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AN ARTINIAN POINT-CONFIGURATION QUOTIENT AND THE STRONG LEFSCHETZ PROPERTY

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Abstract. In this paper, we study an Artinian point-configuration quotient having the SLP. We show that an Artinian quotient of points in $\mathbb{P}^n$ has the SLP when the union of two sets of points has a specific Hilbert function. As an application, we prove that an Artinian linear star configuration quotient $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ has the SLP if $\mathbb{X}$ and $\mathbb{Y}$ are linear star-configurations in $\mathbb{P}^2$ of type s and t for $s \geq \binom{t}{2}-1$ and $t \geq 3$. We also show that an Artinian $\mathbb{k}$-configuration quotient $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ has the SLP if $\mathbb{X}$ is a $\mathbb{k}$-configuration of type $(1,2)$ or $(1,2,3)$ in $\mathbb{P}^2$, and $\mathbb{X}\cup\mathbb{Y}$ is a basic configuration in $\mathbb{P}^2$.

1. Introduction

Ideals of sets of finite points in $\mathbb{P}^n$ have been studied for a long time ([8,9,11]), and in particular we consider an ideal of a special configuration in $\mathbb{P}^n$, so called a *star-configuration* and a $\mathbb{k}$-*configuration* in $\mathbb{P}^n$ ([1–3, 6, 7, 9–11, 15]). In 2006, Geramita, Migliore, and Sabourin introduced the notion of a star-configuration set of points in $\mathbb{P}^2$ (see [10]), the name having been inspired by the fact that 10-points in $\mathbb{P}^2$, defined by 5 general linear forms in $\mathbb{k}[x_0, x_1, x_2]$ resembles a star. In this paper, we refer to this as a "linear star-configuration", as more general definition of star-configurations has evolved through the subsequent literature (see [1, 6, 7, 19]). Indeed, a star-configuration in $\mathbb{P}^n$ has been studied to find the dimension of secant varieties to the variety of reducible forms in $R=\mathbb{k}[x_0, x_1, \ldots, x_n]$, where $\mathbb{k}$ is a field of characteristic 0 (see [4, 5, 20]).

If R/I is a standard graded Artinian algebra and ℓ is a general linear form, we recall that R/I is said to have *the weak Lefschetz property* (*WLP*) if the

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multiplication map by ℓ

$$[R/I]_d \stackrel{\times \ell}{\to} [R/I]_{d+1}$$

has maximal rank for every $d \geq 0$. Over the years, there have been several papers which have devoted to a classification of possible Artinian quotients having the WLP (see [1,8,9,13,14,16–18,21,22]). The *strong Lefschetz property* (*SLP*) says that for every $i \geq 1$ the multiplication map by ℓ^i

$$[R/I]_d \stackrel{\times \ell^i}{\to} [R/I]_{d+i}$$

has maximal rank for every $d \geq 0$ ([13, 14, 17]). In [14] the authors proved that a complete intersection ideal in $\Bbbk[x_0, x_1]$ has the SLP. Moreover, in [13], the authors give a nice description for a graded Artinian ring having the SLP by using the so-called *Jordan type* (see Lemma 2.2). *The Jordan type* is the partition of n specifying the lengths of blocks in the Jordan block matrix determined by the multiplication map by ℓ in a suitable $\Bbbk$-basis for R/I. Here, we apply this result often to show that some Artinian quotients of the ideals of points in $\mathbb{P}^n$ have the SLP.

We use Hilbert functions for many our arguments. Given a homogeneous ideal $I \subset R$, the Hilbert function of R/I, denoted $\mathbf{H}_{R/I}$, is the numerical function $\mathbf{H}_{R/I} : \mathbb{Z}^+ \cup \{0\} \to \mathbb{Z}^+ \cup \{0\}$ defined by

$$\mathbf{H}_{R/I}(i) := \dim_{\Bbbk}[R/I]_i = \dim_{\Bbbk}[R]_i - \dim_{\Bbbk}[I]_i,$$

where $[R]_i$ and $[I]_i$ denote the i-th graded component of R and I, respectively. If $I := I_{\mathbb{X}}$ is the defining ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote

$$\mathbf{H}_{R/I_{\mathbb{X}}}(i) := \mathbf{H}_{\mathbb{X}}(i) \quad \text{for} \quad i \geq 0,$$

and call it the *Hilbert function* of $\mathbb{X}$.

Let $R = \Bbbk[x_0, x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$ of characteristic 0. For positive integers r and s with $1 \leq r \leq \min\{n, s\}$, suppose $F_1, \ldots, F_s$ are general forms in R of degrees $d_1, \ldots, d_s$, respectively. Here s general forms $F_1, \ldots, F_s$ in R means that all subsets of size $1 \leq r \leq \min\{n+1, s\}$ are regular sequences in R, and if $\mathcal{H} = \{\boldsymbol{F}_1, \ldots, \boldsymbol{F}_s\}$ is a collection of distinct hypersurfaces in $\mathbb{P}^n$ corresponding to general $F_1, \ldots, F_s$ respectively, then the hypersurfaces *meet properly*, by which we mean that the intersection of any r of these hypersurfaces with $1 \leq r \leq \min\{n, s\}$ has codimesion r. We call the variety $\mathbb{X}$ defined by the ideal

$$\bigcap_{1 \leq i_1 < \cdots < i_r \leq s} (F_{i_1}, \ldots, F_{i_r})$$

a star-configuration in $\mathbb{P}^n$ of type (r, s). In particular, if $\mathbb{X}$ is a star-configuration in $\mathbb{P}^n$ of type (n, s), then we simply call *a point star-configuration in* $\mathbb{P}^n$ of type s for short.

Notice that each n-forms $F_{i_1}, \ldots, F_{i_n}$ of s-general forms $F_1, \ldots, F_s$ in R define $d_{i_1} \cdots d_{i_n}$ points in $\mathbb{P}^n$ for each $1 \leq i_1 < \cdots < i_n \leq s$. Thus the ideal

$$\bigcap_{1 \leq i_1 < \cdots < i_n \leq s} (F_{i_1}, \ldots, F_{i_n})$$

defines a finite set $\mathbb{X}$ of points in $\mathbb{P}^n$ with

$$\deg(\mathbb{X}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq s} d_{i_1} d_{i_2} \cdots d_{i_n}.$$

Furthermore, if $F_1, \ldots, F_s$ are general linear (quadratic, cubic, quartic, quintic, etc) forms in R, then we call $\mathbb{X}$ *a linear* (*quadratic, cubic, quartic, quintic, etc*) *star-configuration in* $\mathbb{P}^n$ *of type* s, respectively.

To provide some additional focus to this paper, we consider the following questions.

Question 1.1. Let $\mathbb{X}$ and $\mathbb{Y}$ be finite sets of points in $\mathbb{P}^n$ and $R = \Bbbk[x_0, x_1, \ldots, x_n]$.

(a) Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the WLP?
(b) Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the SLP?

Question 1.2. More precisely, let $\mathbb{X}$ and $\mathbb{Y}$ be finite point star configurations in $\mathbb{P}^n$, or $\mathbb{X}$ be a $\Bbbk$-configuration in $\mathbb{P}^n$ such that $\mathbb{X} \cup \mathbb{Y}$ is a basic configuration in $\mathbb{P}^n$.

(a) Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the WLP?
(b) Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the SLP?

In [1], the authors proved that an Artinian linear star-configuration quotient in $\mathbb{P}^2$ has the WLP, which is a partial answer to Question 1.2(a). Indeed, it is still true that any finite number of an Artinian linear point star-configuration quotient in $\mathbb{P}^n$ has the WLP. In [8,9], the authors show that Question 1.2(a) is true in general if $\mathbb{X}$ is a $\Bbbk$-configuration in $\mathbb{P}^n$ and $\mathbb{X} \cup \mathbb{Y}$ is a basic configuration in $\mathbb{P}^n$ with the condition $2\sigma(\mathbb{X}) \leq \sigma(\mathbb{X} \cup \mathbb{Y})$, where

$$\sigma(\mathbb{X}) = \min\{i \mid \mathbf{H}_{\mathbb{X}}(i-1) = \mathbf{H}_{\mathbb{X}}(i)\}.$$

In this paper, we focus on Questions 1.1(b) and 1.2(b). More precisely, we first find a condition in which an Artinian quotient of two sets of points in $\mathbb{P}^n$ has the SLP (see Lemma 2.4 and Proposition 2.5). Next we find some Artinian linear star configuration quotient in $\mathbb{P}^2$ that has the SLP (see Corollary 2.9). Then, we find an Artinian $\Bbbk$-configuration quotient having the SLP (see Proposition 3.4 and Theorem 3.6). Unfortunately, we do not have any counter example of an Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ of two point sets in $\mathbb{P}^n$, which does not have the SLP, and thus we expect Question 1.1(a) and (b) are true in general, especially when $\mathbb{X}$ and $\mathbb{Y}$ are sets of general points in $\mathbb{P}^n$.

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2. Artinian linear star-configuration quotients in $\mathbb{P}^2$

In this section, we shall show that an Artinian ring $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ has the SLP if $\mathbb{X}$ and $\mathbb{Y}$ are linear star-configurations in $\mathbb{P}^2$ of type s and t with $s \geq \binom{t}{2}-1$ and $t \geq 3$, respectively.

We first introduce the following two results of a star-configuration in $\mathbb{P}^n$ in [13, 22].

Remark 2.1. Let $\Bbbk$ be a field of characteristic zero and let $F \in \Bbbk[x_0, x_1, \ldots, x_n] = R = \bigoplus_{i\geq 0} R_i$ $(n \geq 1)$ be a homogeneous polynomial (form) of degree d, i.e., $F \in R_d$. It is well known that in this case each R_i has a basis consisting of i-th powers of linear forms. Thus we may write

$$F = \sum_{i=1}^{r} \alpha_i L_i^d, \qquad \alpha_i \in \Bbbk,\ L_i \in R_1.$$

If $\Bbbk$ is algebraically closed (which we now assume for the rest of the paper), then each $\alpha_i = \beta_i^d$ for some $\beta_i \in \Bbbk$ and so we can write

$$F = \sum_{i=1}^{r} (\beta_i L_i)^d = \sum_{i=1}^{r} M_i^d, \quad M_i \in R_1. \tag{2.1}$$

We call a description of F as in equation (2.1), a *Waring Decomposition* of F. The least integer r such that F has a Waring Decomposition with exactly r summands is called the *Waring Rank* (or simply the *rank*) of F.

Lemma 2.2 ([13])**.** *Assume A is graded and $\mathbf{H}_A$ is unimodal. Then*

(a) *A has the WLP if and only if the number of parts of the Jordan type $J_\ell = \max\{\mathbf{H}_A(i)\}$. (The Sperner number of A);*
(b) *ℓ is a strong Lefschetz element of A if and only if $J_\ell = \mathbf{H}_A^\vee$.*

Proposition 2.3 ([22, Proposition 2.5])**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star-configurations in $\mathbb{P}^2$ of type s and t, respectively, with $3 \leq t$ and $s \geq \lfloor \frac{1}{2}\binom{t}{2} \rfloor$. Then $\mathbb{X}\cup\mathbb{Y}$ has generic Hilbert function.*

Recall that

$$\mathbf{H}_A \ : \ h_0 \quad h_1 \quad \cdots\cdots \quad h_c$$

is said to be unimodal if there exists j such that

$$\begin{cases} h_i \leq h_{i+1} & (i < j), \\ h_i \geq h_{i+1} & (j \leq i). \end{cases}$$

Lemma 2.4. *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^n$ and let A be an Artinian quotient of the coordinate ring of $\mathbb{X}$. Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for every $0 \leq i \leq s-1$ and $A_s = 0$. Then an Artinian ring A has the SLP.*

Proof. First, we assume that the Hilbert function of A is of the form

$$\mathbf{H}_A \ : \ h_0 \quad h_1 \quad \cdots \quad h_{\sigma-1} \quad h_\sigma \quad \cdots \quad h_{s-1} \quad 0,$$

where $h_{\sigma-2} < h_{\sigma-1} = h_\sigma = \cdots = h_{s-1}$.

Let ℓ be a general linear form in A_1. Since ℓ is not a zero divisor of A, we see that the multiplication map by ℓ^{s-1}

$$[R/I_{\mathbb{X}}]_0 = [A]_0 \xrightarrow{\times \ell^{s-1}} [A]_{s-1} = [R/I_{\mathbb{X}}]_{s-1}$$

is injective. Hence we have a string of length s

$$1, \ell, \dots, \ell^{s-1},$$

and so the Jordan type J_ℓ for $\mathbf{H}_A$ is of the form

$$J_\ell = (s, \dots).$$

(i) Let $i = 1$. Then the multiplication map by ℓ^{s-2}

$$[R/I_{\mathbb{X}}]_1 = [A]_1 \xrightarrow{\times \ell^{s-2}} [A]_{s-1} = [R/I_{\mathbb{X}}]_{s-1}$$

is injective. Hence there are $g_1 := (h_1 - h_0) = (h_1 - 1)$ linear forms $F_{1,1}, F_{1,2}, \dots, F_{1,g_1} \in [A]_1$ such that the h_1 linear forms

$$\ell, F_{1,1}, F_{1,2}, \dots, F_{1,g_1}$$

are linearly independent. Hence there are g_1-strings of length $(s-1)$

$$\begin{array}{lll} F_{1,1}, F_{1,1}\ell, & \dots, & F_{1,1}\ell^{s-2}, \quad \text{and} \\ F_{1,2}, F_{1,2}\ell, & \dots, & F_{1,2}\ell^{s-2}, \\ & \vdots & \\ F_{1,g_1}, F_{1,g_1}\ell, & \dots, & F_{1,g_1}\ell^{s-2}. \end{array}$$

(ii) For $1 \le i < \sigma - 1$ and $1 \le j \le i$, define

$$g_j := h_j - h_{j-1}$$

for such j. Assume that there are g_j-forms $F_{j,1}, \dots, F_{j,g_j} \in [A]_j$ and there are g_j-strings of length $(s-j)$

$$\begin{array}{rll} F_{j,1}, F_{j,1}\ell, & \dots, & F_{j,1}\ell^{s-j-1}, \\ F_{j,2}, F_{j,2}\ell, & \dots, & F_{j,2}\ell^{s-j-1}, \\ & \vdots & \\ F_{j,g_j}, F_{j,g_j}\ell, & \dots, & F_{j,g_j}\ell^{s-j} \end{array}$$

such that the $\left(1 + \sum_{k=1}^{j} g_k\right)$-forms

$$\ell^j, \underbrace{F_{1,1}\ell^{j-1}, \dots, F_{1,g_1}\ell^{j-1}}_{g_1\text{-forms}}, \dots, \underbrace{F_{j-1,1}\ell, \dots, F_{j-1,g_{j-1}}\ell}_{g_{j-1}\text{-forms}}, \underbrace{F_{j,1}, \dots, F_{j,g_j}}_{g_j\text{-forms}}$$

are linearly independent for such j.

Since the multiplication map by $\ell^{(s-1)-(i+1)}$

$$[R/I_{\mathbb{X}}]_{i+1} = [A]_{i+1} \xrightarrow{\times \ell^{(s-1)-(i+1)}} [A]_{s-1} = [R/I_{\mathbb{X}}]_{s-1}$$

is injective, there are linearly independent $g_{i+1} := (h_{i+1} - h_i)$-forms $F_{i+1,1}, \ldots, F_{i+1,g_{i+1}} \in [A]_{i+1}$. Then the following $\left(1 + \sum_{k=1}^{i+1} g_k\right)$-forms

$$\ell^{i+1}, \underbrace{F_{1,1}\ell^i, \ldots, F_{1,g_1}\ell^i}_{g_1\text{-forms}}, \ldots, \underbrace{F_{i-1,1}\ell^2, \ldots, F_{i-1,g_{i-1}}\ell^2}_{g_{i-1}\text{-forms}}, \underbrace{F_{i,1}\ell, \ldots, F_{i,g_i}\ell}_{g_i\text{-forms}}, \underbrace{F_{i+1,1}, \ldots, F_{i+1,g_{i+1}}}_{g_{i+1}\text{-forms}}$$

are linearly independent as well. Hence we have g_{i+1}-strings of length $(s - i - 1)$

$$\begin{array}{rcc} F_{i+1,1}, F_{i+1,1}\ell, & \ldots, & F_{i+1,1}\ell^{s-i-2}, \\ F_{i+1,2}, F_{i+1,2}\ell, & \ldots, & F_{i+1,2}\ell^{s-i-2}, \\ & \vdots & \\ F_{i+1,g_{i+1}}, F_{i+1,g_{i+1}}\ell, & \ldots, & F_{i+1,g_{i+1}2}\ell^{s-i-2}. \end{array}$$

It is from (i) $\sim$ (ii) that the Jordan type

$$J_\ell = (s, \underbrace{s-1, \ldots, s-1}_{g_1\text{-times}}, \ldots, \underbrace{s-i, \ldots, s-i}_{g_i\text{-times}}, \ldots, \underbrace{s-\sigma+1, \ldots, s-\sigma+1}_{g_{\sigma-1}\text{-times}}) = \mathbf{H}_A^\vee,$$

as we wished. Therefore, by Lemma 2.2, an Artinian ring has the SLP, which completes the proof. □

The following proposition is immediate from Lemma 2.4.

Proposition 2.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star-configurations in $\mathbb{P}^2$ of type t and s with $t \geq 2$ and $s \geq \binom{t}{2}$. Then an Artinian ring $R/(I_\mathbb{X} + I_\mathbb{Y})$ has the SLP.*

Proof. First, note that the Hilbert functions of $R/I_\mathbb{X}$, $R/I_\mathbb{Y}$, and $R/(I_\mathbb{X} \cap I_\mathbb{Y})$ (see Proposition 2.3) are

$$\begin{array}{lcccccccccl} \mathbf{H}_{R/I_\mathbb{X}} & : & 1 & 3 & \cdots & \overset{(t-2)\text{-nd}}{\binom{t}{2}} & \binom{t}{2} & \to, & & & \\ \mathbf{H}_{R/I_\mathbb{Y}} & : & 1 & 3 & \cdots & \overset{(t-2)\text{-nd}}{\binom{t}{2}} & \binom{t+1}{2} & \cdots & \overset{(s-2)\text{-nd}}{\binom{s}{2}} & \binom{s}{2} & \to, \\ \mathbf{H}_{R/(I_\mathbb{X} \cap I_\mathbb{Y})} & : & 1 & 3 & \cdots & \overset{(t-2)\text{-nd}}{\binom{t}{2}} & \binom{t+1}{2} & \cdots & \overset{(s-2)\text{-nd}}{\binom{s}{2}} & \overset{(s-1)\text{-st}}{\binom{s+1}{2}} = \binom{s}{2} + \binom{t}{2} & \to, \end{array}$$

respectively. Using the exact sequence

$$0 \to R/(I_\mathbb{X} \cap I_\mathbb{Y}) \to R/I_\mathbb{X} \oplus R/I_\mathbb{Y} \to R/(I_\mathbb{X} + I_\mathbb{Y}) \to 0,$$

the Hilbert function of $R/(I_\mathbb{X} + I_\mathbb{Y})$ is

$$\mathbf{H}_{R/(I_\mathbb{X}+I_\mathbb{Y})} \quad : \quad 1 \quad 3 \quad \cdots \quad \overset{(t-2)\text{-nd}}{\binom{t}{2}} \quad \cdots \quad \overset{(s-2)\text{-nd}}{\binom{t}{2}} \quad 0 \quad \to,$$

and so by Lemma 2.4, an Artinian linear star configuration quotient $R/(I_\mathbb{X}+I_\mathbb{Y})$ has the SLP, which completes the proof. □

Example 2.6. Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star-configurations in $\mathbb{P}^2$ of type 5 and 9, respectively. Note that $9 = \binom{5}{2} - 1$. By Proposition 2.3 the Hilbert function of an Artinian ring $A := R/(I_\mathbb{X} + I_\mathbb{Y})$ is

$$(1, 3, 6, 10, 10, 10, 10, 10, \overset{8\text{-th}}{1}).$$

(a) By Waring decomposition, there is a general linear form $\ell \in [A]_1$ such that

$$\ell^8 \in [A]_8,$$

i.e., we have a string of length 9

$$1, \ell, \dots, \ell^8.$$

Hence the Jordan type J_ℓ is of the form

$$J_\ell = (9, \dots).$$

(b) Note that the multiplication map by ℓ^6

$$[A]_1 \overset{\times \ell^6}{\to} [A]_7$$

is injective, and the multiplication map by ℓ^7

$$[A]_1 \overset{\times \ell^7}{\to} [A]_8$$

is surjective. Then we can choose a basis $\{\ell, F_{1,1}, F_{1,2}\}$ for $[A]_1$ such that

$$F_{1,1}\ell^6, F_{1,2}\ell^6 \neq 0, \quad \text{and} \quad F_{1,1}\ell^7, F_{1,2}\ell^7 = 0.$$

Moreover, since $\{F_{1,1}\ell^6, F_{1,2}\ell^6\}$ is linearly independent, we have 2-strings of length 7

$$\begin{aligned} &F_{1,1}, F_{1,1}\ell, \dots, F_{1,1}\ell^6, \quad \text{and} \\ &F_{1,2}, F_{1,2}\ell, \dots, F_{1,2}\ell^6. \end{aligned}$$

(c) Note that the multiplication map by ℓ^5

$$[A]_2 \overset{\times \ell^5}{\to} [A]_7$$

is injective, and the multiplication map by ℓ^6

$$[A]_2 \overset{\times \ell^6}{\to} [A]_8$$

is surjective. Then we can choose a basis $\{\ell^2, F_{1,1}\ell, F_{1,2}\ell, F_{2,1}, F_{2,2}, F_{2,3}\}$ for $[A]_2$ such that

$$F_{2,1}\ell^5, F_{2,2}\ell^5, F_{2,3}\ell^5 \neq 0, \quad \text{and} \quad F_{2,1}\ell^6, F_{2,2}\ell^6, F_{2,3}\ell^6 = 0.$$

Moreover, since $\{F_{2,1}\ell^5, F_{2,2}\ell^5, F_{2,3}\ell^5\}$ is linearly independent, we have 3-strings of length 6

$$\begin{aligned} &F_{2,1}, F_{2,1}\ell, \dots, F_{2,1}\ell^5, \\ &F_{2,2}, F_{2,2}\ell, \dots, F_{2,2}\ell^5, \quad \text{and} \\ &F_{2,3}, F_{2,3}\ell, \dots, F_{2,3}\ell^5. \end{aligned}$$

(d) Note that the multiplication map by ℓ^4

$$[A]_3 \overset{\times \ell^4}{\to} [A]_7$$

is injective, and the multiplication map by ℓ^6

$$[A]_3 \overset{\times \ell^5}{\to} [A]_8$$

is surjective. Then we can choose a basis $\{\ell^3, F_{1,1}\ell^2, F_{1,2}\ell^2, F_{2,1}\ell, F_{2,2}\ell, F_{2,3}\ell, F_{3,1}, \dots, F_{3,4}\}$ for $[A]_3$ such that

$$F_{3,1}\ell^4, \dots, F_{3,4}\ell^4 \neq 0, \quad \text{and} \quad F_{3,1}\ell^5, \dots, F_{3,4}\ell^5 = 0.$$

Moreover, since $\{F_{3,1}\ell^4, \dots, F_{3,4}\ell^4\}$ is linearly independent, we have 4-strings of length 5

$$\begin{array}{l} F_{3,1}, F_{3,1}\ell, \dots, F_{3,1}\ell^4, \\ F_{3,2}, F_{3,2}\ell, \dots, F_{3,2}\ell^4, \\ F_{3,3}, F_{3,3}\ell, \dots, F_{3,3}\ell^4, \quad \text{and} \\ F_{3,4}, F_{3,4}\ell, \dots, F_{3,4}\ell^4. \end{array}$$

This shows that the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is

$$J_\ell = (9, 7, 7, 6, 6, 6, 5, 5, 5, 5) = \mathbf{H}^\vee_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}.$$

Thus, by Lemma 2.2, an Artinian quotient of two linear star-configurations in $\mathbb{P}^2$ of type 5 and 9 has the SLP, as we wished.

Example 2.6 motivates the following proposition.

Proposition 2.7. *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^n$ and let A be an Artinian quotient of the coordinate ring of $\mathbb{X}$. Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for every $0 \le i \le s-2$ with $A_s = 0$, and the Hilbert function of A is of the form*

$$\mathbf{H}_A \quad : \quad h_0 \quad h_1 \quad \cdots \quad h_{\sigma-1} \quad h_\sigma \quad \cdots \quad \overset{(s-2)\text{-}nd}{h_\sigma} \quad h_{s-1} \quad 0$$

where $h_{\sigma-2} < h_{\sigma-1} = h_\sigma$ and $h_{s-1} = 1$. Then an Artinian ring A has the SLP.

Proof. We first define

$$g_i := h_i - h_{i-1} \quad \text{for} \quad i = 1, \dots, \sigma-1.$$

(a) By Waring decomposition, there is a linear form $\ell \in [A]_1$ such that

$$\ell^{s-1} \in [A]_{s-1}.$$

In other words, there is a string of length s as

$$1, \ell, \dots, \ell^{s-1}.$$

Hence Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_\ell = (s, \dots).$$

(b) Note that the multiplication map by ℓ^{s-3}

$$[R/I_{\mathbb{X}}]_1 = [A]_1 \overset{\times \ell^{s-3}}{\to} [A]_{s-2} = [R/I_{\mathbb{X}}]_{s-2}$$

is injective, and the multiplication map by ℓ^{s-2}

$$[A]_1 \overset{\times \ell^{s-2}}{\to} [A]_{s-1}$$

is surjective. Then we can choose a basis $\{\ell, F_{1,1}, F_{1,2}, \dots, F_{1,g_1}\}$ for $[A]_1$ such that

$$F_{1,1}\ell^{s-3}, F_{1,2}\ell^{s-3}, \dots, F_{1,g_1}\ell^{s-3} \neq 0, \text{ and } F_{1,1}\ell^{s-2}, F_{1,2}\ell^{s-2}, \dots, F_{1,g_1}\ell^{s-2} = 0.$$

Moreover, since $\{F_{1,1}\ell^{s-3}, F_{1,2}\ell^{s-3}, \dots, F_{1,g_1}\ell^{s-3}\}$ is linearly independent, we have g_1-strings of length $(s-2)$

$$\begin{array}{rcl} F_{1,1}, F_{1,1}\ell, & \dots, & F_{1,1}\ell^{s-3}, \\ F_{1,2}, F_{1,2}\ell, & \dots, & F_{1,2}\ell^{s-3}, \\ & \vdots & \\ F_{1,g_1-1}, F_{1,g_1-1}\ell, & \dots, & F_{1,g_1-1}\ell^{s-3}, \quad \text{and} \\ F_{1,g_1}, F_{1,g_1}\ell, & \dots, & F_{1,g_1}\ell^{s-3}. \end{array}$$

This means that Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_\ell = (s, \underbrace{s-2, \dots, s-2}_{g_1\text{-times}}, \dots).$$

(c) Let $1 \leq i \leq \sigma - 1$. Note that the multiplication map by ℓ^{s-i-2}

$$[R/I_{\mathbb{X}}]_i = [A]_i \overset{\times \ell^{s-i-2}}{\to} [A]_{s-2} = [R/I_{\mathbb{X}}]_{s-2}$$

is injective, and the multiplication map by ℓ^{s-i-1}

$$[R/I_{\mathbb{X}}]_i = [A]_i \overset{\times \ell^{s-i-1}}{\to} [A]_{s-1}$$

is surjective. Then we can choose a basis $\mathcal{B}_i$

$$\begin{aligned} \mathcal{B}_i \quad = \quad & \{\ell^i, \underbrace{F_{1,1}\ell^{i-1}, \dots, F_{1,g_1}\ell^{i-1}}_{g_1\text{-times}}, \underbrace{F_{2,1}\ell^{i-2}, \dots, F_{2,g_2}\ell^{i-2}}_{g_2\text{-times}}, \dots, \\ & \underbrace{F_{i-1,1}\ell, \dots, F_{i-1,g_{i-1}}\ell}_{g_{i-1}\text{-times}}, \underbrace{F_{i,1}, \dots, F_{i,g_i}}_{g_i\text{-times}}\} \end{aligned}$$

for $[A]_i$ such that

$$F_{i,1}\ell^{s-i-2}, \dots, F_{i,g_i}\ell^{s-i-2} \neq 0, \quad \text{and} \quad F_{i,1}\ell^{s-i-1}, \dots, F_{i,g_i}\ell^{s-i-1} = 0.$$

Moreover, since $\{F_{i,1}\ell^{s-i-2}, \dots, F_{i,g_i}\ell^{s-i-2}\}$ is linearly independent, we have g_i-strings of length $(s-i-1)$

$$\begin{array}{rcl} F_{i,1}, F_{i,1}\ell, & \dots, & F_{i,1}^{s-i-2}, \\ F_{i,2}, F_{i,2}\ell, & \dots, & F_{i,2}^{s-i-2}, \\ & \vdots & \\ F_{i,g_1-1}, F_{i,g_1-1}\ell, & \dots, & F_{i,g_1-1}^{s-i-2}, \qquad \text{and} \\ F_{i,g_i}, F_{i,g_i}\ell, & \dots, & F_{i,g_i}\ell^{s-i-2}. \end{array}$$

Hence Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_\ell \quad = \quad (s, \underbrace{s-2, s-2, \dots, s-2}_{g_1\text{-times}}, \dots, \underbrace{s-i-1, s-i-1, \dots, s-i-1}_{g_i\text{-times}}, \dots)$$

for such i.

It is from (a) $\sim$ (c) that the Jordan type J_ℓ of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is

$$\begin{aligned} J_\ell &= \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}{}^\vee \\ &= \big(s, \underbrace{s-2, s-2, \ldots, s-2}_{g_1\text{-times}}, \ldots, \underbrace{s-i-1, s-i-1, \ldots, s-i-1}_{g_i\text{-times}}, \ldots, \\ &\quad \underbrace{s-\sigma, s-\sigma, \ldots, s-\sigma}_{g_{\sigma-1}\text{-times}}\big). \end{aligned}$$

Therefore, by Lemma 2.2, an Artinian ring $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ has the SLP, as we wished. $\square$

The following two corollaries are immediate from Proposition 2.7.

Corollary 2.8. *Let $\mathbb{X}$ and $\mathbb{Y}$ be finite sets of general points in $\mathbb{P}^n$ with $n \geq 2$ and $s \geq t \geq n$. Assume that*

$$\binom{s}{n} \leq \deg(\mathbb{X}) < \binom{s+1}{n}, \quad \binom{t}{n} \leq \deg(\mathbb{Y}) < \binom{t+1}{n},$$

and

$$\deg(\mathbb{X}) + \deg(\mathbb{Y}) = \binom{s+1}{n} + 1.$$

Then an Artinian ring $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ has the SLP.

Proof. Since $\mathbb{X}$ and $\mathbb{Y}$ are finite sets of general points in $\mathbb{P}^n$, we get that the Hilbert functions of $R/I_{\mathbb{X}}$, $R/I_{\mathbb{Y}}$, and $R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})$ are

$$\begin{array}{llllllllll} \mathbf{H}_{R/I_{\mathbb{X}}} & : & 1 & \binom{1+n}{n} & \cdots & \overset{(t-n)\text{-th}}{\binom{t}{n}} & \binom{t+1}{n} & \cdots & \overset{(s-n)\text{-th}}{\binom{s}{n}} & \deg(\mathbb{X}) & \rightarrow, \\ \mathbf{H}_{R/I_{\mathbb{Y}}} & : & 1 & \binom{1+n}{n} & \cdots & \overset{(t-n)\text{-th}}{\binom{t}{n}} & \deg(\mathbb{Y}) & \cdots & \deg(\mathbb{Y}) & \deg(\mathbb{Y}) & \rightarrow, \\ \mathbf{H}_{R/(I_{\mathbb{X}}\cap I_{\mathbb{Y}})} & : & 1 & \binom{1+n}{n} & \cdots & \overset{(t-n)\text{-th}}{\binom{t}{n}} & \binom{t+1}{n} & \cdots & \overset{(s-n)\text{-th}}{\binom{s}{n}} & \binom{s+1}{n} = \big[\deg(\mathbb{X}) + \deg(\mathbb{Y})\big] - 1 & \rightarrow, \end{array}$$

respectively. Using the exact sequence

$$0 \rightarrow R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}}) \rightarrow R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \rightarrow R/(I_{\mathbb{X}}+I_{\mathbb{Y}}) \rightarrow 0,$$

the Hilbert function of $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} \;:\; 1 \quad 3 \quad \cdots \quad \overset{(t-n)\text{-th}}{\binom{t}{n}} \quad \deg(\mathbb{Y}) \quad \cdots \quad \overset{(s-n)\text{-th}}{\deg(\mathbb{Y})} \quad 1 \quad \rightarrow,$$

and so by Proposition 2.7, an Artinian ring $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ has the SLP, which completes the proof. $\square$

Corollary 2.9. *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star-configurations in $\mathbb{P}^2$ of type s and t with $s \geq \binom{t}{2} - 1$ and $t \geq 3$. Then an Artinian linear star-configuration quotient $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ has the SLP.*

Proof. By Proposition 2.5, it holds for $s \geq \binom{t}{2}$. So we assume that $s = \binom{t}{2} - 1$. First note that

$$\begin{aligned}\big[\deg(\mathbb{X}) + \deg(\mathbb{Y})\big] - \binom{s+1}{2} &= \left[\binom{s}{2} + \binom{t}{2}\right] - \binom{s+1}{2} \\ &= \left[\binom{s}{2} + s + 1\right] - \binom{s+1}{2} = 1.\end{aligned}$$

Hence the Hilbert functions of $R/I_{\mathbb{X}}$, $R/I_{\mathbb{Y}}$, and $R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})$ (see Proposition 2.3) are

$$\begin{array}{lcccccccccc}
\mathbf{H}_{R/I_{\mathbb{X}}} & : & 1 & 3 & \cdots & \overset{(t-2)\text{-nd}}{\binom{t}{2}} & \binom{t+1}{2} & \cdots & \overset{(s-2)\text{-nd}}{\binom{s}{2}} & \binom{s}{2} & \rightarrow, \\
\mathbf{H}_{R/I_{\mathbb{Y}}} & : & 1 & 3 & \cdots & \overset{(t-2)\text{-nd}}{\binom{t}{2}} & \binom{t}{2} & \cdots & \binom{t}{2} & \binom{t}{2} & \rightarrow, \\
\mathbf{H}_{R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})} & : & 1 & 3 & \cdots & \overset{(t-2)\text{-nd}}{\binom{t}{2}} & \binom{t+1}{2} & \cdots & \overset{(s-2)\text{-nd}}{\binom{s}{2}} & \overset{(s-1)\text{-st}}{\binom{s+1}{2}} = \left[\binom{s}{2} + \binom{t}{2}\right] - 1 & \rightarrow,
\end{array}$$

respectively. Using the exact sequence

$$0 \to R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}}) \to R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \to R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \to 0,$$

the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\begin{array}{lcccccccc}
\mathbf{H}_{R/(I_{\mathbb{X}} + I_{\mathbb{Y}})} & : & 1 & 3 & \cdots & \overset{(t-2)\text{-nd}}{\binom{t}{2}} & \cdots & \overset{(s-2)\text{-nd}}{\binom{t}{2}} & 1 & \rightarrow,
\end{array}$$

and so by Proposition 2.7, an Artinian linear star-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP, as we wished. $\square$

3. Artinian $\Bbbk$-configuration quotients in $\mathbb{P}^2$

In this section, we shall introduce another Artinian quotient having the SLP. We first recall a definition of a $\Bbbk$-configuration in $\mathbb{P}^2$ and some preliminary result.

Definition 3.1. A $\Bbbk$-configuration of points in $\mathbb{P}^2$ is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfy the following conditions: there exist integers $1 \leq d_1 < \cdots < d_m$, and subsets $\mathbb{X}_1, \ldots, \mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \ldots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that

(a) $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{X}_i$,
(b) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \ldots, m$, and
(c) $\mathbb{L}_i$ $(1 < i \leq m)$ does not contain any points of $\mathbb{X}_j$ for all $j < i$.

In this case, the $\Bbbk$-configuration in $\mathbb{P}^2$ is said to be of type $(d_1, \ldots, d_m)$.

Recall that a finite complete intersection set of points $\mathbb{Z}$ in $\mathbb{P}^n$ is said to be a *basic configuration* in $\mathbb{P}^n$ (see [11, 12]) if there exist integers $r_1, \ldots, r_n$ and distinct hyperplanes $\mathbb{L}_{ij} (1 \leq i \leq n, 1 \leq j \leq r_i)$ such that

$$\mathbb{Z} = \mathbb{H}_1 \cap \cdots \cap \mathbb{H}_n \text{ as schemes, where } \mathbb{H}_i = \mathbb{L}_{i1} \cup \cdots \cup \mathbb{L}_{ir_i}.$$

In this case $\mathbb{Z}$ is said to be of type $(r_1, \ldots, r_n)$.

Before we prove our main theorem, we first introduce two lemmas.

Lemma 3.2. *Let $\mathbb{X}$ be a $\Bbbk$-configuration in $\mathbb{P}^2$ of type $(1, 2, \ldots, d)$ (see Figure 1), and let $\mathbb{L}_i$ and $\mathbb{M}_j$ be lines in $\mathbb{P}^2$ defined by linear forms $x_0 - (i-1)x_2$ and $x_1 - (j-1)x_2$ for $1 \le i, j \le d-1$, respectively. Then the multiplication map by $L_1 := x_0$*

$$[R/I_{\mathbb{X}}]_i \stackrel{\times L_1}{\to} [R/I_{\mathbb{X}}]_{i+1}$$

is injective for $i \ge 0$. In particular, for $j \ge 1$, the multiplication map by L_1^j

$$[R/I_{\mathbb{X}}]_i \stackrel{\times L_1^j}{\to} [R/I_{\mathbb{X}}]_{i+j}$$

is injective for every $i \ge 0$.

$$\begin{array}{cccccc|c}
\bullet & & & & & & \mathbb{L}_d \\
\bullet & \bullet & & & & & \mathbb{L}_{d-1} \\
\vdots & \vdots & \cdot & & & & \vdots \\
\bullet & \bullet & \bullet & \cdot & & & \mathbb{L}_3 \\
\bullet & \bullet & \bullet & \cdots & \bullet & & \mathbb{L}_2 \\
\bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \mathbb{L}_1 \\
\mathbb{M}_1 & \mathbb{M}_2 & \mathbb{M}_3 & \cdots & \mathbb{M}_{d-1} & \mathbb{M}_d &
\end{array}$$

FIGURE 1

Proof. If $d = 1$, then $\mathbb{X}$ is a set of a single point in $\mathbb{P}^2$, so it is immediate. Hence we assume that $d > 1$.

Note that

$I_{\mathbb{X}} = (L_1 \cdots L_d, M_1 L_2 \cdots L_d, M_1 M_2 L_3 \cdots L_d, \ldots, M_1 \cdots M_{d-1} L_d, M_1 M_2 \cdots M_d)$ (see [9, 11]) and the Hilbert function of $R/I_{\mathbb{X}}$ is

$$\mathbf{H}_{\mathbb{X}} \quad : \quad 1 \quad \binom{1+2}{2} \quad \cdots \quad \overset{(d-1)\text{-st}}{\binom{(d-1)+2}{2}} \quad \binom{d+1}{2} \quad \to$$

(see Theorems 2.7 and 3.6 in [9]).

First, it is obvious that the multiplication map by $L_1 := x_0$

$$[R/I_{\mathbb{X}}]_i \stackrel{\times L_1}{\to} [R/I_{\mathbb{X}}]_{i+1}$$

is injective for $0 \le i \le d-2$.

Let $i = d - 1 = j_1 + j_2 + j_3$ with $0 \le j_1, j_2, j_3 \le d$.

(i) Assume $j_2 = 0$ and

$$\begin{aligned} x_0^{j_1} x_2^{j_3} L_1 \in [I_{\mathbb{X}}]_d = \langle & L_1 \cdots L_d, M_1 L_2 \cdots L_d, M_1 M_2 L_3 \cdots L_d, \ldots, \\ & M_1 \cdots M_{d-1} L_d, M_1 M_2 \cdots M_d \rangle, \end{aligned}$$

that is,

$$x_0^{j_1} x_2^{j_3} L_1 = \alpha_1 L_1 \cdots L_d + \alpha_2 M_1 L_2 \cdots L_d + \alpha_3 M_1 M_2 L_3 \cdots L_d + \cdots$$

$$+ \alpha_d M_1 \cdots M_{d-1} L_d + \alpha_{d+1} M_1 M_2 \cdots M_d$$

for some $\alpha_i \in \Bbbk$. Let $\wp_{i,j}$ be a point defined by two linear forms L_i and M_j. Since two linear forms L_1 and M_2 vanish on a point $\wp_{1,2}$, we get that

$$\alpha_2 = 0.$$

Moreover, since two forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we have

$$\alpha_3 = 0.$$

By continuing this procedure, one can show that

$$\alpha_2 = \cdots = \alpha_d = 0.$$

Hence

$$x_0^{j_1} x_2^{j_3} L_1 = \alpha_1 L_1 \cdots L_d + \alpha_{d+1} M_1 M_2 \cdots M_d,$$

that is,

$$L_1 \mid \alpha_{d+1} M_1 M_2 \cdots M_d \quad \text{and so,} \quad \alpha_{d+1} = 0.$$

It follows that

$$x_0^{j_1} x_2^{j_3} L_1 = \alpha_1 L_1 \cdots L_d, \quad \text{and thus,} \quad \alpha_1 = 0.$$

(ii) Assume $j_2 > 0$ and

$$\begin{aligned} x_0^{j_1} x_1^{j_2} x_2^{j_3} L_1 = {} & \alpha_1 L_1 \cdots L_d + \alpha_2 M_1 L_2 \cdots L_d + \alpha_3 M_1 M_2 L_3 \cdots L_d \\ & + \cdots + \alpha_d M_1 \cdots M_{d-1} L_d + \alpha_{d+1} M_1 M_2 \cdots M_d \end{aligned}$$

for some $\alpha_i \in \Bbbk$. Recall that $M_1 := x_1$. Thus

$$M_1 \mid \alpha_1 L_1 \cdots L_d, \quad \text{and hence,} \quad \alpha_1 = 0.$$

By the analogous argument as in (i), one can show that

$$\alpha_2 = \cdots = \alpha_d = \alpha_{d+1} = 0.$$

It is from (i) and (ii) that

$$x_0^{j_1} x_1^{j_2} x_2^{j_3} L_1 \notin [I_{\mathbb{X}}]_d,$$

which means that the multiplication map by L_1

$$[R/I_{\mathbb{X}}]_{d-1} \overset{\times L_1}{\to} [R/I_{\mathbb{X}}]_d$$

is injective, and surjective as well. Thus the multiplication map by L_1

$$[R/I_{\mathbb{X}}]_i \overset{\times L_1}{\to} [R/I_{\mathbb{X}}]_{i+1}$$

is injective and surjective for every $i \geq d-1$, as we wished.

So it follows that the multiplication map by L_1^j

$$[R/I_{\mathbb{X}}]_i \overset{\times L_1^j}{\to} [R/I_{\mathbb{X}}]_{i+j}$$

is injective for every $i \geq 0$. This completes the proof. $\square$

The following lemma is immediate from Proposition 2.7. But we introduce another elementary proof here.

Lemma 3.3. *Let $\mathbb{X}$ be a $\Bbbk$-configuration in $\mathbb{P}^2$ of type $(1,2)$ in a basic configuration $\mathbb{Z}$ in $\mathbb{P}^2$ of type $(a,2)$ with $a \geq 2$, and let $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$, ($\mathbb{X}$ is a set of solid 3-points in $\mathbb{Z}$ in Figure 2). Then an Artinian $\Bbbk$-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.*

$$\begin{array}{ccccccc}
\bullet & \circ & \circ & \cdots & \circ & \circ & \mathbb{L}_2 \\
\bullet & \bullet & \circ & \cdots & \circ & \circ & \mathbb{L}_1 \\
\mathbb{M}_1 & \mathbb{M}_2 & \mathbb{M}_3 & \cdots & \mathbb{M}_{a-1} & \mathbb{M}_a &
\end{array}$$

FIGURE 2

Proof. First, if $a = 2$, then the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} \quad : \quad 1 \quad 1 \quad 0,$$

(see [12, Theorem 2.1]) and so it follows that $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

Now suppose $a \geq 3$ and assume that $\mathbb{L}_i$ and $\mathbb{M}_j$ are lines defined by linear forms $L_i = x_0 - (i-1)x_2$ and $M_j = x_1 - (j-1)x_2$ for i and j, respectively. Let $\wp_{i,j}$ be a point defined by two linear forms L_i and M_j. Then

$$\begin{aligned}
I_{\mathbb{X}} &= (L_1L_2, L_1M_1, M_1M_2), \\
I_{\mathbb{Y}} &= (L_1L_2, L_2M_3M_4 \cdots M_a, M_2M_3M_4 \cdots M_a)
\end{aligned}$$

(see [9, 11]) and an ideal $I_{\mathbb{X}} + I_{\mathbb{Y}}$ has 5-minimal generators, i.e.,

$$I_{\mathbb{X}} + I_{\mathbb{Y}} = (L_1L_2, L_1M_1, M_1M_2, L_2M_3M_4 \cdots M_a, M_2M_3M_4 \cdots M_a).$$

By [12, Theorem 2.1], the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} \quad : \quad 1 \quad 3 \quad 3 \quad \cdots \quad \overset{(a-2)\text{-nd}}{3} \quad 1 \quad 0 \quad \rightarrow .$$

Note that

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(i) = \mathbf{H}_{R/I_{\mathbb{X}}}(i)$$

for $0 \leq i \leq a-2$.

(i) Assume $x_0L_1^{a-2} = L_1^{a-1} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-1}$. Then

$$\begin{aligned}
x_0L_1^{a-2} = L_1^{a-1} = {} & F_1L_1L_2 + F_2L_1M_1 + F_3M_1M_2 + \beta_1L_2M_3M_4 \cdots M_a \\
& + \beta_2M_2M_3M_4 \cdots M_a
\end{aligned}$$

for some $F_i \in R_{a-3}$ and $\beta_j \in \Bbbk$. Since two linear forms L_1 and M_2 vanish on a point $\wp_{1,2}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = 0$ as well. This means that

$$x_0L_1^{a-2} = L_1^{a-1} = F_1L_1L_2 + F_2L_1M_1 + F_3M_1M_2 \in [I_{\mathbb{X}}]_{a-1},$$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a, \dots).$$

(ii) Similarly, it is from Lemma 3.2 that

$$x_1L_1^{a-3}, x_2L_1^{a-3} \notin [I_{\mathbb{X}}]_{a-2} = [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-2}.$$

Furthermore, it is obvious that two forms $x_1L_1^{a-3}, x_2L_1^{a-3}$ are linearly independent in $[R/(I_{\mathbb{X}} + I_{\mathbb{Y}})]_{a-2} = [R/I_{\mathbb{X}}]_{a-2}$. So it is from (i) and (ii) that the Jordan type J_{L_1} of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is

$$J_{L_1} = \mathbf{H}^{\vee}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} = (a, a-2, a-2).$$

Therefore, by Lemma 2.2, an Artinian $\Bbbk$-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP. □

The following proposition can be obtained using Proposition 2.7. However, we also introduce a different proof here.

Proposition 3.4. *Let $\mathbb{X}$ be a $\Bbbk$-configuration of type $(1,2)$ contained in a basic configuration $\mathbb{Z}$ in $\mathbb{P}^2$ of type (a,b) with $2 \le b \le a$. Define $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$. ($\mathbb{X}$ is a set of solid 3-points in Figure 3.) Then an Artinian $\Bbbk$-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.*

$$\begin{array}{ccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \mathbb{L}_b \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\circ & \circ & \circ & \cdots & \circ & \circ & \mathbb{L}_3 \\
\bullet & \circ & \circ & \cdots & \circ & \circ & \mathbb{L}_2 \\
\bullet & \bullet & \circ & \cdots & \circ & \circ & \mathbb{L}_1 \\
\mathbb{M}_1 & \mathbb{M}_2 & \mathbb{M}_3 & \cdots & \mathbb{M}_{a-1} & \mathbb{M}_a &
\end{array}$$

FIGURE 3

Proof. First, if $a = b = 2$, then it is immediate. If $a \ge 3$ and $b = 2$, by Lemma 3.3 it holds.

Now suppose $a \ge b \ge 3$ and assume that $\mathbb{L}_i$ is a line defined by a linear form $L_i = x_0 - (i-1)x_2$ and $\mathbb{M}_j$ is a line defined by a linear form $M_j = x_1 - (j-1)x_2$ for i and j. Let $\wp_{i,j}$ be a point defined by two linear forms L_i and M_j. Then it is from [9, 11] that

$$I_{\mathbb{X}} = (L_1L_2, L_1M_1, M_1M_2), \quad \text{and}$$
$$I_{\mathbb{Y}} = (L_1L_2\cdots L_b, L_2L_3\cdots L_bM_3\cdots M_a, L_3\cdots L_bM_2M_3\cdots M_a, M_1M_2\cdots M_a).$$

Then an ideal $I_{\mathbb{X}} + I_{\mathbb{Y}}$ has 5-minimal generators, i.e.,

$$I_{\mathbb{X}} + I_{\mathbb{Y}} = (L_1L_2, L_1M_1, M_1M_2, L_2L_3\cdots L_bM_3\cdots M_a, L_3\cdots L_bM_2M_3\cdots M_a),$$

and by [12, Theorem 2.1] the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} \ : \ 1 \quad 3 \quad 3 \quad \cdots \quad 3 \quad \overset{(a+b-4)\text{-st}}{3} \quad 1 \quad 0 \quad \rightarrow .$$

(i) Assume $x_0L_1^{a+b-4} = L_1^{a+b-3} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a+b-3}$. Then

$$\begin{aligned} x_0L_1^{a+b-4} = L_1^{a+b-3} &= F_1L_1L_2 + F_2L_1M_1 + F_3M_1M_2 \\ &\quad + \beta_1L_2L_3\cdots L_bM_3\cdots M_a + \beta_2L_3\cdots L_bM_2M_3\cdots M_a \end{aligned}$$

for some $F_i \in R_{a+b-5}$ and $\beta_j \in \Bbbk$. Since two linear forms L_1 and M_2 vanish on a point $\wp_{1,2}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = 0$ as well. This means that

$$x_0L_1^{a+b-4} = L_1^{a+b-3} = F_1L_1L_2 + F_2L_1M_1 + F_3M_1M_2 \in [I_{\mathbb{X}}]_{a+b-3},$$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+b-2, \dots).$$

(ii) Similarly, it is from Lemma 3.2 that the following 3-forms

$$x_0L_1^{a+b-5}, x_1L_1^{a+b-5}, x_2L_1^{a+b-5}$$

are linearly independent. In particular, the following 2-forms

$$x_1L_1^{a+b-5}, x_2L_1^{a+b-5}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is

$$J_{L_1} = \mathbf{H}^{\vee}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} = (a+b-2, a+b-4, a+b-4).$$

It is from (i) and (ii) with Lemma 2.2 that an Artinian $\Bbbk$-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP, which completes the proof. $\square$

We now slightly extend the previous result.

Lemma 3.5. *Let $\mathbb{X}$ be a $\Bbbk$-configuration of type $(1, 2, 3)$ in a basic configuration $\mathbb{Z}$ in $\mathbb{P}^2$ of type $(a, 3)$ with $a \geq 3$ such that $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$. ($\mathbb{X}$ is a set of solid 6-points in Figure 4.) Then an Artinian $\Bbbk$-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.*

$$\begin{array}{cccccccc} \bullet & \circ & \circ & \circ & \cdots & \circ & & \mathbb{L}_3 \\ \bullet & \bullet & \circ & \circ & \cdots & \circ & & \mathbb{L}_2 \\ \bullet & \bullet & \bullet & \circ & \cdots & \circ & & \mathbb{L}_1 \\ \mathbb{M}_1 & \mathbb{M}_2 & \mathbb{M}_3 & \mathbb{M}_4 & \cdots & \mathbb{M}_a & & \end{array}$$

FIGURE 4

Proof. If $a = 3$, then in Proposition 3.4, $\mathbb{Z}$ is a basic configuration of type $(3, 3)$ and hence, $\mathbb{Y}$ is a set of 6 points, lemma holds. So we suppose that $a > 3$. First note that the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} \quad : \quad 1 \quad 3 \quad 6 \quad \cdots \quad \overset{(a-2)\text{-nd}}{6} \quad 3 \quad 1 \quad 0.$$

We assume that $\mathbb{L}_i$ is a line defined by a linear form $L_i = x_0 - (i-1)x_2$ and $\mathbb{M}_j$ is a line defined by a linear form $M_j = x_1 - (j-1)x_2$ for i and j. Let $\wp_{i,j}$ be a point defined by two linear forms L_i and M_j. Then

$$\begin{aligned} I_{\mathbb{X}} &= (L_1L_2L_3, L_1L_2M_1, L_1M_1M_2, M_1M_2M_3), \quad \text{and} \\ I_{\mathbb{Y}} &= (L_1L_2L_3, L_2L_3M_4\cdots \mathbb{M}_a, L_3M_3M_4\cdots M_a, M_2M_3\cdots M_a). \end{aligned}$$

So an ideal $I_{\mathbb{X}} + I_{\mathbb{Y}}$ has 7-minimal generators, i.e.,

$$\begin{aligned} I_{\mathbb{X}} + I_{\mathbb{Y}} = (&L_1L_2L_3, L_1L_2M_1, L_1M_1M_2, M_1M_2M_3, \\ &L_2L_3M_4\cdots M_a, L_3M_3M_4\cdots M_a, M_2M_3\cdots M_a). \end{aligned}$$

Note that

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(i) = \mathbf{H}_{R/I_{\mathbb{X}}}(i)$$

for $0 \le i \le a-2$.

(i) Assume $x_0L_1^{a-1} = L_1^a \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_a$. Then

$$\begin{aligned} x_0L_1^{a-1} = L_1^a = {}& F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \\ &+ \beta_1L_2L_3M_4\cdots M_a + \beta_2L_3M_3M_4\cdots M_a + \beta_3M_2M_3\cdots M_a \end{aligned}$$

for some $F_i \in R_{a-3}$ and $\beta_j \in \Bbbk$. Since two linear forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$x_0L_1^{a-1} = L_1^a = F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \in [I_{\mathbb{X}}]_a,$$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+1, \dots).$$

(ii) By the analogous argument as in (i), one can show that

$$x_1L_1^{a-2}, x_2L_1^{a-2} \notin [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-1}.$$

We now suppose that

$$\alpha x_1L_1^{a-2} + \beta x_2L_1^{a-2} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-1}$$

for some $\alpha, \beta \in \Bbbk$. Then

$$\begin{aligned} &\alpha x_1L_1^{a-2} + \beta x_2L_1^{a-2} \\ &= F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \\ &\quad + \beta_1L_2L_3M_4\cdots M_a + \beta_2L_3M_3M_4\cdots M_a + \beta_3M_2M_3\cdots M_a \end{aligned}$$

for some $F_i \in R_{a-3}$ and $\beta_j \in \Bbbk$. Since two linear forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$\begin{aligned} &\alpha x_1L_1^{a-2} + \beta x_2L_1^{a-2} \\ &= F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \in [I_{\mathbb{X}}]_{a-1}. \end{aligned}$$

By Lemma 3.2, we get that

$$\alpha x_1 + \beta x_2 = 0, \quad \text{i.e.,} \quad \alpha = \beta = 0,$$

which implies that two forms

$$x_1 L_1^{a-2}, x_2 L_1^{a-2}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+1, a-1, a-1, \dots).$$

(iii) It is from Lemma 3.2 that

$$x_1^2 L_1^{a-4}, x_1 x_2 L_1^{a-4}, x_2^2 L_1^{a-4} \notin [I_{\mathbb{X}}]_{a-2} = [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-2}$$

and the following set of 6-forms

$$\begin{aligned} & \{x_0 L_1^{a-3}, x_1 L_1^{a-3}, x_2 L_1^{a-3}, x_1^2 L_1^{a-4}, x_1 x_2 L_1^{a-4}, x_2^2 L_1^{a-4}\} \\ = & \{x_0^2 L_1^{a-4}, x_0 x_1 L_1^{a-4}, x_0 x_2 L_1^{a-4}, x_1^2 L_1^{a-4}, x_1 x_2 L_1^{a-4}, x_2^2 L_1^{a-4}\} \end{aligned}$$

is linearly independent. In particular, the 3-forms

$$x_1^2 L_1^{a-4}, x_1 x_2 L_1^{a-4}, x_2^2 L_1^{a-4}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+1, a-1, a-1, a-3, a-3, a-3).$$

It is from (i) $\sim$ (iii) that the Jordan type J_{L_1} is

$$J_{L_1} = \mathbf{H}^{\vee}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} = (a+1, a-1, a-1, a-3, a-3, a-3).$$

Therefore, by Lemma 2.2, an Artinian $\Bbbk$-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP. $\square$

Theorem 3.6. *Let $\mathbb{X}$ be a $\Bbbk$-configuration of type $(1,2,3)$ in a basic configuration $\mathbb{Z}$ in $\mathbb{P}^2$ of type (a,b) with $a \geq 4$ and $b \geq 3$, and let $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$. ($\mathbb{X}$ is a set of solid 6-points in Figure 5.) Then an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.*

$$\begin{array}{ccccccc} \circ & \circ & \circ & \circ & \cdots & \circ & \mathbb{L}_b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bullet & \circ & \circ & \circ & \cdots & \circ & \mathbb{L}_3 \\ \bullet & \bullet & \circ & \circ & \cdots & \circ & \mathbb{L}_2 \\ \bullet & \bullet & \bullet & \circ & \cdots & \circ & \mathbb{L}_1 \\ \mathbb{M}_1 & \mathbb{M}_2 & \mathbb{M}_3 & \mathbb{M}_4 & \cdots & \mathbb{M}_a & \end{array}$$

FIGURE 5

Proof. If $b = 3$, then, by Lemma 3.5, it holds. So we suppose that $b > 3$. Note that, by [12, Theorem 2.1], the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} \quad : \quad 1 \quad 3 \quad 6 \quad \cdots \quad \overset{(a+b-5)\text{-nd}}{6} \quad 3 \quad 1 \quad 0.$$

We assume that $\mathbb{L}_i$ is a line defined by a linear form $L_i = x_0 - (i-1)x_2$ and $\mathbb{M}_j$ is a line defined by a linear form $M_j = x_1 - (j-1)x_2$ for i and j. Let $\wp_{i,j}$ be a point defined by two linear forms L_i and M_j. Then

$$\begin{aligned} I_{\mathbb{X}} &= (L_1L_2L_3, L_1L_2M_1, L_1M_1M_2, M_1M_2M_3), \quad \text{and} \\ I_{\mathbb{Y}} &= (L_1L_2\cdots L_b, L_2\cdots L_bM_4\cdots M_a, L_3\cdots L_bM_3\cdots M_a, \\ &\qquad L_4\cdots L_bM_2\cdots M_a, M_1M_2M_3\cdots M_a). \end{aligned}$$

So an ideal $I_{\mathbb{X}} + I_{\mathbb{Y}}$ has 7-minimal generators, i.e.,

$$\begin{aligned} I_{\mathbb{X}} + I_{\mathbb{Y}} = (&L_1L_2L_3,\ L_1L_2M_1,\ L_1M_1M_2,\ M_1M_2M_3, \\ &L_2\cdots L_bM_4\cdots M_a,\ L_3\cdots L_bM_3\cdots M_a,\ L_4\cdots L_bM_2\cdots M_a). \end{aligned}$$

Note that

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(i) = \mathbf{H}_{R/I_{\mathbb{X}}}(i)$$

for $0 \le i \le a+b-5$.

(i) Assume $x_0L_1^{a+b-4} = L_1^{a+b-3} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a+b-3}$. Then

$$\begin{aligned} x_0L_1^{a+b-4} = L_1^{a+b-3} = {} & F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \\ & + \beta_1L_2\cdots L_bM_4\cdots M_a + \beta_2L_3\cdots L_bM_3\cdots M_a \\ & + \beta_3L_4\cdots L_bM_2\cdots M_a \end{aligned}$$

for some $F_i \in R_{a+b-6}$ and $\beta_j \in \Bbbk$. Since two linear forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$\begin{aligned} x_0L_1^{a+b-4} &= L_1^{a+b-3} \\ &= F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \in [I_{\mathbb{X}}]_{a+b-3}, \end{aligned}$$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+b-2, \dots).$$

(ii) By the analogous argument as in (i), one can show that

$$x_1L_1^{a+b-5}, x_2L_1^{a+b-5} \notin [I_{\mathbb{X}}]_{a+b-4} = [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a+b-4}.$$

We now suppose that the following 3-forms

$$\alpha x_0L_1^{a+b-5} + \beta x_1L_1^{a+b-5} + \beta x_2L_1^{a+b-5} \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a+b-4}$$

for some $\alpha, \beta, \gamma \in \Bbbk$, that is,

$$\begin{aligned} &\alpha x_0L_1^{a+b-5} + \beta x_1L_1^{a+b-5} + \beta x_2L_1^{a+b-5} \\ &= F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \end{aligned}$$

$$+ \beta_1 L_2 \cdots L_b M_4 \cdots M_a + \beta_2 L_3 \cdots L_b M_3 \cdots M_a + \beta_3 L_4 \cdots L_b M_2 \cdots M_a$$

for some $F_i \in R_{a+b-6}$ and $\beta_j \in \mathbb{k}$. Since two linear forms L_1 and M_3 vanish on a point $\wp_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$\begin{aligned} &\alpha x_0 L_1^{a+b-5} + \beta x_1 L_1^{a+b-5} + \beta x_2 L_1^{a+b-5} \\ = &F_1 L_1 L_2 L_3 + F_2 L_1 L_2 M_1 + F_3 L_1 M_1 M_2 + F_4 M_1 M_2 M_3 \in [I_{\mathbb{X}}]_{a+b-4}. \end{aligned}$$

Hence, Lemma 3.2, $\alpha = \beta = \gamma = 0$, as we wished. This implies that the 3-forms

$$x_0 L_1^{a+b-5}, x_1 L_1^{a+b-5}, x_2 L_1^{a+b-5}$$

are linearly independent. In particular, the 2-forms

$$x_1 L_1^{a+b-5}, x_2 L_1^{a+b-5}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+b-2, a+b-4, a+b-4, \dots).$$

(iii) It is from Lemma 3.2 that the following 6-forms

$$x_0^2 L_1^{a+b-7}, x_0 x_1 L_1^{a+b-7}, x_0 x_2 L_1^{a+b-7}, x_1^2 L_1^{a+b-7}, x_1 x_2 L_1^{a+b-7}, x_2^2 L_1^{a+b-7}$$

are linearly independent. In particular, the following 3-forms

$$x_1^2 L_1^{a+b-7}, x_1 x_2 L_1^{a+b-7}, x_2^2 L_1^{a+b-7}$$

are linearly independent. Hence the Jordan type of $\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}$ is of the form

$$J_{L_1} = (a+b-2, a+b-4, a+b-4, a+b-6, a+b-6, a+b-6).$$

It is from (i) $\sim$ (iii) that the Jordan type J_{L_1} is

$$\mathbf{H}^{\vee}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} = (a+b-2, a+b-4, a+b-4, a+b-6, a+b-6, a+b-6).$$

Therefore, by Lemma 2.2, an Artinian $\mathbb{k}$-configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP, which completes the proof of this theorem. □

Remark 3.7. Theorem 3.6 has been proved if $\mathbb{X}$ is a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(1,2)$ or $(1,2,3)$ in a basic configuration in $\mathbb{P}^2$. However, if $\mathbb{X}$ is a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(1,2,\dots,d)$ in a basic configuration in $\mathbb{P}^2$ with $d \geq 4$, then it cannot be proved by the same method as in the proof of Theorem 3.6.

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