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A SYMBOLIC POWER OF THE IDEAL OF A STANDARD $\Bbbk$-CONFIGURATION IN $\mathbb{P}^2$

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Abstract. In [4], the authors show that if $\mathbb{X}$ is a $\Bbbk$-configuration in $\mathbb{P}^2$ of type $(d_1, \ldots, d_s)$ with $d_s > s \geq 2$, then $\Delta\mathbf{H}_{m\mathbb{X}}(md_s - 1)$ is the number of lines containing exactly d_s-points of $\mathbb{X}$ for $m \geq 2$. They also show that if $\mathbb{X}$ is a $\Bbbk$-configuration in $\mathbb{P}^2$ of type $(1, 2, \ldots, s)$ with $s \geq 2$, then $\Delta\mathbf{H}_{m\mathbb{X}}(m\mathbb{X} - 1)$ is the number of lines containing exactly s-points in $\mathbb{X}$ for $m \geq s+1$. In this paper, we explore a *standard* $\Bbbk$-configuration in $\mathbb{P}^2$ and find that if $\mathbb{X}$ is a standard $\Bbbk$-configuration in $\mathbb{P}^2$ of type $(1, 2, \ldots, s)$ with $s \geq 2$, then $\Delta\mathbf{H}_{m\mathbb{X}}(m\mathbb{X} - 1) = 3$, which is the number of lines containing exactly s-points in $\mathbb{X}$ for $m \geq 2$ instead of $m \geq s+1$.

1. Introduction

Let $\mathbb{X} = \{\wp_1, \ldots, \wp_s\}$ be a set of distinct points in $\mathbb{P}^n$. If $I_{\wp_i}$ is the ideal associated to $\wp_i$ in $R = \Bbbk[x_0, x_1, \ldots, x_n]$, where $\Bbbk$ is an infinite field of any characteristic, then the homogeneous ideal associated to $\mathbb{X}$ is the ideal $I_{\mathbb{X}} = I_{\wp_1} \cap \cdots \cap I_{\wp_s}$. Given s positive integers $m_1, \ldots, m_s$ (not necessarily distinct), the subscheme in $\mathbb{P}^n$ defined by the ideal $I_{\mathbb{Z}} = I_{\wp_1}^{m_1} \cap \cdots \cap I_{\wp_s}^{m_s}$ is called a set of *fat points*. We say that m_i is the *multiplicity* of the point $\wp_i$. If $m_1 = \cdots = m_s = m$, then $\mathbb{Z}$ is a *homogeneous set of fat points* of multiplicity m, which we are interested in this article. In this case, we write $m\mathbb{X}$ for $\mathbb{Z}$, and $I_{m\mathbb{X}}$ for $I_{\mathbb{Z}}$. It is well known that $I_{m\mathbb{X}} = I_{\mathbb{X}}^{(m)}$, the m-th symbolic power of the ideal $I_{\mathbb{X}}$ (see [1, 2, 3, 4]).

Let I be a homogeneous ideal of R. The *Hilbert function* of R/I, denoted $\mathbf{H}_{R/I}$, is the numerical function $\mathbf{H}_{R/I} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ defined by

$$\mathbf{H}_{R/I}(i) := \dim_{\Bbbk} R_i - \dim_{\Bbbk} I_i \quad \text{for} \quad i \geq 0,$$

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where R_i, respectively I_i, denotes the i-th graded component of R, respectively I. If $I = I_{\mathbb{X}}$ is a defining ideal of a subscheme $\mathbb{X}$ of $\mathbb{P}^n$, then we denote the Hilbert function $\mathbf{H}_{R/I_{\mathbb{X}}}$ by $\mathbf{H}_{\mathbb{X}}$.

In [9], Roberts and Roitman introduced special configurations of points in $\mathbb{P}^2$, which they named $\mathbb{k}$-*confi- gurations*. In the late 1990's, this definition was extended to $\mathbb{P}^n$ by Geramita, Harima, and Shin (see [7, 8]). In [7], the authors prove that there is a one to one correspondence between $\mathbb{k}$-configurations in $\mathbb{P}^n$ and 0-dimensional differentiable O-sequences, i.e., Hilbert functions of sets of points in $\mathbb{P}^n$. They also find a graded minimal free resolution of a $\mathbb{k}$-configuration in $\mathbb{P}^n$, and so the Hilbert function of a $\mathbb{k}$-configuration in $\mathbb{P}^n$. Interestingly, a graded minimal free resolution or the Hilbert function of a $\mathbb{k}$-configuration in $\mathbb{P}^n$ depends upon only the type (see [8, Corollary 3.7]). However, $\mathbb{k}$-configurations of the same type can have different geometric properties. In other words, with notation as in Definition 2.1 we cannot distinguish how many lines among the s-lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$ can contain exactly d_s-points in $\mathbb{X}$. In [4], the authors show the following theorem.

Theorem 1.1 ([4, Theorems 3.1 and 4.7]). *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a $\mathbb{k}$-configuration of type $d = (d_1, \ldots, d_s) \neq (1)$. Then there exists an integer m_0 such that for all $m \geq m_0$,*

$$\Delta\mathbf{H}_{m\mathbb{X}}(md_s - 1) = \text{number of lines containing exactly } d_s \text{ points of } \mathbb{X},$$

where $\Delta\mathbf{H}_{m\mathbb{X}}(-)$ is the first difference function of the Hilbert function of fat points of multiplicity m supported on $\mathbb{X}$. Furthermore, if $d_s > s$, then $m_0 = 2$, and if $d_s = s$, then $m_0 = s + 1$.

Indeed, we don't doubt that the above theorem holds for $m_0 = 2$ instead of $m_0 = s + 1$ even when $d_s = s$. In this article, we prove that this holds for $m \geq 2$ if a $\mathbb{k}$-configuration in $\mathbb{P}^2$ is standard.

2. A Symbolic Power of The Ideal of A Standard $\mathbb{k}$-configuration in $\mathbb{P}^2$

We recall the definition of a $\mathbb{k}$-configuration in $\mathbb{P}^2$.

Definition 2.1 ([7, 8, 9]). A $\mathbb{k}$-configuration of points in $\mathbb{P}^2$ is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfies the following conditions: there exist integers $1 \leqslant d_1 < \cdots < d_s$, subsets $\mathbb{X}_1, \ldots, \mathbb{X}_s$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \ldots, \mathbb{L}_s \subseteq \mathbb{P}^2$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^{s} \mathbb{X}_i$;

(2) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subseteq \mathbb{L}_i$ for each $i = 1, \ldots, s$, and;
(3) $\mathbb{L}_i$ $(1 < i \leqslant s)$ does not contain any points of $\mathbb{X}_j$ for all $1 \leq j < i$.

In this case, the $\Bbbk$-configuration is said to be of type $(d_1, \ldots, d_s)$.

Let $(d_1, \ldots, d_s)$ be the parameters of a $\Bbbk$-configuration $\mathbb{X}$ in $\mathbb{P}^2$. We shall construct a set of points which realizes these parameters and which are located in the following lines.

$$\mathbb{L}_1 : x_1 = 0;\ \mathbb{L}_2 : x_1 = 1;\ \cdots;\ \mathbb{L}_s : x_1 = s - 1.$$

(Note that this is a family of lines parallel to the x_0-axis.) In each of these lines we shall place points as follows: in a line $\mathbb{L}_i$, we place the d_i-points in $\mathbb{X}$ in the following way.

$$\begin{array}{rll} d_1 & \text{points with coordinates} & (1, s-1, 1), \ldots, (d_1, s-1, 1), \\ d_2 & \text{points with coordinates} & (1, s-2, 1), \ldots, (d_2, s-2, 1), \\ & \vdots & \\ d_{s-1} & \text{points with coordinates} & (1, 1, 1), \ldots, (d_{s-1}, 1, 1), \\ d_s & \text{points with coordinates} & (1, 0, 1), \ldots, (d_s, 0, 1). \end{array}$$

A $\Bbbk$-configuration of points in $\mathbb{P}^2$ constructed as above will be called a *standard* $\Bbbk$-*configuration in* $\mathbb{P}^2$.

Before we prove our main theorem, we introduce a result in [2], which we shall often use in this section. Let $\mathbb{Z} = \mathbb{Z}_0$ be a fat point subscheme of $\mathbb{P}^2$. Choose a sequence of lines $\mathbb{L}_1, \ldots, \mathbb{L}_r$ and define $\mathbb{Z}_i$ to be the residual of $\mathbb{Z}_{i-1}$ with respect to the line $\mathbb{L}_i$. Define the associated *reduction vector* $\mathbf{v} = (v_1, \ldots, v_r)$ by taking $v_i = \deg(\mathbb{L}_i \cap \mathbb{Z}_{i-1})$. In particular, v_i is the sum of multiplicities of the points in $\mathbb{L}_i \cap \mathbb{Z}_{i-1}$. Given $\mathbf{v} = (v_1, \ldots, v_r)$, we define two functions

$$f_{\mathbf{v}}(t) = \textstyle\sum_{i=0}^{r-1} \min(t - i + 1, v_{i+1}), \quad \text{and} \tag{2.1}$$

$$F_{\mathbf{v}}(t) = \min_{0 \leq i \leq r} \left(\binom{t+2}{2} - \binom{t-i+2}{2} + \textstyle\sum_{j=i+1}^{r} v_j\right). \tag{2.2}$$

Theorem 2.2 ([2, Theorem 1.1])**.** *Let $\mathbb{Z} = \mathbb{Z}_0$ be a fat point scheme in $\mathbb{P}^2$ with reduction vector $\mathbf{v} = (v_1, \ldots, v_r)$ such that $\mathbb{Z}_{r+1} = \varnothing$. Then the Hilbert function $\mathbf{H}_{\mathbb{Z}}(t)$ of $\mathbb{Z}$ is bounded by $f_{\mathbf{v}}(t) \leq \mathbf{H}_{\mathbb{Z}}(t) \leq F_{\mathbf{v}}(t)$.*

Example 2.3. Consider a standard $\Bbbk$-configuration $\mathbb{X}$ in $\mathbb{P}^2$ of type $(1, 2, 3, 4, 5, 6, 7)$ with $m = 3$ (see Figure 1).

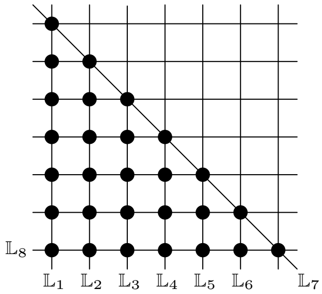

Figure 1. a standard $\mathbb{k}$-configuration of type $(1,2,3,4,5,6,7)$

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$19-i+1$	**20**	**19**	**18**	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
v_{i+1}	21	20	19	**16**	**13**	**10**	**7**	**4**	**12**	**10**	**8**	**6**	**4**	**2**	**1**	**5**	**4**	**3**	**2**	**1**
	$\mathbb{L}_7$	$\mathbb{L}_8$	$\mathbb{L}_1$	$\mathbb{L}_2$	$\mathbb{L}_3$	$\mathbb{L}_4$	$\mathbb{L}_5$	$\mathbb{L}_6$	$\mathbb{L}_1$	$\mathbb{L}_2$	$\mathbb{L}_3$	$\mathbb{L}_4$	$\mathbb{L}_5$	$\mathbb{L}_6$	$\mathbb{L}_7$	$\mathbb{L}_1$	$\mathbb{L}_2$	$\mathbb{L}_3$	$\mathbb{L}_4$	$\mathbb{L}_5$

By Theorem 2.2, one can see that

$$\begin{aligned} f_{\mathbf{v}}(19) &= \textstyle\sum_{i=0}^{19} \min(19-i+1, v_{i+1}) \\ &= \left[\textstyle\sum_{i=0}^{19} v_{i+1}\right] - 3 = \deg(3\mathbb{X}) - 3. \end{aligned}$$

Moreover, if we take $i = 3$, then

$$\begin{aligned} F_{\mathbf{v}}(19) &\le \tbinom{19+2}{2} - \tbinom{19-3+2}{2} + \textstyle\sum_{j=4}^{20} v_j \\ &= \deg(3\mathbb{X}) - 3, \end{aligned}$$

and so

$$\mathbf{H}_{3\mathbb{X}}(19) = \deg(3\mathbb{X}) - 3.$$

Moreover, since $\text{reg}(3\mathbb{X}) = 3 \cdot 7 = 21$, we get that

$$\mathbf{H}_{3\mathbb{X}}(20) = \deg(3\mathbb{X}).$$

Thus

$$\Delta\mathbf{H}_{3\mathbb{X}}(20) = 3,$$

which is the number of lines containing exactly 7-points in $\mathbb{X}$.

Using the same idea as in Example 2.3, we can obtain the following theorem. Indeed, in [4], the following theorem was mentioned without any proof (see [4, Remark 4.8]), so we attempt a precise proof and calculation with a visualization of construction here.

Theorem 2.4. *Let $\mathbb{X}$ be a standard $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(1,2,3,\ldots,s)$ with $s \ge 2$. Then*

$$\Delta\mathbf{H}_{m\mathbb{X}}(ms-1) = 3,$$

which is the number of lines containing exactly s-points in $\mathbb{X}$ for $m \ge 2$.

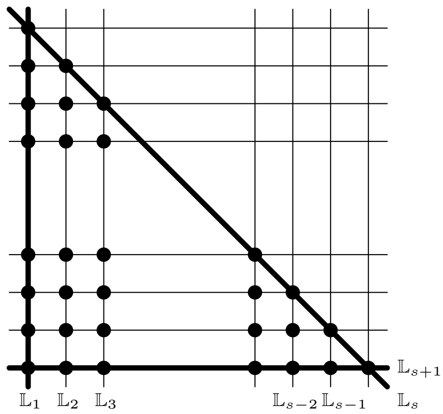

Figure 2. a standard $\Bbbk$-configuration in $\mathbb{P}^2$ of type $(1, 2, \ldots, s)$

Proof. We define the lines as follows (see Figure 2).

- For $1 \le i \le s-1$, $\mathbb{L}_i$ is a vertical line containing $(s-i+1)$-points.
- $\mathbb{L}_s$ is a diagonal line containing s-points.
- $\mathbb{L}_{s+1}$ is a bottom horizontal line containing s-points.

We now describe how to construct the components of a reduction vector $\mathbf{v}$.

(1) For $1 \le i \le s+1$,

$$\begin{aligned} v_1 &= ms = \text{the number of points on a line } \mathbb{L}_s,\\ v_2 &= ms-1 = \text{the number of points on a line } \mathbb{L}_{s+1},\\ v_3 &= m(s-2)+(m-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_1,\end{aligned}$$

$$\begin{aligned} v_4 &= m(s-3)+(m-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_2,\\ &\vdots \qquad \vdots\\ v_{i+2} &= m(s-i-1)+(m-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_i,\\ &\vdots \qquad \vdots\\ v_s &= m\cdot 1+(m-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_{s-2},\\ v_{s+1} &= (m-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_{s-1}.\end{aligned}$$

(2) If $0 \le i \le 2$, then

$$\min(ms-i-1, v_{i+1}) = ms-i-1.$$

(3) If $3 \leq i \leq s$, then

$$\begin{array}{rcl}(ms-i-1)-v_{i+1} & = & (ms-i-1)-\big[m(s-i)-(m-1)\cdot 2\big] \\ & = & (m-1)i+2m-3>0.\end{array}$$

So

$$\min(ms-i-1, v_{i+1}) = v_{i+1},$$

for such i.

(1) For $\ell s+2 \leq i+1 = \ell s+j+1 \leq \ell s+s+1$ with $1 \leq \ell \leq m-2$, and $1 \leq j \leq s$,

$$v_{\ell s+2} = (m-\ell)(s-2)+(m-\ell-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_1,$$

$$v_{\ell s+3} = (m-\ell)(s-3)+(m-\ell-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_2,$$

$$v_{\ell s+4} = (m-\ell)(s-4)+(m-\ell-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_3,$$

$$v_{\ell s+5} = (m-\ell)(s-5)+(m-\ell-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_4,$$

$$\vdots \qquad \vdots$$

$$v_{\ell s+j+1}(m-\ell)(s-j-1)+(m-\ell-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_j,$$

$$\vdots \qquad \vdots$$

$$v_{(\ell+1)s-1} = (m-\ell)\cdot 1+(m-\ell-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_{s-2},$$

$$v_{(\ell+1)s} = (m-\ell-1)\cdot 2 = \text{the number of points on a line } \mathbb{L}_{s-1},$$

$$v_{(\ell+1)s+1} = (m-\ell-1)\cdot 1 = \text{the number of points on a line } \mathbb{L}_s.$$

So, for $1 \leq j \leq s-1$,

$$\begin{aligned}(ms-i-1)-v_{i+1} &= (ms-(\ell s+j)-1)-\big[(m-\ell)(s-j-1) \\ &\qquad +(m-\ell-1)\cdot 2\big] \\ &= (m-\ell-1)(j-1) \geq 0,\end{aligned}$$

and for $j=s$, i.e., $i=\ell s+s$

$$\begin{array}{rcl}(ms-(\ell s+s)-1)-v_{\ell(s+1)+1} & = & (ms-(\ell s+s)-1)-(m-\ell-1)\cdot 1 \\ & = & (s-1)(m-\ell)-s \\ & \geq & 2(s-1)-s, \quad (\text{since } m-\ell \geq 2) \\ & \geq & 0.\end{array}$$

Therefore,

$$\min(ms-i-1, v_{i+1}) = v_{i+1},$$

for such i.

(2) For $(m-1)s+2 \leq i+1 = (m-1)s+j+1 \leq ms-1$ with $1 \leq j \leq s-2$,

$$\begin{array}{lllll} v_{(m-1)s+2} & = & 1\cdot(s-2) & = & \text{the number of points on a line } \mathbb{L}_1, \\ v_{(m-1)s+3} & = & 1\cdot(s-3) & = & \text{the number of points on a line } \mathbb{L}_2, \\ \vdots & & \vdots & & \\ v_{(m-1)s+j+1} & = & 1\cdot(s-j-1) & = & \text{the number of points on a line } \mathbb{L}_j, \\ \vdots & & \vdots & & \\ v_{(m-1)s-1} & = & 1\cdot 1 & = & \text{the number of points on a line } \mathbb{L}_{s-2}. \end{array}$$

So

$$\begin{array}{rcl} (ms-i-1)-v_{i+1} & = & (ms-((m-1)s+j)-1)-(s-j-1) \\ & = & 0. \end{array}$$

Thus,

$$\min(ms-i-1, v_{i+1}) = v_{i+1},$$

for such i.

Moreover, one can easily show that

$$\min(ms-i, v_{i+1}) = v_{i+1}, \quad \text{for every } i \geq 0.$$

We now calculate the total sum of components of the reduction vector.

$$\begin{array}{rl} & (2ms-1) + \sum_{\ell=1}^{m-1}\sum_{i=1}^{s-1}\big[(m-\ell+1)(s-i-1)+(m-\ell)\cdot 2\big] \\ & +\frac{(s-1)(s-2)}{2} + \frac{(m-1)(m-2)}{2} \\ = & \frac{m(m+1)}{2}\cdot\frac{s(s+1)}{2} \\ = & \deg(m\mathbb{X}). \end{array}$$

By Theorem 2.2, one can obtain

$$\begin{array}{lllll} f_{\mathbf{v}}(ms-2) & = & F_{\mathbf{v}}(ms-2) & = \mathbf{H}_{m\mathbb{X}}(ms-2) = \deg(m\mathbb{X})-3, & \text{and,} \\ f_{\mathbf{v}}(ms-1) & = & F_{\mathbf{v}}(ms-1) & = \mathbf{H}_{m\mathbb{X}}(ms-2) = \deg(m\mathbb{X}). & \end{array}$$

Thus, we have

$$\Delta\mathbf{H}_{m\mathbb{X}}(ms-1) = 3,$$

as we wished. □

Remark 2.5. In the proof of Theorem 2.4, we precisely calculate the two total sums of components of the reduction vectors, and show that those two numbers exactly match to $\Delta\mathbf{H}_{m\mathbb{X}}(m\mathbb{X}-2)$ and $\Delta\mathbf{H}_{m\mathbb{X}}(m\mathbb{X}-1)$, respectively. However, in [4], the authors do not mention the total sum of components of the reduction vectors for the proofs of their theorems.

It is known that if either $\mathbb{X}$ is a standard $\Bbbk$-configuration in $\mathbb{P}^2$ or $s = 2$, then

$$\Delta\mathbf{H}_{m\mathbb{X}}(ms-1) = \text{the number of lines containing exactly } s\text{-points in } \mathbb{X}.$$

(see Theorem 2.4). So the following question is still open in general.

Question 2.6. Let $\mathbb{X}$ be a $\Bbbk$-configuration in $\mathbb{P}^2$ of type $(1, 2, \ldots, s)$ with $s \geq 2$. Is it true that

$$\Delta\mathbf{H}_{m\mathbb{X}}(ms-1) = \text{the number of lines containing exactly } s\text{-points in } \mathbb{X} \text{ for } m \geq 2?$$

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