

ON FUNCTIONALLY CONVEX SETS AND FUNCTIONALLY CLOSED SETS IN REAL BANACH SPACES

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ABSTRACT. We have introduced two new notions of convexity and closedness in functional analysis. Let X be a real normed space, then $C(\subseteq X)$ is functionally convex (briefly, F -convex), if $T(C) \subseteq \mathbb{R}$ is convex for all bounded linear transformations $T \in B(X, \mathbb{R})$; and $K(\subseteq X)$ is functionally closed (briefly, F -closed), if $T(K) \subseteq \mathbb{R}$ is closed for all bounded linear transformations $T \in B(X, \mathbb{R})$. By using these new notions, the Alaoglu-Bourbaki-Eberlein-Šmuljan theorem has been generalized. Moreover, we show that X is reflexive if and only if the closed unit ball of X is F -closed. James showed that for every closed convex subset C of a Banach space X , C is weakly compact if and only if every $f \in X^*$ attains its supremum over C at some point of C . Now, we show that if A is an F -convex subset of a Banach space X , then A is bounded and F -closed if and only if every element of X^* attains its supremum over A at some point of A .

1. INTRODUCTION

Convexity is an important tool in many fields of Mathematics, having applications in different areas. Various generalizations of the convexity were given in the literature, including nearly convexity, closely convexity, convexlike, quasiconvex, approximately convex and so forth. Furthermore, generalizing of convexity is a difficult task. Several generalizations have appeared to be mere formal extensions of convexity, most of which deal with invexity.

In this work, by two notions functionally convex sets and functionally closed sets which we introduced in [6] and [7], we improve some basic theorems in functional analysis. Among other things, a generalization of the Alaoglu-Bourbaki-Eberlein-Šmuljan theorem is proved. In fact, we show that a real Banach space X is reflexive

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if and only if the closed unit ball of X is F -closed. Moreover, a weak form of the James theorem is proved. Indeed, we show that if A is an F -convex subset of a Banach space X then, the subset A is bounded and F -closed if and only if every element of X^* attains its supremum over A at some point of A .

From now on, we suppose that all normed spaces and Banach spaces are real.

Definition 1.1 ([6]). In a normed space X , we say that $K(\subseteq X)$ is *m-functionally convex* (briefly, *m-F-convex*) (for $m \in \mathbb{N}$) if for every bounded linear transformation $T \in B(X, \mathbb{R}^m)$, the subset $T(K)$ of \mathbb{R}^m is convex. A 1- F -convex set is called *F-convex*. A subset K of X is called *permanently F-convex* if K is m - F -convex for all $m \in \mathbb{N}$.

It is easy to see that every convex set is permanently F -convex.

Proposition 1.2. *Every $m + 1$ - - F -convex set is m - - F -convex.*

Proof. For every $T \in B(X, \mathbb{R}^m)$, we define $S : X \longrightarrow \mathbb{R}^{m+1}$ by $S(x) = (Tx, 0)$. Note that, $S \in B(X, \mathbb{R}^{m+1})$ and for every $A \subseteq X$, the set $T(A)$ is convex if and only if $S(A)$ is convex. \square

Proposition 1.3 ([6]). *If T is a bounded linear mapping from a normed space X into a normed space Y , and K is F -convex in X , then $T(K)$ is F -convex in Y .*

Corollary 1.4 ([6]). *Let A, B be two F -convex subsets of a normed space X and λ be a real number, then*

$$A + B = \{a + b : a \in A, b \in B\}, \quad \lambda A = \{\lambda.a : a \in A\}$$

are F -convex.

Proposition 1.5 ([6]). *Let A and B be F -convex subsets of a linear space X , which have nonempty intersection. Then $A \cup B$ is F -convex.*

Definition 1.6 ([6]). Let X be a normed space and let $A \subseteq X$. We say that A is *functionally closed set* (briefly, *F-closed*), if $f(A)$ is closed for all $f \in X^*$.

Note that every compact set is F -closed. Also, every closed subset of real numbers \mathbb{R} is F -closed. In $X = \mathbb{R}^2$, the set $A = \{(x, y) : x, y \geq 0\}$ is (non-compact) F -closed whereas, the set $A = \mathbb{Z} \times \mathbb{Z}$ is closed but it is not F -closed (by taking $f(x, y) = x + \sqrt{2}y$, the set $f(A)$ is not closed in \mathbb{R}). In fact, we know that a subgroup G of \mathbb{R} is dense or else there exists $a \in \mathbb{R}$ such that $G = \{na : n \in \mathbb{Z}\}$.

Now, since the set $f(A)$ is not cyclic so, the set $f(A)$ is not closed. By taking $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$ a nonconvex F-closed and F-convex set is obtained. Also, the set $B = \{(x, y) : x \in [0, \frac{\pi}{2}), y \geq \tan(x)\}$ is a closed convex set which is not F-closed. On the other hand, $A = \{(x, y) : 1 < x^2 + y^2 \leq 4\}$ is a non-compact and F-closed set. The two last examples show that weakly closed(weakly compact) and F-closed sets are different.

Now, we mention the following theorem, which help us to find a big class of F-convex sets.

Theorem 1.7 ([6]). *Every arcwise connected subset of a normed space X is F-convex.*

Remark 1.8. The converse of the above theorem is not valid. Hence, by taking $S = \{(x, \sin(\frac{1}{x}) : 0 < x \leq 1\}$, the set \bar{S} which is called the sine's curve of topologist is connected and so for any linear functional $f \in (\mathbb{R} \times \mathbb{R})^*$, the set $f(\bar{S})$ is an interval. Thus, \bar{S} is an F-convex set which is not arcwise connected.

2. MAIN RESULTS

Theorem 2.1. *Let A be an F-convex subset of Banach space X . Then \bar{A} , the closure of A is F-convex.*

Proof. For every $f \in X^*$, we have $f(A) \subseteq f(\bar{A}) \subseteq \overline{f(A)}$. Hence, by assumption, $f(\bar{A})$ is an interval. This completes the proof. \square

Remark 2.2. In contrary the case of convex sets, interior of an F-convex set, necessarily is not F-convex. For instance, take $X = \mathbb{R} \times \mathbb{R}$ and let $B = \{(x, y) : x^2 + y^2 \leq 1\}$. Then if A is all elements surrounded by B and $B + \frac{1}{2}$ is F-convex, but the interior of A is not F-convex. Since, by taking f as projection on x -axis we have $f(A^\circ) = (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$, which is not convex.

In the following, for a subset A of a Banach space X , a necessary and sufficient condition for F-convexity is proved.

Theorem 2.3. *Let X be a Banach space, $A \subseteq X$ is F-convex if and only if*

$$co(A) \subseteq \bigcap_{f \in X^*} A + Ker(f).$$

Proof. The set $A \subseteq X$ is F -convex if and only if for all $f \in X^*$, the element $\sum_{i=1}^n \lambda_i f(a_i)$ belongs to $f(A)$ which, $\lambda_i \geq 0$, $a_i \in A$ and $\sum_{i=1}^n \lambda_i = 1$. This is equivalent that for all $f \in X^*$, there is $a \in A$ such that $a - \sum_{i=1}^n \lambda_i a_i \in Ker(f)$. \square

Remark 2.4. Note that in special case $X = \mathbb{R}$, since every nonzero functional is one to one, we have $\bigcap_{f \in X^*} A + Ker(f) = A$. Thus $A \subseteq \mathbb{R}$ is F -convex if and only if $co(A) \subseteq A$. Also, we have $A \subseteq co(A)$. Then we obtain $A \subseteq \mathbb{R}$ is F -convex if and only if A is convex.

Let X be a vector space. A hyperplane in X (through $x_0 \in X$) is a set of the form $H = x_0 + Ker(f) \subseteq X$, where f is a non-zero linear functional on X . Equivalently, $H = f^{-1}(\gamma)$, where $\gamma = f(x_0)$. So, we have

$$\bigcap_{f \in X^*} A + Ker(f) = \bigcap_{f \in X^*} \bigcup_{a \in A} a + Ker(f) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Hence, $A \subseteq X$ is F -convex if and only if

$$co(A) \subseteq \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Proposition 2.5. *Let A be a subset of Banach space X . The set $U = \bigcap_{B \in \Gamma} \bigcap_{f \in X^*} f^{-1}(f(B))$ is F -convex, where $\Gamma = \{B : A \subseteq B, B \text{ is } F\text{-convex}\}$.*

Proof. By the above remark and F -convexity of B we have $co(B) \subseteq \bigcap_{f \in X^*} f^{-1}(f(B))$. Intersecting on all $B \in \Gamma$, implies that

$$co(A) = \bigcap_{B \in \Gamma} co(B) \subseteq U \subseteq \bigcap_{f \in X^*} f^{-1}(f(co(A))).$$

On the other hand, for every $g \in X^*$,

$$g(co(A)) \subseteq g(U) \subseteq g(g^{-1}(g(co(A)))) \subseteq g(co(A)).$$

Hence, for every $g \in X^*$, $g(U) = g(co(A))$. So U is F -convex. \square

Theorem 2.6 ([3]). *If K_1 and K_2 are disjoint closed convex subsets of a locally convex linear topological space X and if K_1 is compact, then there exist constants c and $\epsilon > 0$ and a continuous linear functional f on X such that*

$$f(K_2) \leq c - \epsilon < c \leq f(K_1).$$

Lemma 2.7 ([6]). *If A is a subset of a Banach space X , then*

$$\bigcap_{f \in X^*} f^{-1}(f(A)) \subseteq \overline{co}(A).$$

Corollary 2.8 ([6]). *Let A be an F -closed subset of a Banach space X . Then A is F -convex if and only if*

$$\overline{\text{co}}(A) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Corollary 2.9. *A compact subset A in a Banach space X is convex if and only if A is F -convex and X^* separates A and every element of $X - A$.*

Proof. If A is a compact convex subset of X , then by Theorem 2.6, the assertion holds. Conversely, assume that A is a compact F -convex subset of X . Hence, $\overline{\text{co}}(A) = \bigcap_{f \in X^*} f^{-1}(f(A))$. On the other hand, there is $f \in X^*$ such that for every $x \in X - A$, we have $f(A) < f(x)$. This implies that x is outside of $f^{-1}(f(A))$. Thus $f^{-1}(f(A)) = A$ and $\overline{\text{co}}(A) = A$. \square

Remark 2.10. If X is a Hilbert space, then by Riesz representation theorem for every $f \in X^*$, there exists a unique $z \in X$ such that for all $x \in X$, $f(x) = \langle x, z \rangle$, the inner product of x and z . Then

$$\text{Ker}(f) = \{x \in X : \langle x, z \rangle = 0\} \doteq z^\perp.$$

In this case, we have

$$(2.1) \quad \bigcap_{f \in X^*} f^{-1}(f(A)) = \bigcap_{f \in X^*} A + \text{Ker}(f) = \bigcap_{z \in X} A + z^\perp.$$

Thus, in a Hilbert space X , every F -closed subset A of X is F -convex if and only if

$$\overline{\text{co}}(A) = \bigcap_{z \in X} A + z^\perp.$$

Corollary 2.11. *Let A and B be F -closed and F -convex subsets of a Banach space X which have nonempty intersection. Then*

$$\overline{\text{co}}(A \cup B) = \overline{\text{co}}(A) \cup \overline{\text{co}}(B).$$

Proof. By Proposition 1.5, $A \cup B$ is F -convex. Then we have

$$\begin{aligned} \overline{\text{co}}(A \cup B) &= \bigcap_{f \in X^*} f^{-1}(f(A \cup B)) \\ &= \left(\bigcap_{f \in X^*} f^{-1}(f(A)) \right) \cup \left(\bigcap_{f \in X^*} f^{-1}(f(B)) \right) \\ &= \overline{\text{co}}(A) \cup \overline{\text{co}}(B). \end{aligned}$$

\square

Corollary 2.12. *Let A and B be F -closed and F -convex subsets of a Banach space X . Then*

$$\overline{\text{co}}(A + B) = \overline{\text{co}}(A) + \overline{\text{co}}(B).$$

Proof. Obviously, we have

$$\overline{\text{co}}(A + B) \subseteq \overline{\text{co}}(A) + \overline{\text{co}}(B).$$

Let x be an arbitrary element of $\overline{\text{co}}(A) + \overline{\text{co}}(B)$. Then there are $x_1 \in \overline{\text{co}}(A)$ and $x_2 \in \overline{\text{co}}(B)$ such that $x = x_1 + x_2$. Then for every $f \in X^*$, we have $f(x_1) \in f(A)$ and $f(x_2) \in f(B)$. This implies that $f(x_1 + x_2) \in f(A + B)$ and hence, $x \in f^{-1}(f(A + B))$. It follows that

$$\overline{\text{co}}(A) + \overline{\text{co}}(B) \subseteq \bigcap_{f \in X^*} f^{-1}(f(A + B)) = \overline{\text{co}}(A + B).$$

Note that if A and B are F -convex and F -closed, then $A + B$ is F -closed. \square

There are several statements equivalent with reflexivity in real Banach spaces. Some of them are collected in the following theorem which is named the Alaoglu-Bourbaki-Eberlein-Šmuljan theorem.

Theorem 2.13 ([5, p. 777]). *For a Banach space X the following five statements are equivalent:*

- (i) X is reflexive.
- (ii) Every bounded sequence $\{x_n\}$ in X has a functionally convergent subsequence.
- (iii) The closed unit ball in X is weakly compact.
- (iv) Every bounded closed convex set in X is weakly compact.
- (v) Every bounded weakly closed set in X is weakly compact.

The equivalence of (i) and (ii) is called the Eberlein-Šmuljan theorem. The equivalence of (i) and (iii) is called the Alaoglu theorem. In the following, we add a new statement to the above list. Indeed, we generalized the Alaoglu theorem by replacing the assumption of weakly compactness by F -closedness of the closed unit ball of X which is weaker than that condition. To this end, we need the following theorem.

Theorem 2.14. *For a norm space X , we have the following.*

1. *The topology $\sigma(X^{**}, X^*)$ induces $\sigma(X, X^*)$ on X .*

2. The closed unit ball U of X is $\sigma(X^{**}, X^*)$ -dense in the closed unit ball U^{**} of X^{**} .

3. The vector space X is $\sigma(X^{**}, X^*)$ -dense in X^{**} .

Where, $\sigma(X, X^*)$ refers to the weak topology on X and X^* is the dual of X .

Proof. Here [1]. □

Theorem 2.15. *Let X be a Banach spaces. Then X is reflexive if and only if the closed unit ball of X is F -closed.*

Proof. Let U , the closed unit ball of X , be F -closed. Assume that $\{x_n\} \subset U$ is $\sigma(X^{**}, X^*)$ -convergent to x . Then, for every $f \in X^*$, we have $\lim \hat{x}_n(f) = \hat{x}(f)$. Consequently, for every $f \in X^*$, we have $\lim f(x_n) = f(x)$. On the other hand, U is F -closed, then $f(x) \in f(U)$ for all $f \in X^*$. It follows that $x \in \bigcap_{f \in X^*} f^{-1}(f(U)) \subseteq \overline{\text{co}}(U) = U$. This means that U is $\sigma(X^{**}, X^*)$ -closed. Now, it follows from Theorem 2.14 (2) that $U = U^{**}$. Hence, $X = X^{**}$. Conversely, if X is reflexive, then U is weakly compact. If f is a bounded linear functional on X , then it is weakly continuous. Therefore, the set $f(U)$ is a compact set of \mathbb{R} . Hence, it is closed. and so U is F -closed. □

Remark 2.16. As a consequence of the above theorem, the closed unit ball of the space c_0 is not F -closed. Note that this set is a closed convex and bounded subset of c_0 .

James [4] showed that for every closed convex subset C of a Banach space X , C is weakly compact if and only if every $f \in X^*$ attains its supremum over C at some point of C . Now, we have the following theorem.

Theorem 2.17. *Let A be an F -convex subset of a Banach space X . Then A is bounded and F -closed if and only if every element of X^* attains its supremum over A at some point of A .*

Proof. Suppose that every $f \in X^*$ attains its supremum over A at some point of A . For every $x \in A$, we have

$$|f(x)| \leq |f(a_0)| < \infty, \quad (f \in X^*)$$

and so the uniform boundedness principle implies that A is bounded. On the other hand, since A is F -convex, $f(A)$ is a bounded interval. By assumption, f

attains its supremum over A at some point of A . Hence, the right side of this interval is closed. By taking $-f$, one may show the closedness of the left side of the interval.

If A is bounded and F -closed then, the set $f(A)$ is a compact subset of \mathbb{R} . \square

Theorem 2.18. *Let X be an infinite dimensional Banach space with separable conjugate and A be an F -closed subset of X . Then, A is F -convex if and only if $\overline{A}^w = \overline{co}(A)$, where \overline{A}^w is the weak closure of A .*

Proof. Let $\overline{A}^w = \overline{co}(A)$, then every $x \in \overline{co}(A)$ belongs to \overline{A}^w . Hence, there is a sequence $\{a_n\} \subseteq A$ which weakly tends to x , and so for every $f \in X^*$ we have $f(x) = \lim_{n \rightarrow \infty} f(a_n)$. This implies that $f(x) \in \overline{f(A)} = f(A)$. Therefore, $x \in f^{-1}(f(A))$. Applying Theorem 2.8, the F -convexity of A is obtained.

On the other hand, let A be an F -convex subset of X . For every $x \in \overline{A}^w$, there is a sequence $\{a_n\} \subseteq A$ such that $f(x) \in \overline{f(A)} = f(A)$. Therefore, $x \in f^{-1}(f(A))$. Applying Lemma 2.7, we have $\overline{A}^w \subseteq \overline{co}(A)$. If x is an arbitrary element of $\overline{co}(A)$, then there is a sequence $\{y_n\} \subseteq co(A)$ such that y_n tends to x . So, for every $f \in X^*$, the real number $f(y_n)$ tends to $f(x)$. Let B be a dense countable set of X^* so, there is a sequence $\{f_i\}_1^\infty \subseteq B$ such that $f = \lim_{i \rightarrow \infty} f_i$. Since A is F -convex, for every $f \in X^*$ we have $f(A) = f(co(A))$. Hence, for each $i \in \mathbb{N}$, there is a sequence $\{a_n^i\} \subseteq A$ which, for every $n \in \mathbb{N}$, the equality $f_i(y_n) = f_i(a_n^i)$ holds. By a diagonal method, one may find a sequence $a_n^* = a_n^n$ in A such that $\lim_{n \rightarrow \infty} f_m(a_n^*) = f_m(x)$. Applying the identity, $f = \lim_{i \rightarrow \infty} f_i$ we have $\lim_{n \rightarrow \infty} f(a_n^*) = f(x)$. This implies that $x \in \overline{A}^w$. \square

Corollary 2.19. *Let X be an infinite dimensional Banach space with separable conjugate and A be a weakly compact subset of X . Then, A is F -convex if and only if A is convex.*

At the end, we offer some problems.

1- Is a Banach space X reflexive if and only if every closed bounded F -convex subset of X is F -closed?

2- Let X be a real Banach space. If A is a bounded F -closed and F -convex subset of X , then

$$\overline{co}(A) = \overline{co}(Ext(A)) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

By this the Krein-Millman theorem will be generalized. In other words the assumption of weakly compactness is replaced by a weaker condition.

3-Let X be a Hilbert space and A be a nonempty closed and F -closed and F -convex subset of X and $x \in X - co(A)$. Is there $a_0 \in A$ such that

$$\|x - a_0\| = \inf_{a \in A} \|x - a\|?$$

This means that the nearest point theorem will be extended.

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