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RINGS WITH REFLEXIVE IDEALS

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Abstract. Let R be a ring with identity. A right ideal ideal I of a ring R is called *reflexive* (resp. *completely reflexive*) if $aRb \subseteq I$ implies that $bRa \subseteq I$ (resp. if $ab \subseteq I$ implies that $ba \subseteq I$) for any $a, b \in R$. R is called *reflexive* (resp. *completely reflexive*) if the zero ideal of R is a reflexive ideal (resp. a completely reflexive ideal). Let $K(R)$ (called the *reflexive radical* of R) be the intersection of all reflexive ideals of R. In this paper, the following are investigated: (1) Some equivalent conditions on an reflexive ideal of a ring are obtained; (2) reflexive (resp. completely reflexive) property is Morita invariant; (3) For any ring R, we have $K(M_n(R)) = M_n(K(R))$ where $M_n(R)$ is the ring of all n by n matrices over R; (4) For a ring R, we have $K(R)[x] \subseteq K(R[x])$; in particular, if R is quasi-Armendaritz, then R is reflexive if and only if $R[x]$ is reflexive.

1. Introduction and basic definitions

Throughout this paper, all rings are associative with identity unless otherwise specified. Let R be a ring. Let $J(R)$ and $P(R)$ denote the Jacobson radical and the prime radical of R respectively. Denote the n by n full (resp. upper triangular) matrix ring over R by $M_n(R)$ (resp. $U_n(R)$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). $R[x]$ denotes the polynomial ring with an indeterminate x over R.

Mason [7] called a right ideal N of a ring R *reflexive* if $aRb \subseteq N$ implies $bRa \subseteq N$ for $a, b \in R$, and assign the term *completely reflexive* to those N for which $ab \in N$ implies $ba \in N$. If the zero ideal is reflexive (resp. completely reflexive), then R is usually called *reflexive* (resp. *completely reflexive*); while a completely reflexive ring is called *reversible* by Cohn [2] (also refer [6]).

It is obvious that both any prime ideal and semiprime ideal of a ring R is reflexive. However, the converse need not be true by the following examples:

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Example 1. Let $\mathbb{Z}$ be the ring of integers. Then for all non-prime integers n, $n\mathbb{Z}$ are clearly reflexive, but not semiprime ideals of $\mathbb{Z}$.

Example 2. Let $\mathbb{H}$ be the Hamilton quaternion of real numbers. Consider a ring

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{H} \right\}.$$

Then R is a noncommutative local ring with $J^2 \neq 0 = J^3$. Note that $\{R, J, J^2, 0\}$ is the set of all ideals of R, and so all ideals of R are reflexive. But 0 and J^2 are not semiprime ideals of R.

In [7], it was shown that if $1 \in R$, then N is a reflexive right ideal iff whenever A, B are right ideals with $AB \subseteq N$, then $BA \subseteq N$. In section 2, we will show that a right ideal N of a ring R is reflexive iff $AB \subseteq N$ implies $BA \subseteq N$ for any right (left) ideals A, B in R iff $ARB \subseteq N$ implies that $BRA \subseteq N$ for any nonempty subsets A, B of R. Narbonne [8] called a ring R *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. We call a right ideal N of a ring R *semicommutative* if $ab \in N$ implies $aRb \subseteq N$ for $a, b \in R$. It is shown that any completely reflexive ideal of a ring is reflexive (resp. semicommutative).

It was shown in [4, Proposition 2.2] that a ring R is reflexive and semicommutative iff R is reversible (equivalently, completely reflexive). In section 2, we will show that an ideal N of a ring R is reflexive and semicommutative iff N is completely reflexive. We will also show that (1) the reflexive (resp. completely reflexive) property is Morita invariant by obtaining that an ideal N is reflexive (resp. completely reflexive) iff R/N is reflexive (resp. completely reflexive) ring; (2) for given ideals N, I of a ring R with $\{a \in R | aI \subseteq N\} = N$, if N is reflexive in R, then $N \cap I$ is a reflexive ideal of I (as a ring).

We call the intersection of all reflexive ideals of a ring R the $reflexive\ radical$ of R and denote it by $K(R)$. It is evident that $K(R)$ is the smallest reflexive ideal of R. If R has no proper reflexive ideals, then $K(R) = R$. It is clear that $K(R) \subseteq P(R) \subseteq J(R)$ since every prime ideal of R is reflexive and every maximal ideal is prime. In section 3, we will show that (1) if I is an ideal of a ring R such that $I \subseteq K(R)$, then $K(R/I) = K(R)/I$; (2) $K(R)$ is the smallest ideal of R among all ideals J of R satisfying $K(R/J) = 0$; (3) $K(M_n(R)) = M_n(K(R))$; (4) $K(R)[x] \subseteq K(R[x])$.

2. ideal-reversible ideals of rings

Proposition 2.1. *Let N be a right ideal N of a ring R. Then N is reflexive if and only if $IJ \subseteq N$ implies $JI \subseteq N$ for any right ideals I, J of R.*

Proof. Refer [7, Proposition 2.3]. □

Corollary 2.2. *Any reflexive right ideal of a ring is two-sided ideal.*

Proof. Let N be a reflexive right ideal of a ring R. Since N is a right ideal of R, $NR \subseteq N$ for two right ideals N, R of R. Since N is reflexive, $RN \subseteq N$ by Proposition 2.1, and so N is a two-sided ideal of R. □

Proposition 2.3. *For a right ideal N of a ring R, the following are equivalent:*
(1) *N is reflexive;*
(2) *$IJ \subseteq N$ implies $JI \subseteq N$ for any right ideals I, J of R;*
(3) *$IJ \subseteq N$ implies $JI \subseteq N$ for any ideals I, J of R;*
(4) *$ARB \subseteq N$ implies $BRA \subseteq N$ for any nonempty subsets A, B of R.*

Proof. (1) ⇔ (2) It follows from Proposition 2.1.

(4) ⇒ (1) and (4) ⇒ (2) ⇒ (3) are clear.

(1) ⇒ (4) Suppose that N is reflexive. Let A, B be two nonempty subsets of R with $ARB \subseteq N$. Then $aRb \subseteq N$ for any $a \in A$ and $b \in B$, and so $bRa \subseteq N$ by assumption. Thus $BRA = \sum_{a\in A, b\in B} bRa \subseteq N$.

(3) ⇒ (2) Suppose that (3) holds. Let I, J be two right ideals of R with $IJ \subseteq N$. Since N is two-sided ideal of R by Corollary 2.2, $(RI)(RJ) \subseteq (RI)J \subseteq RN \subseteq N$ for some ideals RI, RJ of R, and then $JI \subseteq (RJ)(RI) \subseteq N$ by assumption. □

Corollary 2.4. *For a ring R, the following are equivalent:*
(1) *R is reflexive;*
(2) *$IJ = 0$ implies $JI = 0$ for any right ideals I, J of R;*
(3) *$IJ = 0$ implies $JI = 0$ for any ideals I, J of R;*
(4) *$ARB = 0$ implies $BRA = 0$ for any nonempty subsets A, B of R.*

Proof. It follows form the Proposition 2.3. □

Proposition 2.5. *Let N be a right ideal of a ring R. Then we have the following:*
(1) *If N is completely reflexive, then N is reflexive;*
(2) *If N is completely reflexive, then N is semicommutative.*

Proof. (1) Suppose that $AB \subseteq N$ for any right ideals A, B in R, and let $\alpha \in BA$ be arbitrary. Then $\alpha = \sum_{i=1}^{n} b_i a_i$ where $a_i \in A, b_i \in B$. Since each $a_i b_i \in AB \subseteq N$ and N is completely reflexive, $b_i a_i \in N$, yielding that $\alpha \in N$, and so N is reflexive.

(2) Let $ab \in N$ for $a, b \in R$. Since N is a right ideal of R, $a(br) \in N$ for all $r \in R$, and so $(br)a \in N$ because N is completely reflexive. Thus $bRa \subseteq N$, and so $(bR)(aR) \subseteq N$. Since N is reflexive by (1), we have that $aRb \subseteq (aR)(bR) \subseteq N$. Therefore, N is semicommutative. □

Note that the converses of Proposition 2.5 do not hold by the following examples:

Example 3. Let $\mathbb{Z}_4$ be the rings of integers modulo 4 and $R = Mat_2(\mathbb{Z}_4)$. Let $N = Mat_2(2\mathbb{Z}_4)$ of R be an ideal of R. Note that N is not completely reflexive because $pq \in N$, but $qp \notin N$ for some $p = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, q = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \in R$. Next, we will show that N is reflexive. Since all the two-sided ideals of R are 0, N and R, it is easy to show that N is reflexive by Proposition 2.3.

Example 4. By [4, Example 2.3. (1)], there exists a semicommutative ring R but not reflexive. Hence we can take a semicommutative ring R_1 which is not completely reflexive. Consider $R = R_1 \times R_2$ for some ring R_2, and let $N = \{0\} \times R_2$ be an ideal of R. Note that R/N is isomorphic to R_1. Since R_1 is semicommutative, R/N is semicommutative, and so N is semicommutative by the below Theorem 2.11. On the other hand, since R_1 is not completely reflexive, R/N is not completely reflexive, and then N is not completely reflexive by the below Corollary 2.12.

Corollary 2.6. *For a ring R we have the following:*
(1) *If R is completely reflexive, then R is reflexive;*
(2) *If R is completely reflexive, then R is semicommutative.*

Proof. It follows from Proposition 2.5. □

Corollary 2.7. *Any completely reflexive right ideal of a ring is two-sided ideal.*

Proof. It follows from Corollary 2.2 and Proposition 2.5. □

Proposition 2.8. *Let N be a right ideal of a ring R. Then N is both reflexive and semicommutative if and only if N is completely reflexive.*

Proof. Suppose that N is both reflexive and semicommutative. Let $ab \in N$ for any $a, b \in R$. Since N is semicommutative, $aRb \subseteq N$. Since N is reflexive, $ba \in bRa \subseteq N$, and so N is completely reflexive. The converse follows from Proposition 2.5. □

Corollary 2.9. *A ring R is both reflexive and semicommutative if and only if R is completely reflexive.*

Proof. It follows from Proposition 2.8. □

Theorem 2.10. *N is a reflexive ideal of a ring R if and only if R/N is a reflexive ring.*

Proof. Suppose that N is a reflexive ideal of R. Let I, J be ideals of R/N such that $IJ \subseteq N$, a zero of R/N. Then there exists ideals I_0, J_0 of R such that $I_0, J_0 \supseteq N$ and $I = I_0/N, J = J_0/N$. Since $IJ = (I_0/N)(J_0/N) = (I_0J_0)/N \subseteq N$, $I_0J_0 \subseteq N$. Since N is reflexive, $J_0I_0 \subseteq N$ by Proposition 2.3. Thus $JI = (J_0I_0)/N = N$, which yields that R/N is a reflexive ring.

Suppose that R/N is a reflexive ring. Let A, B be ideals of N such that $AB \subseteq N$. Thus $AB + N = N$. Note that $(A + N)(B + N) \subseteq AB + N = N$, and so $((A+N)/N)((B+N)/N) = (A+N)(B+N)/N = N$. Since R/N is a reflexive ring, $((B+N)/N)((A+N)/N) = (B+N)(A+N)/N) = N$, yielding that $(B+N)(A+N) \subseteq N$, and so $BA \subseteq (B+N)(A+N) \subseteq N$, which means that N is a reflexive ideal of R. □

Theorem 2.11. *N is a semicommutative ideal of a ring R if and only if R/N is a semicommutative ring.*

Proof. Let $\overline{R} = R/N$. Suppose that N is a semicommutative ideal of R. Let $\overline{a}\overline{b} = \overline{0}(= N)$, the zero of R/N, for $\overline{a} = a+N, \overline{b} = b+N \in R/N$. Let $\overline{r} = r+N \in R/N$ be arbitrary. Since N is semicommutative and $ab \in N$, $arb \in N$, and so $\overline{arb} = (\overline{a})(\overline{r})(\overline{b}) = \overline{0}$, i.e., $\overline{a}\overline{R}\overline{b} = \overline{0}$. Thus $\overline{R}$ is a semicommutative ring.

Suppose that $\overline{R}$ is a semicommutative ring. Let $ab \in N$ for $a, b \in R$ and $r \in R$ be arbitrary. Then $\overline{a}\overline{b} = \overline{0}$. Since $\overline{R}$ is semicommutative, $(\overline{a})(\overline{r})(\overline{b}) = \overline{0}$, and so $arb \in N$, i.e., $aRb \subseteq N$. Thus N is a semicommutative ideal of R. □

Corollary 2.12. *For an ideal N of a ring R, N completely reflexive if and only if R/N is a completely reflexive ring.*

Proof. It follows from Proposition 2.5, Theorem 2.10 and Theorem 2.11. □

Theorem 2.13. *Let N be an ideal of a ring R. Then we have the following:*

(1) *If N is reflexive in R, then so is eNe in eRe for each $e^2 = e \in R$.*

(2) *N is reflexive in R if and only if $M_n(N)$ is reflexive in $M_n(R)$ for all $n \geq 1$.*

Proof. (1) Suppose that N is reflexive in R. Let $a, b \in eRe$ such that $a(eRe)b \subseteq eNe$. Since $a(eRe)b \subseteq eNe \subseteq N$ and N is reflexive, we have that $b(eRe)a \subseteq N$, and clearly $b(eRe)a \subseteq eNe$, and so eNe is reflexive in eRe.

(2) Suppose that N is reflexive in R. Let A, B be ideals of $M_n(R)$ such that $AB \subseteq M_n(N)$. Note that there exist ideals I, J such that $A = M_n(I), B = M_n(J)$. Note that $AB = M_n(I)M_n(J) = M_n(IJ)$ and then $IJ \subseteq N$. Since N is reflexive, $JI \subseteq N$, and so $BA = M_n(J)M_n(I) = M_n(JI) \subseteq M_n(N)$. Thus $M_n(N)$ is reflexive in $M_n(R)$.

Conversely, if $M_n(N)$ is reflexive in $M_n(R)$, then $e_{11}M_n(N)e_{11}$ is reflexive in $e_{11}M_n(R)e_{11}$ by (1) where e_{11} is the matrix in $M_n(R)$ with (1,1)-entry 1 and elsewhere 0. Since $N \cong e_{11}M_n(N)e_{11}$ and $R \cong e_{11}M_n(R)e_{11}$, N is reflexive in R. □

Corollary 2.14. *Let R be a ring. Then we have the following:*

(1) *If R is reflexive, then so is eRe for each $e^2 = e \in R$.*

(2) *R is reflexive if and only if $M_n(R)$ is reflexive for all $n \geq 1$.*

Proof. It follows from Theorem 2.13. □

Remark 1. Let N be an ideal of a ring R. By the similar argument given in the proof of Theorem 2.13, we have that (1) if N is a completely reflexive ideal of R, then so is eNe in eRe for each $e^2 = e \in R$; (2) N is completely reflexive in R if and only if $M_n(N)$ is completely reflexive in $M_n(R)$ for all $n \geq 1$.

Corollary 2.15. *Let N be a reflexive ideal of a ring R. Then $\overline{e}\overline{R}\overline{e}$ is reflexive for an idempotent $\overline{e} \in \overline{R}$ where $\overline{e} = e + N$ and $\overline{R} = R/N$.*

Proof. It follows from Theorem 2.10 and Corollary 2.14. □

Proposition 2.16. *If N is a semicommutative ideal of a ring R, then so is eNe in eRe for each $e^2 = e \in R$.*

Proof. Let $a, b \in eRe$ such that $ab \subseteq eNe$. Since N is semicommutative, $aRb \in N$, and then $aRb = e(aRb)e \subseteq eNe$, yielding that eNe is semicommutative. □

Even though the reflexive (resp. completely reflexive) property of any ideal of a ring is Morita invariant by Theorem 2.13 (resp. Remark 1), the semicommutative property of any ideal of a ring does not satisfy Morita invariant property by the following example:

Example 5. Let $\mathbb{Z}_4$ be the rings of integers modulo 4 and $R = Mat_2(\mathbb{Z}_4)$. Then clearly, $2\mathbb{Z}_4$ is a semicommutative ideal of $\mathbb{Z}_4$. Observe that the ideal $N = Mat_2(2\mathbb{Z}_4)$ of R is not semicommutative. Indeed, take $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \in R$. Then $ab = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \in N$, but $arb = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \notin N$ for some $r = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \in R$.

Proposition 2.17. *Let I be a reflexive right ideal of a ring R. If I is semiprime (as a ring without identity), then R is a reflexive ring.*

Proof. Since I is reflexive, I is two-sided ideal of R by Corollary 2.2, and then R/I is a reflexive ring by Theorem 2.10. Suppose that $aRb = 0$ for $a, b \in R$. Then $\overline{a}\overline{R}\overline{b} = \overline{0} = I$ where $\overline{R} = R/I$, $\overline{a} = a + I, \overline{b} = b + I \in \overline{R}$. Since $\overline{R}$ is a reflexive ring, $\overline{b}\overline{R}\overline{a} = \overline{0}$, and so $bRa \subseteq I$. Note that $(bRa)^2 = (bRa)(bRa) = 0$ because $ab = 0$, yielding that $bRa = 0$ because $bRa \subseteq I$ and I is semiprime, and so R is a reflexive ring. □

Note that a subring of a reflexive ring could not be reflexive by the following example:

Example 6. Let R be a reflexive ring and consider $U_2(R)$ (2×2 upper triangular matrix ring over R), which is a subring of $Mat_2(R)$. By Corollary 2.14, $Mat_2(R)$ is a reflexive ring. Let $A = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & R \\ 0 & R \end{pmatrix}$ be two ideals of $U_2(R)$. Since $AB \neq 0 = BA$, $U_2(R)$ is not reflexive.

Now we raise a question:

Question 1. If N is a reflexive ideal of a ring R, then is $N \cap I$ a reflexive ideal of I (as a ring) for any ideal I of R?

The answer is negative by the following example:

Example 7. Let $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$ be a ring and $R_{11} = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$ be the minor matrix of R obtained by crossing out first row and first column of R. Then all the nonzero ideals of R_{11} are obtained as follows:

$P = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$, $K_n = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & n\mathbb{Z} \end{pmatrix}$ and $T_n = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & n\mathbb{Z} \end{pmatrix}$ for all positive integers n.

By the simple computation, we have the following:

$$\begin{gathered} PJ = PK_n = JK_n = JT_n = T_nJ = J - (1), \\ JP = K_nP = K_nJ = 0 - (2), \\ PT_n = T_nP = P - (3), \\ K_mK_n = K_mT_n = T_mK_n = K_{mn}, \\ T_mT_n = T_nT_m = T_{mn} \end{gathered}$$

for all positive integers m, n. We observe that all the nonzero ideals of R_{11} are reflexive by the above equalities, and the smallest reflexive ideal of R_{11} is J.

Next, consider an ideal $N = \begin{pmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{pmatrix}$ of R. Then it is easy to check that N is a reflexive ideal of R because $\begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$ is a reflexive ideal of R_{11}. Let $N_0 = N \cap I = \begin{pmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ where $I = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Observe that N_0 is not a reflexive ideal of R. Indeed, taking two ideals

$A = \begin{pmatrix} 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = I$ of I, we have that $AB = 0$ but $BA = A \nsubseteq N_0$, yielding that N_0 is not a reflexive ideal of I (as a ring).

Let I be a subset of a ring R and N be an ideal of R. The set $\{a \in R | aI \subseteq N\}$ is a left ideal of R, which is actually an ideal if I is a left ideal. The set $\{a \in R | aI \subseteq N\}$ is called the *left annihilator* of I in N and is denoted $ann_\ell(I; N)$. Similarly, the set

$$ann_r(I; N) = \{a \in R | Ia \subseteq N\}$$

is an ideal of R if I is a right ideal. The set $ann_r(I; N)$ is called the *right annihilator* of I in N. It is evident that $N \subseteq ann_\ell(I; N) \cap ann_r(I; N)$. When $ann_\ell(I; N) = ann_r(I; N)$, it is denoted $ann(I; N)$, and called *annihilator* of I in N. In particular, if $N = 0$, then $ann_\ell(I; 0)$ (resp. $ann_r(I; 0)$) is called *left* *annihilator* of I (resp. *right annihilator* of I), and is simply denoted $ann_\ell(I)$ (resp. $ann_r(I)$). When $ann_\ell(I) = ann_r(I)$, it is denoted $ann(I)$.

Lemma 2.18. *Let N, I be ideals of a ring R. If N is reflexive in R, then $ann_r(I; N) = ann_\ell(I; N)$.*

Proof. Let $K_1 = ann_r(I; N), K_2 = ann_\ell(I; N)$. Note that K_1 and K_2 are ideals of R because N and I are ideals of R. Since $IK_1 \subseteq N$ and N is reflexive in R, $K_1 I \subseteq N$, which yields that $K_1 \subseteq K_2$. Similarly, we get $K_2 \subseteq K_1$, and so $K_1 = K_2$, as desired. □

Proposition 2.19. *Let N, I be ideals of a ring R. Suppose that $ann(I; N) \subseteq N$. If N is a reflexive ideal of R, then $N \cap I$ is a reflexive ideal of I (as a ring).*

Proof. By Lemma 2.18, we have that $ann_r(I; N) = ann_\ell(I; N)(= ann(I; N))$. Let A, B be ideals of I such that $AB \subseteq N \cap I$. Clearly $BA \subseteq I$. Consider the case $AB \subseteq N$. Note that $(RA)(IB) \subseteq N$ for some two left ideals RA, IB of R. Since N is a reflexive ideal of R, $(I(BA) \subseteq)(IB)(RA) \subseteq N$, yielding that $BA \subseteq ann(I; N) \subseteq N$ by assumption, and so $BA \subseteq N \cap I$, yielding that $N \cap I$ is a reflexive ideal of I. □

Note that the assumption in Proposition 2.19 is not surplus by the Example 2.22. Indeed, for given ideals $N = \begin{pmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{pmatrix}, I = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ of a ring R as in Example 2.22, we have that N is a reflexive ideal of R but $N \cap I$ is not reflexive ideal of R with $ann(I; N) = \begin{pmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & \mathbb{Z} \end{pmatrix} \nsubseteq N$.

Corollary 2.20. *Let I be any ideal of a ring R. If N is a prime ideal of R, then $N \cap I$ is a reflexive ideal of I (as a ring).*

Proof. Since N is a prime ideal of R, N is a reflexive ideal of R, and so $ann_r(I; N) = ann_\ell(I; N)$ by Lemma 2.18. Let $A = ann(I; N)$. Then $IA \subseteq N$. Since N is a prime ideal of R, $I \subseteq N$ or $A \subseteq N$. If $I \subseteq N$, then $N \cap I = I$ is clearly reflexive in I. If $A \subseteq N$, then $N \cap I$ is reflexive in I (as a ring) by Proposition 2.19. □

3. the reflexive radicals of rings

In this section, we begin with the following Lemma:

Lemma 3.1. (1) *The intersection of two reflexive ideals of a ring R is reflexive.*
(2) *The intersection of all reflexive ideals of a ring R is reflexive.*

Proof. Clear. □

We call the intersection of all reflexive ideals of a ring R the *reflexive radical* of R and denote it by $K(R)$. It is evident that $K(R)$ is the smallest reflexive ideal of R, and R is a ring such that $K(R) = 0$ if and only if R is a reflexive ring.

Corollary 3.2. *For any ring R, $K(R/K(R)) = 0$.*

Proof. Since $K(R)$ is a reflexive ideal of R by Lemma 3.1, $R/K(R)$ is a reflexive ring by Theorem 2.10, and so $K(R/K(R)) = 0$. □

Corollary 3.3. *If $K(R)$ is semiprime (as a ring without identity) for a ring R, then R is reflexive.*

Proof. Since $K(R)$ is a reflexive ideal of R, it follows from Proposition 2.17. □

Example 8. Consider a ring $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$ and $R_{11} = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$ discussed in Example

2.22. Then we have that $K(R_{11}) = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$.

Now we will find $K(R)$. To do this, it is enough to consider the following ideals among all nonzero ideals of R:

$$I_1 = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{pmatrix}, I_3 = \begin{pmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{pmatrix}, I_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that I_1, I_2, I_3 are reflexive ideals of R. But I_4 is not a reflexive ideal of R

because $AB = 0, BA \nsubseteq I_4$ for some ideals $A = \begin{pmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}, B = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{pmatrix}$.

Therefore, we have that $K(R) = I_1 \cap I_2 \cap I_3 = I_3 = \begin{pmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{pmatrix}$.

Now we also raise a question:

Question 2. For an ideal I (as a ring) of a ring R, $K(I) = I \cap K(R)$?

The answer is negative by the following example:

Example 9. Let $R = U_2(F)$ (2×2 upper triangular matrix ring over a field F) and let $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ be an ideal of R. By Example 6, $U_2(F)$ is not a reflexive ring (i.e., the zero ideal of R is not reflexive). Since $R/I \cong F \times F$, which is reflexive, I is reflexive by Theorem 2.10, i.e., $K(I) = 0$. On the other hand, observe that all nonzero ideals of R are I, $I_1 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $I_2 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ which are clearly reflexive. Hence $K(R) = I \cap I_1 \cap I_2 = I$, and so $I \cap K(R) = I \neq 0 = K(I)$.

Lemma 3.4. *Let I, N be ideals of a ring R such that $I \subseteq N$. Then N is a reflexive ideal of R if and only if N/I is a reflexive ideal of a ring R/I.*

Proof. Suppose that N is a reflexive ideal of R. Let $AB \subseteq N/I$ for ideals A, B of R/I. Then $A = A_0/I, B = B_0/I$ for some ideals $A_0, B_0 \supseteq I$ of R. Since $AB = (A_0/I)(B_0/I) = (A_0B_0)/I \subseteq N/I$, we have that $A_0B_0 \subseteq N$, and then $B_0A_0 \subseteq N$ by assumption. Thus $BA = (B_0/I)(A_0/I) = (B_0A_0)/I \subseteq N/I$, yielding that N/I is a reflexive ideal of R/I. Conversely, suppose that N/I is a reflexive ideal of R/I, and let $P_0Q_0 \subseteq N$ for ideals P_0, Q_0 of R. Let $P_1 = P_0 + I, Q_1 = Q_0 + I$ be ideals of R. Since $P_1Q_1 = (P_0 + I)(Q_0 + I) \subseteq P_0Q_0 + I$, $(P_1/I)(Q_1/I) = (P_1Q_1)/I \subseteq (P_0Q_0 + I)/I \subseteq N/I$. Since N/I is a reflexive ideal of R/I, $(B_1/I)(P_1/I) \subseteq N/I$, and so $Q_1P_1 \subseteq N$. Therefore, $Q_0P_0 \subseteq Q_1P_1 \subseteq N$, yielding that N is a reflexive ideal of R. □

Theorem 3.5. *Let R be a ring. Then we have the following:*
(1) *If I is an ideal of R such that $I \subseteq K(R)$, then $K(R/I) = K(R)/I$;*
(2) *If $K(R/J) = 0$ for any ideal J of R, then $J \supseteq K(R)$.*

Proof. (1) Since $K(R)$ is a reflexive ideal of R, $K(R)/I$ is a reflexive ideal of R/I by Lemma 3.4, and so $K(R/I) \subseteq K(R)/I$. To show $K(R)/I \subseteq K(R/I)$, let A be any reflexive ideal of R/I. Then $A = A_0/I$ for some ideal $A_0 \supseteq I$ of R. Since A is a reflexive ideal of R/I, A_0 is a reflexive ideal of R by Lemma 3.4. Thus $K(R) \subseteq A_0$, and so $K(R)/I \subseteq A_0/I$ $(= A)$, yielding that $K(R)/I \subseteq K(R/I)$.

(2) Assume that $K(R) \supset J$ $(K(R) \neq J)$. Since $K(R/J) = 0$, $K(R)/J = K(R/J) = 0$ by (1), and then $K(R) \subseteq J$, which is a contradiction. Hence $J \supseteq K(R)$. □

Theorem 3.6. *For any ring R, we have $K(M_n(R)) = M_n(K(R))$.*

Proof. Let $N = K(R)$. By Lemma 3.1, N is a reflexive ideal of R, and so $M_n(N)$ is a reflexive ideal of $M_n(R)$ by Theorem 2.13. Since $M_n(N)$ is a reflexive ideal of $M_n(R)$, $K(M_n(R)) \subseteq M_n(N)$. Next, we will show that $M_n(N) \subseteq K(M_n(R))$. Let A be any reflexive ideal of $M_n(R)$. Then there exists an ideal A_0 of R such that $A = M_n(A_0)$. Since A is reflexive, A_0 is reflexive by Theorem 2.13, and so $N \subseteq A_0$. Thus $M_n(N) \subseteq M_n(A_0) = A$, yielding that $M_n(N) \subseteq K(M_n(R))$. Therefore, we have $K(M_n(R)) = M_n(K(R))$. □

Proposition 3.7. *For any ring R, we have the following:*
(1) $K(R)[x] \subseteq K(R[x])$;
(2) *If $R[x]$ is reflexive, then R is reflexive.*

Proof. (1) Let $N = K(R)$. It is enough to show that $N[x] \subseteq A$ for any reflexive ideal A of $R[x]$. We note that $A \cap R$ is a reflexive ideal of R. Indeed, if $aRb \subseteq A \cap R$ for $a, b \in R$, then $aR[x]b = (aRb)[x] \subseteq A$, and then $(bRa)[x] = bR[x]a \subseteq A$ because A is reflexive, and so $bRa \subseteq A \cap R$, yielding that $A \cap R$ is reflexive. Since $A \cap R$ is reflexive, $N \subseteq A \cap R \subseteq A$. Therefore, $N[x] \subseteq A$, as desired.

(2) It follows from (1). □

Lambek [5] called a right ideal I of a ring R *symmetric* if $rst \in I$ implies $rts \in I$ for all $r, s, t \in R$. It is obvious that every symmetric right ideal of a unital ring is completely reflexive (and hence reflexive). Note that the converse of (2) of proposition 3.7 could not be true by the following example:

Example 10. ([1, Example 2.4]). Let $P = \mathbb{Z}_2\{a_0, a_1, a_2, b_0, b_1, b_2, c\}$ be the free algebra of polynomials with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Consider an ideal of the ring $\mathbb{Z}_2 + P$, say I, generated by the following elements:

$$a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,$$

$$b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,$$

$$(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2), \text{and}, r_1r_2r_3r_4.$$

where $r, r_1, r_2, r_3, r_4 \in P$. Set $R = (\mathbb{Z}_2+P)/I$. It was shown that R is symmetric (and hence reflexive), but $R[x]$ is not reflexive by considering two ideals A, B of R generated by $a_0 + a_1x + a_2x^2, (b_0 + b_1x + b_2x^2)c$, respectively, satisfying that $AB = 0$ but $0 \neq (b_0 + b_1x + b_2x^2)c(a_0 + a_1x + a_2x^2) \in BA$.

A ring R is called *quasi-Armendariz* [3] provided that $a_iRb_j = 0$ for all i, j whenever $f = \sum_{i=0}^{m} a_ix^i, g = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $fR[x]g = 0$. In [1], V. Camilo et al. have shown that for a quasi-Armendariz ring R, R is ideal-symmetric if and only if $R[x]$ is ideal-symmetric. We also have the following:

Theorem 3.8. *Let R be a quasi-Armendariz ring. Then R is reflexive if and only if $R[x]$ is reflexive.*

Proof. The implication ($\Leftarrow$) holds by Proposition 3.7. To show the reverse inclusion, suppose that R is reflexive. Consider $fR[x]g = 0$ for $f = \sum_{i=0}^{m} a_ix^i, g = \sum_{j=0}^{n} b_jx^j \in R[x]$. Since R is quasi-Armendariz, $a_iRb_j = 0$ for all i, j, and then $b_jRa_i = 0$ for all i, j because R is reflexive. To show $gR[x]f = 0$, consider $ghf \in gR[x]f$ for any arbitrary $h = \sum_{k=0}^{\ell} c_kx^k \in R[x]$. Then $ghf = \sum_{k=0}^{\ell}(gc_kf)x^k$. Note that for each k, $gc_kf = \sum_{t=0}^{m+n} \alpha_tx^t$, where $\alpha_t = \sum_{i,j=0}^{t} b_jc_ka_i$ $(i + j = t)$. Since $b_jc_ka_i \in b_jRa_i = 0$ for all i, j, $\alpha_t = 0$ for each t, and so $gc_kf = 0$ for each k, yielding that $ghf = 0$. Thus $gR[x]f = 0$, and so $R[x]$ is reflexive. □

Corollary 3.9. *If R is a ring such that $R/K(R)$ is quasi-Armendariz, then $K(R)[x] = K(R[x])$.*

Proof. By Proposition 3.7, we have $K(R)[x] \subseteq K(R[x])$. To show the reverse inclusion, let $N = K(R)$. Since N is reflexive, R/N is a reflexive ring by Theorem 2.10. Since R/N is quasi-Armendariz, $(R/N)[x]$ is reflexive by Theorem 3.8. Since $(R/N)[x] \cong R[x]/N[x]$ is reflexive, $N[x]$ is a reflexive ideal of $R[x]$ by Theorem 2.10, and so $N[x] \supseteq K(R[x])$ as desired. □

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