^s(G), \widehat{H}^t(K)\right)$$

and it converges to $\widehat{H}^n(B^*(G)\widehat{\otimes}B^*(K))$ *for each* $n = p + q$ *with* $0 \leq n \leq s_0 + t_0 - 2$.

Proof. Since $\widehat{H}^s(G) = 0$ for every $s > s_0$ and $\widehat{H}^t(K) = 0$ for every $t > t_0$, $\bigoplus_{s+t=q} \mathrm{Tor}^p \left(\widehat{H}^s(G), \widehat{H}^t(K)\right) = 0$ for all s and t such that $s > s_0$ or $t > t_0$. So, we only consider for each $n = p + q$ with $0 \leq n \leq s_0 + t_0 - 2$. Let $m = s_0 + t_0 - 1$. Consider the following sequences:

$$0 \to \mathbb{R} \to B^1(G) \xrightarrow{d^1} B^2(G) \to \cdots \to B^m(G) \xrightarrow{d^m} \mathrm{Im}\ d^m \to 0 \quad \text{and}$$

$$0 \to \mathbb{R} \to B^1(K) \xrightarrow{\partial^1} B^2(K) \to \cdots \to B^m(K) \xrightarrow{\partial^m} \mathrm{Im}\ \partial^m \to 0. \tag{3.12.1}$$

Then the sequences (3.12.1) are positive and bounded above complexes of Banach spaces. Set $\mathbf{U} = B^*(G)$ and $\mathbf{V} = B^*(V)$. Then the corollary follows from Theorem 3.7. □

As one of the simplest examples, let G be a discrete amenable group. We consider $\{B^*(G), d^*\}$. Then it is shown that every $B^n(G) = \mathbb{R}$ for $n \geq 0$ in [4]. Also, $\widehat{H}^n(G) = 0$ for $n > 0$ and $\widehat{H}^0(G) = \mathbb{R}$. So, every Im d^* and $\widehat{H}^n(G)$ are flat. Then, for a discrete group K, consider $\{B^*(K), \partial^*\}$ such that every Im ∂^n is either closed or finite dimensional. Recall that $\mathbb{R}\widehat{\otimes}U = U$ for a Banach space U. Hence $B^*(G)\widehat{\otimes}B^*(K) = B^*(K)$ and so

$$H^n(B^*(G)\widehat{\otimes}B^*(K)) \cong H^n(B^*(K)) = \widehat{H}^n(K).$$

Also, notice that

$$\begin{aligned} E_2^{p,q} &= \bigoplus_{s+t=q} \mathrm{Tor}^p \left(\widehat{H}^s(G), \widehat{H}^t(K)\right) \\ &= \mathrm{Tor}^p \left(\mathbb{R}, \widehat{H}^q(G)\right) = \mathbb{R}\widehat{\otimes}\widehat{H}^n(K) = \widehat{H}^n(K) \end{aligned}$$

and

$$\bigoplus_{p+q=n} H^p(B^*(G))\widehat{\otimes}H^q(B^*(K)) = \mathbb{R}\widehat{\otimes}H^n(B^*(K)) = H^n(B^*(K)) = \widehat{H}^n(K).$$

References

[1] T. Bühler, *On the algebraic foundations of bounded cohomology*, Mem. Amer. Math. Soc. **214** (2011), no. 1006, xxii+97 pp.

[2] J. Cigler, V. Losert, and P. Michor, *Banach Modules and Functors on Categories of Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, **46**, Marcel Dekker, Inc., New York, 1979.

[3] L. Frerick and D. Sieg, *Exact categories in functional analysis*, http://www.researchgate.net/publication/265264916, 2010.

[4] N. Ivanov, *Foundation of theory of bounded cohomology*, J. Soviet Math. **37** (1987), 1090–1114.
[5] V. I. Kuzminov and I. A. Shvedov, *Homological aspects of the theory of Banach complexes*, Siberian Math. J. **40** (1999), no. 4, 754–763; translated from Sibirsk. Mat. Zh. **40** (1999), no. 4, 893–904, iii.
[6] S. Mac Lane, *Homology*, reprint of the 1975 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
[7] J. McCleary, *A User's Guide to Spectral Sequences*, second edition, Cambridge Studies in Advanced Mathematics, **58**, Cambridge University Press, Cambridge, 2001.
[8] R. E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics, **183**, Springer-Verlag, New York, 1998.
[9] Y. Mitsumatsu, *Bounded cohomology and l^1-homology of surfaces*, Topology **23** (1984), no. 4, 465–471.
[10] G. A. Noskov, *The Hochschild-Serre spectral sequence for bounded cohomology*, in Proceedings of the International Conference on Algebra, Part 1 (Novosibirsk, 1989), 613–629, Contemp. Math., 131, Part 1, Amer. Math. Soc., Providence, RI.
[11] J. Rotman, *Introduction to Homological Algebra*, Academic Press, Inc. 1979.

HeeSook Park
Department of Mathematics Education
Sunchon National University
Sunchon 57922, Korea
Email address: hseapark@scnu.ac.kr