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AN APPROXIMATED EUROPEAN OPTION PRICE UNDER STOCHASTIC ELASTICITY OF VARIANCE USING MELLIN TRANSFORMS

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Abstract. In this paper, we derive a closed-form formula of a second-order approximation for a European corrected option price under stochastic elasticity of variance model mentioned in Kim et al. (2014) [1] [J.-H. Kim, J Lee, S.-P. Zhu, S.-H. Yu, A multiscale correction to the Black-Scholes formula, Appl. Stoch. Model. Bus. 30 (2014)]. To find the explicit-form correction to the option price, we use Mellin transform approaches.

1. The review of stochastic elasticity of variance(SEV) model

This paper is a continuation of the research of Kim et al. [1] on the pricing of a European option under stochastic elasticity of variance(SEV) model. First of all, we review an underlying asset price model given by the following stochastic differential equations(SDEs)

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t^{1-\gamma f(Y_t)} dW_t^x, && (1)\\ dY_t &= \alpha(m - Y_t)dt + \beta dW_t^y && (2)\end{aligned}$$

under a market probability measure, where μ is a return rate, γ, m, α and β are some constants, f is a smooth function with $0 \le c_1 \le f \le c_2 \le \frac{1}{2\gamma}$ for some constants c_1 and c_2 (cf. Karatzas & Shreve, 1991), and W_t^x and W_t^y are correlated Brownian motions. As we can see in Kim et. al. [1], to illustrate a stochastic elasticity of variance(SEV) model, which is an extended version of a constant elasticity of variance(CEV) model, we choose a fast mean-reverting Ornstein-Uhlenbeck(OU) process Y_t. The process Y_t is an ergodic process whose

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typical time to return to the mean level of its long-run distribution. Here, the term "mean reverting" refers to the characteristic time it takes for a process to get back to the mean level of its invariant distribution. We call the α *the rate of mean-reversion.* As mean reversion rate α gets larger, the process Y_t in (1.2) has a tendency to revert the long-run mean level m regardless of the time. We call the model (2.1) mean-reverting OU process because the volatility is a monotonic function of a process Y_t whose drift pulls it towards the mean value m. The volatility is correspondingly pulled towards approximately $f(m)$. In this case, the process Y_t has the invariant distribution which is normal with $\mathcal{N}(m,\nu^2)$, where $\nu^2 = \frac{\beta^2}{2\alpha}$ which can be mentioned by the standard deviation of the invariant distribution of Y_t. Meanwhile, if mean reversion rate α goes to infinity, the underlying asset price X_t is close to the CEV diffusion. In addition, if γ goes to zero, the given model approaches the geometric Brownian motion model. So, we have two small parameters representing the inverse of mean reversion rate α and the parameter γ, that is, ϵ and δ satisfying $\epsilon = \frac{1}{\alpha}$ and $\delta = \gamma^2$, respectively.

No-arbitrage pricing theory describes that option prices have the expectation representation of discounted payoffs in terms of a risk-neutral measure. Since (1.1)-(1.2) represent incomplete markets, there is a chance of more than one equivalent martingale measure. By utilizing ς (the market prices of volatility risk), the processes defined by

$$\begin{aligned} dW_t^{x*} &= dW_t^x + \frac{\mu - r}{\sigma X_t^{-\gamma f(Y_t)}}dt, \\ dW_t^{y*} &= dW_t^y + \varsigma(Y_t)dt, \end{aligned}$$

are standard Brownian motions under a risk-neutral measure Q. Here, ς is assumed to be smooth bounded function of y only. If we assume that $(\frac{\mu-r}{\sigma X_t^{-\gamma f(Y_t)}}, \varsigma(Y_t))$ satisfies the Novikov condition, by using the Girsanov theorem, the above model dynamics can be transformed into

$$dX_t = rX_t dt + \sigma X_t^{1-\gamma f(Y_t)} dW_t^{x*}, \tag{3}$$

$$dY_t = \left[\frac{1}{\epsilon}(m - Y_t) - \frac{1}{\sqrt{\epsilon}}\nu\sqrt{2}\Lambda(Y_t)\right]dt + \frac{1}{\sqrt{\epsilon}}\nu\sqrt{2}\, dW_t^{y,*} \tag{4}$$

under the equivalent martingale measure Q, where r is a risk-free interest rate, the correlation of two standard Brownian motions, $W_t^{x,*}$ and $W_t^{y,*}$ is given by $d\langle W^{x,*}, W^{y,*}\rangle_t = \rho_{xy}dt$ and $\Lambda(Y_t) = \frac{\rho_{xy}(\mu-r)}{f(Y_t)} + \varsigma(Y_t)\sqrt{1-\rho_{xy}{}^2}$.
The risk neutral valuation is the pricing of a contingent claim in the equivalent martingale measure. Hence, the option price under the stochastic elasticity of variance model is evaluated as the expected discounted payoff of the contingent claim under the equivalent martingale measure Q, which is given by the formula

$$P^{\epsilon,\gamma}(t,x,y) = E^Q[e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y],$$

where $h(X_T)$ is the payoff of the option at time T and r is the risk free rate of interest over $[t, T]$.

Then, by the application of the Feynman-Kac formual(cf. Oksendal, [2]), the European put option price with exercise price K at the expiration T expressed by $P(t, x, y)$ has the following partial differential equation(PDE)

$$\begin{aligned}
&\frac{\partial P^{\epsilon,\gamma}}{\partial t} + \frac{1}{2}\sigma^2 x^{2(1-f(y)\sqrt{\delta})}\frac{\partial^2 P^{\epsilon,\gamma}}{\partial x^2} + rx\frac{\partial P^{\epsilon,\gamma}}{\partial x} - rP^{\epsilon,\gamma} \\
&\quad + \frac{1}{\sqrt{\epsilon}}\left(\rho_{xy}\nu\sqrt{2}\sigma x^{1-f(y)\sqrt{\delta}}\frac{\partial^2 P^{\epsilon,\gamma}}{\partial x \partial y} - \nu\sqrt{2}\Lambda(x,y)\frac{\partial P^{\epsilon,\gamma}}{\partial y}\right) \\
&\quad + \frac{1}{\epsilon}\left(\nu^2\frac{\partial^2 P^{\epsilon,\gamma}}{\partial y^2} + (m-y)\frac{\partial P^{\epsilon,\gamma}}{\partial y}\right) = 0,
\end{aligned}$$

where $P^{\epsilon,\gamma}(T, x, y) = h(x) = (K - x)^+$.

Based on the framework introduced by [1], if we use operators defined by

$$\begin{aligned}
&\mathcal{L}_0 := \nu^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \qquad \mathcal{L}_{10} := \nu\sqrt{2}\left(\rho_{xy}x\frac{\partial^2}{\partial x\partial y} - \Lambda(y)\frac{\partial}{\partial y}\right), \\
&\mathcal{L}_{11} := -\sigma\rho_{xy}\nu\sqrt{2}xf(y)\ln x\frac{\partial^2}{\partial x\partial y}, \qquad \mathcal{L}_{12} := \sigma\rho_{xy}\frac{\nu}{\sqrt{2}}xf^2(y)(\ln x)^2\frac{\partial^2}{\partial x\partial y}, \\
&\mathcal{L}_{20} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r\cdot := \mathcal{L}_{BS}, \\
&\mathcal{L}_{21} := -\sigma^2x^2f(y)\ln x\frac{\partial^2}{\partial x^2}, \qquad \mathcal{L}_{22} := \sigma^2x^2f^2(y)(\ln x)^2\frac{\partial^2}{\partial x^2},
\end{aligned} \tag{5}$$

we can rewrite the above equation as follows :

(6)

$$\begin{aligned}
&\mathcal{L}^{\epsilon,\delta}P^{\epsilon,\delta}(t,x,y) = 0, \qquad P^{\epsilon,\delta}(T,x,y) = h(x), \quad t < T, \\
&\mathcal{L}^{\epsilon,\delta} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_{10} + \sqrt{\delta}\mathcal{L}_{11} + \delta\mathcal{L}_{12} + \cdots\right) + \left(\mathcal{L}_{20} + \sqrt{\delta}\mathcal{L}_{21} + \delta\mathcal{L}_{22} + \cdots\right).
\end{aligned}$$

Note that $\mathcal{L}_{20}$ is exactly an operator corresponding to the Black-Scholes model. Notation $\mathcal{L}_{BS}$ may be used instead of $\mathcal{L}_{20}$ for emphasis.

2. European option pricing with stochastic elasticity of variance(SEV) model

In this section, we review European option pricing based on the stochastic elasticity of variance(SEV) model. As you can see in Kim et al. [1], they presented the analytic formula of the correction terms, $P^{\delta}_{1,0}$, $P^{\delta,\epsilon}_{1,1}$ and $P^{\delta}_{2,0}$, which are given by the double integral forms. By using multiscale analysis, the theory of Poisson equation and the change of variable to solve the given partial differential equations(PDEs), they solved the analytic solutions of the leading order term and the correction terms on European put option price under SEV model.

Remark 1 As mentioned in Kim et al. [1], they have derived an asymptotic solution of European put option price $P^{\epsilon,\delta}(t,x,y)$ by using the asymptotic expansion $P^{\epsilon,\delta}(t,x,y)=\sum_{i=0,j=0}^{\infty}\delta^{\frac{i}{2}}\epsilon^{\frac{j}{2}}P_{i,j}(t,x,y)$ and the theory of the Poisson equation. Then, the asymptotic solution of European put option price $P^{\epsilon,\delta}(t,x,y)$ is expressed by

(7)
$$\begin{aligned} P^{\epsilon,\delta}(t,x,y) \approx \tilde{P}(t,x) &= P_{0,0}(t,x)+\sqrt{\delta}P_{1,0}(t,x)+\sqrt{\delta\epsilon}P_{1,1}(t,x)+\delta P_{2,0}(t,x),\\ &= P_{0,0}(t,x)+P^{\delta}_{1,0}(t,x)+P^{\delta,\epsilon}_{1,1}(t,x)+P^{\delta}_{2,0}(t,x),\end{aligned}$$

where $P_{0,0}(t,x)$ is the solution of Black-Scholes PDE satisfying

$$\frac{\partial P_{0,0}}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 P_{0,0}}{\partial x^2}+rx\frac{\partial P_{0,0}}{\partial x}-P_{0,0}=0, \tag{8}$$

then the explicit formula of Black-Scholes put option with the volatility σ given by

$$\begin{aligned} P_{0,0}(t,x) &= Ke^{-r(T-t)}\mathcal{N}(-d_2)-x\mathcal{N}(-d_1),\\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{x}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)\right],\\ d_2 &= d_1-\sigma\sqrt{T-t}.\end{aligned} \tag{9}$$

and $P^{\delta}_{1,0}$, $P^{\delta,\epsilon}_{1,1}$ and $P^{\delta}_{2,0}$ are the solutions that satisfy the following nonhomogenous PDEs:

$$\begin{aligned} \frac{\partial P^{\delta}_{1,0}}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 P^{\delta}_{1,0}}{\partial x^2}+r\left(x\frac{\partial P^{\delta}_{1,0}}{\partial x}-P^{\delta}_{1,0}\right) =& V^{\delta}_{1,0}x^2\ln x\frac{\partial^2 P_{0,0}}{\partial x^2},\\ & P^{\delta}_{1,0}(T,x)=0, 0\le t\le T,\\ \frac{\partial P^{\delta,\epsilon}_{1,1}}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 P^{\delta,\epsilon}_{1,1}}{\partial x^2}+r\left(x\frac{\partial P^{\delta,\epsilon}_{1,1}}{\partial x}-P^{\delta,\epsilon}_{1,1}\right) =& V^{\delta,\epsilon}_{1,1}x\frac{\partial}{\partial x}\left(x^2\ln x\frac{\partial^2 P_{0,0}}{\partial x^2}\right)\\ & +U^{\delta,\epsilon}_{1,1}x^2\ln x\frac{\partial^2 P_{0,0}}{\partial x^2},\\ & P^{\delta,\epsilon}_{1,1}(T,x)=0, 0\le t\le T, \text{and}\\ \frac{\partial P^{\delta}_{2,0}}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 P^{\delta}_{2,0}}{\partial x^2}+r\left(x\frac{\partial P^{\delta}_{2,0}}{\partial x}-P^{\delta}_{2,0}\right) =& V^{\delta}_{2,0}(x\ln x)^2\frac{\partial^2 P_{BS}}{\partial x^2}+U^{\delta}_{2,0}x^2\ln x\frac{\partial^2 P_{1,0}}{\partial x^2},\\ & P^{\delta}_{2,0}(T,x)=0, 0\le t\le T,\end{aligned}$$

respectively, where $V^{\delta}_{1,0}=-\sqrt{\delta}\sigma^2\langle f\rangle$, $V^{\delta,\epsilon}_{1,1}=-\sqrt{\delta\epsilon}\rho\sigma^3\nu\sqrt{2}\langle\psi'\rangle$, $U^{\delta,\epsilon}_{1,1}=\sqrt{\delta\epsilon}\sigma^2\nu\sqrt{2}\langle\Lambda\psi'\rangle$, $V^{\delta}_{2,0}=-\delta\sigma^2\langle f^2\rangle$ and $U^{\delta}_{2,0}=\delta\sigma^2\langle f\rangle$. Also, $P^{\delta}_{1,0}=\sqrt{\delta}P_{1,0}$, $P^{\delta,\epsilon}_{1,1}=\sqrt{\delta\epsilon}P_{1,1}$ and $P^{\delta}_{2,0}=\delta P_{2,0}$.

3. The derivation of the explicit-closed form solutions of the leading order price and the correction terms

In this section, we are trying to find the explicit-form solution of the leading order price $P_{0,0}$, the first correction term $P_{1,0}^{\delta}$ and the second correction terms, $P_{1,1}^{\delta,\epsilon}$ and $P_{2,0}^{\delta}$ on the approximated European put option price described in (2.1). In fact, the existence of the explicit-closed solution is very important not only in analyzing the behavior of the option price with regard to the model parameter but also in implementing option's data fittings from option's market data. This is because it decreases the gap of the relative error between the market option price and the model option price as well as reduces the computing time as the option pricing model is calibrated in practice. Hence, to obtain the explicit-form solution, we utilize Mellin transform techniques.

3.1. A review of the Mellin Transforms

To derive the explicit-form solution of the leading order price and the correction terms, we use the Mellin transform stated in [3]. For a locally Lebesgue integrable function $f(x),\ x \in \mathbb{R}^+$, the Mellin transform $\mathcal{M}(f(x), w),\ w \in \mathbb{C}$ is defined by

$$\mathcal{M}(f(x), w) := \hat{f}(w) = \int_0^{\infty} f(x)x^{w-1}dx,$$

and if $a < Re(w) < b$ and c such that $a < c < b$ exists, the inverse of the Mellin transform is expressed by

$$f(x) = \mathcal{M}^{-1}(\hat{f}(w)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(w)x^{-w}dw.$$

3.2. The derivation of the leading order price $P_{0,0}$

Most of all, the solution of $P_{0,0}$ is the put option price of Black-Scholes model and was given by (2.3). However, we are trying to solve the closed solution with Mellin transform approaches. By denoting the Mellin transform of $P_{0,0}(t,x)$ as $\hat{p}_0(t,w)$, the Black-Scholes equation stated in (2.2) can be transformed into

$$\frac{d\hat{p}_0}{dt} + \left(\frac{\sigma^2}{2}(w^2+w) - rw - r\right)\hat{p}_0 = 0, \tag{10}$$

where $\hat{p}_0(t,w) = \int_0^{\infty} P_{0,0}(t,x)x^{w-1}dx$. Also, the solution of the ODE (3.1) is expressed by $\hat{p}_0(t,w) = \hat{\theta}(w)e^{-\frac{1}{2}\sigma^2 q(w)t}$, where $q(w) = w^2 + (1-k_1)w - k_1$, $k_1 = \frac{2r}{\sigma^2}$ and $\hat{\theta}$ is defined by the Mellin transform of the payoff function $\theta(x) =$

$(K-x)^+$ as follows :

$$\begin{aligned}\hat{\theta}(w) &= \int_0^\infty (K-x)^+ x^{w-1}dx = \int_0^K (K-x)x^{w-1}dx \\ &= \frac{K^{w+1}}{w} - \frac{K^{w+1}}{w+1} = \frac{K^{w+1}}{w(w+1)}\end{aligned}$$

Then, the inverse of the Mellin is

$$P_{0,0}(t,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2 q(w)(T-t)}\hat{\theta}(w)x^{-w}dw, \tag{11}$$

and by applying the formula of integral transform and the convolution property of the Mellin transform as shown in [4], we have the well-known Black-Scholes formula of the European put option as follows :

$$\begin{aligned} P_{0,0}(t,x) &= Ke^{-r(T-t)}\mathcal{N}(-d_2) - x\mathcal{N}(-d_1), \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{X}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)\right], \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

3.3. The derivation of the correction terms $P^\delta_{1,0}$, $P^{\delta,\epsilon}_{1,1}$ and $P^\delta_{2,0}$

Next, to find the solution of $P^\delta_{1,0}(t,x)$ from the following PDE

$$\frac{\partial P^\delta_{1,0}}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P^\delta_{1,0}}{\partial x^2} + r\left(x\frac{\partial P^\delta_{1,0}}{\partial x} - P^\delta_{1,0}\right) = V^\delta_{1,0}x^2 \ln x \frac{\partial^2 P_{0,0}}{\partial x^2},$$
$$P^\delta_{1,0}(T,x) = 0, \quad 0 \le t \le T.$$

As mentioned above in the subsection (3.1), if we apply the inverse Mellin transform to the right-hand side in (3.3), then

$$\begin{aligned} V^\delta_{1,0}x^2 \ln x \frac{\partial^2 P_{0,0}}{\partial x^2} &= \frac{V^\delta_{1,0}x^2\ln x}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2 q(w)(T-t)}\hat{\theta}(w)\frac{\partial^2 x^{-w}}{\partial x^2}dw \\ &= \frac{V^\delta_{1,0}}{2\pi i}\int_{c-i\infty}^{c+i\infty} w(w+1)e^{\frac{1}{2}\sigma^2 q(w)(T-t)}\hat{\theta}(w)x^{-w}(\ln x)dw \\ &= -\frac{V^\delta_{1,0}}{2\pi i}\left[w(w+1)e^{\frac{1}{2}\sigma^2 q(w)(T-t)}\hat{\theta}(w)x^{-w}\right]_{c-i\infty}^{c+i\infty} \\ &+ \frac{V^\delta_{1,0}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\partial w(w+1)e^{\frac{1}{2}\sigma^2 q(w)(T-t)}\hat{\theta}(w)}{\partial w}x^{-w}dw \\ &= \frac{V^\delta_{1,0}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\partial e^{\frac{1}{2}\sigma^2 q(w)(T-t)}K^{w+1}}{\partial w}x^{-w}dw. \end{aligned}$$

Similarly, by taking the inverse Mellin transform to the left-hand side of (3.3), we obtain

$$\frac{d\hat{p}_{10}}{dt} + \left(\frac{\sigma^2}{2}(w^2+w) - rw - r\right)\hat{p}_{10} = V_{1,0}^{\delta}\frac{\partial e^{\frac{1}{2}\sigma^2 q(w)(T-t)}K^{w+1}}{\partial w}, \tag{12}$$

where $\hat{p}_{10}(t,w)$ is the Mellin transform of $P_{1,0}^{\delta}(t,x)$.

Lemma 1 The solution of the following ODE equation

$$\frac{d\hat{p}_{10}}{dt} + \left(\frac{\sigma^2}{2}(w^2+w) - rw - r\right)\hat{p}_{10} = f(t,w) \tag{13}$$

is given by

$$\hat{p}_{10}(t,w) = -\int_t^T f(x,w)e^{\frac{1}{2}\sigma^2 q(w)(x-t)}dx.$$

Therefore, by Lemma 1, we have

$$\begin{aligned}
\hat{p}_{10}(t,w) &= -V_{1,0}^{\delta}\int_t^T \frac{\partial e^{\frac{1}{2}\sigma^2 q(w)(T-x)}K^{w+1}}{\partial w}e^{\frac{1}{2}\sigma^2 q(w)(x-t)}dx \\
&= -V_{1,0}^{\delta}\int_t^T \frac{\partial\left(\frac{1}{2}\sigma^2 q(w)(T-x) + (w+1)\log K\right)}{\partial w}e^{\frac{1}{2}\sigma^2 q(w)(T-x)}K^{w+1}e^{\frac{1}{2}\sigma^2 q(w)(x-t)}dx \\
&= -V_{1,0}^{\delta}\int_t^T \left(\frac{1}{2}\sigma^2(2w+(1-k_1))(T-x) + \ln K\right)e^{\frac{1}{2}\sigma^2 q(w)(T-x)}K^{w+1}e^{\frac{1}{2}\sigma^2 q(w)(x-t)}dx \\
&= -V_{1,0}^{\delta}e^{\frac{1}{2}\sigma^2 q(w)(T-t)}K^{w+1}\int_t^T \left(\frac{1}{2}\sigma^2(2w+(1-k_1))(T-x) + \ln K\right)dx \\
&= -V_{1,0}^{\delta}e^{\frac{1}{2}\sigma^2 q(w)(T-t)}K^{w+1}\left(\frac{1}{4}\sigma^2(2w+(1-k_1))(T-t)^2 + \ln K(T-t)\right).
\end{aligned}$$

Lemma 2 Let $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{f}(w)x^{-w}dw$. For $Re(\alpha) \geq 0$, the inverse Mellin transforms of $\hat{f}(w) = e^{\alpha(w+\beta)^2}$, $\hat{f}(w) = we^{\alpha(w+\beta)^2}$, $\hat{f}(w) = w^2e^{\alpha(w+\beta)^2}$,

$\hat{f}(w) = w^3 e^{\alpha(w+\beta)^2}$ and $\hat{f}(w) = w^4 e^{\alpha(w+\beta)^2}$ are given by the following equations, respectively,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha(w+\beta)^2}x^{-w}dw &= \frac{1}{2}\pi^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}x^{\beta}e^{-\frac{1}{4\alpha}(\ln x)^2}\\
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} we^{\alpha(w+\beta)^2}x^{-w}dw &= \frac{1}{2}\pi^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}\left(\frac{\ln x}{2\alpha}-\beta\right)x^{\beta}e^{-\frac{1}{4\alpha}(\ln x)^2}\\
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^2e^{\alpha(w+\beta)^2}x^{-w}dw &= \frac{1}{2}\pi^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}\left(\frac{(\ln x)^2}{4\alpha^2}-\frac{\beta}{\alpha}\ln x-\frac{1}{2\alpha}+\beta^2\right)x^{\beta}e^{-\frac{1}{4\alpha}(\ln x)^2},\\
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^3e^{\alpha(w+\beta)^2}x^{-w}dw &= \frac{1}{2}\pi^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}\left\{\frac{(\ln x)^3}{8\alpha^3}-\frac{3\beta}{4\alpha^2}(\ln x)^2+\left(\frac{3\beta^2}{2\alpha}-\frac{3}{4\alpha^2}\right)\ln x\right.\\
&\quad\left.+\frac{3\beta}{2\alpha}-\beta^3\right\}x^{\beta}e^{-\frac{1}{4\alpha}(\ln x)^2},\\
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^4e^{\alpha(w+\beta)^2}x^{-w}dw &= \frac{1}{2}\pi^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}\left\{\frac{(\ln x)^4}{16\alpha^4}-\frac{\beta}{2\alpha^3}(\ln x)^3+\left(\frac{3\beta^2}{2\alpha^2}-\frac{3}{4\alpha^2}\right)(\ln x)^2\right.\\
&\quad\left.+\left(\frac{3\beta}{\alpha^2}-\frac{2\beta^2}{\alpha}\right)\ln x+\frac{3}{4\alpha^2}-\frac{3\beta}{\alpha}+\beta^4\right\}x^{\beta}e^{-\frac{1}{4\alpha}(\ln x)^2},
\end{aligned}$$

where β is a complex number.

Proof The proof of the computation of $\hat{f}(w) = e^{\alpha(w+\beta)^2}$ is presented by Yoon [5]. By the similar method, we can obtain the rest of the above folmulas.

Hence, by using the inverse Mellin transform mentioned in the Subsection 3.1 and Lemma 2, we can obtain the solution of the first correction $P_{1,0}^{\delta}(t,x)$ as follows :

$$P_{1,0}^{\delta}(t,x) = \frac{-V_{1,0}^{\delta}\sqrt{T-t}(\ln x+\ln K)}{2\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2(T-t)}\left(\ln x-\ln K-\frac{\sigma^2(T-t)}{2}(1-k_1)\right)^2},$$

where $k_1 = \frac{2r}{\sigma^2}$. By the similar method, if we use the procedure of the derivation of $P_{1,0}^{\delta}(t,x)$, the Subsection 3.1 and the above Lemma 2, the explicit closed-form solutions of the second correction terms, $P_{1,1}^{\delta,\epsilon}$ and $P_{2,0}^{\delta}(t,x)$ on the PDEs of $P_{1,1}^{\delta,\epsilon}$ and $P_{2,0}^{\delta}(t,x)$ mentioned in (2.4) are obtained as follows :

$$\begin{aligned}
P_{1,1}^{\delta,\epsilon}(t,x) &= \frac{1}{\sigma\sqrt{2\pi(T-t)}}\left[V_{1,1}^{\delta,\epsilon}\left\{\frac{1}{2\sigma^2}\left((\ln x)^2-(\ln K)^2\right)-\frac{1}{4}(T-t)(1-k_1)(\ln x+\ln K)\right.\right.\\
&- \left.\left.\frac{1}{2}(T-t)\right\}-U_{1,1}^{\delta,\epsilon}\frac{1}{2}(T-t)(\ln x+\ln K)\right]e^{-\frac{1}{2\sigma^2(T-t)}\left(\ln x-\ln K-\frac{\sigma^2(T-t)}{2}(1-k_1)\right)^2},
\end{aligned}$$

and

$$
\begin{aligned}
P^{\delta}_{2,0}(t,x) \;=\;& \frac{1}{\sigma\sqrt{2\pi(T-t)}}\Bigg[-V^{\delta}_{2,0}\left\{\frac{(T-t)}{3}\Big((\ln x)^2+\ln x\ln K\right.\\
&+\left.(\ln K)^2\Big)-\frac{\sigma^2(T-t)^2}{6}\right\}\left(V^{\delta}_{1,0}\right)^2\left\{\frac{1}{8\sigma^4}(\ln x)^4+\frac{(T-t)k_1}{8\sigma^2}(\ln x)^3\right.\\
&+\Bigg(-\frac{(\ln K)^2}{4\sigma^4}+\frac{k_1(T-t)\ln K}{8\sigma^2}\\
&\qquad+(T-t)^2\left(\frac{(1-k_1)^2}{32}-\frac{(1-k_1)}{16}-\frac{3}{8}\right)\Bigg)(\ln x)^2\\
&+\Bigg(\frac{(\ln K)^3}{2\sigma^4}+\frac{(T-t)(1-3k_1)}{4\sigma^2}(\ln K)^2+\Bigg(\frac{3(1-k_1)^2}{8}-\frac{9(1-k_1)}{16}\\
&\qquad-\frac{k_1}{16}\left(1-k_1+\frac{2\ln K}{\sigma^2(T-t)}\right)\Bigg)(T-t)^2\ln K\\
&-\frac{\sigma^2(T-t)^3}{16}(1-k_1)^2(1+k_1)-\frac{3k_1(T-t)}{8}\Bigg)\ln x+\frac{(\ln K)^4}{8\sigma^4}\\
&-\frac{(T-t)(\ln K)^3}{8\sigma^2}+\left(\frac{(T-t)}{\sigma^2}-\frac{(1-k_1)(1+k_1)(T-t)^2}{32}\right)(\ln K)^2
\end{aligned}
$$

respectively.

4. Concluding Remarks

This article verifies that explicit closed-form solutions for second order approximation option price under a stochastic elasticity of variance (SEV) model mentioned in Kim et al. [1] can be derived by making use of Mellin transform approaches. Then, the Mellin transform enables us to change the complicated homogeneous or nonhomogeneous PDEs into a simpler ODE or a reduced PDE so that we can find the solution of the given PDEs more easily and effectively. In addition, by obtaining the explicit-form solution, we can notice that the existence of the closed solution has a significant influence on the accuracy of the option pricing error as well as the speed of option's data fitting. Finally, the Mellin transform methods help us to resolve the complexity of the calculation of the PDE by comparison with the probabilistic techniques, Fourier transforms, and the method of change of variables. Therefore, a lot of studies of the Mellin transforms on financial instruments continue to be works in progress by researchers.

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