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STRONG CONVERGENCE OF HYBRID ITERATIVE SCHEMES WITH ERRORS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. In this paper, we prove a strong convergence result under an iterative scheme for N finite asymptotically k_i-strictly pseudo-contractive mappings and a firmly nonexpansive mappings S_r. Then, we modify this algorithm to obtain a strong convergence result by hybrid methods. Our results extend and unify the corresponding ones in [1, 2, 3, 8]. In particular, some necessary and sufficient conditions for strong convergence under Algorithm 1.1 are obtained.

1. Introduction and Preliminaries

In [1], Ceng et al. introduced an implicit iterative scheme for finding a common element of a set of solutions of an equilibrium problem and a set of the fixed points of a strict pseudo-contraction in the setting of real Hilbert spaces and proved a strong convergence result for the following iterative scheme;
Let C be a nonempty closed convex subset of a Hilbert space H, $\phi : C \times C \to \mathbb{R}$ be a function, $T : C \to C$ be a mapping and $x_1 \in H$, define sequences $\{x_n\}$ and $\{v_n\}$ as follows;

$$\begin{cases} \phi(v_n, y) + \frac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \geq 0 \text{ for } y \in C \\ x_{n+1} = a_n v_n + (1 - a_n)Tv_n \text{ for } n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}$ is a sequence in $(0, 1)$ and $\{r_n\}$ is a sequence in $(0, \infty)$.

Recently, Kumam et al. [3] introduced the following iterative scheme for finding a common element of a set of solutions of an equilibrium problem and a set of common fixed points of a finite family of asymptotically k-strict pseudo-contraction in Hilbert

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spaces;

$$\begin{cases} \phi(v_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - v_{n-1}, v_{n-1} - x_{n-1}\rangle \geq 0 \ \text{ for } \ y \in C \\ x_n = a_{n-1}v_{n-1} + (1 - a_{n-1})T_{i(n)}^{h(n)}v_{n-1} \ \text{ for } \ n \geq 0, \end{cases}$$

where $\{a_n\}$, $\{r_n\}$, C and ϕ are the same first dealt, and $i(n) \equiv n(mod\ N)$, $h(n) = \lceil \frac{n}{N} \rceil$ with a ceiling function $\lceil \cdot \rceil$.

And, they modified the above iterative scheme to get strong convergence theorems by the hybrid method as follows;

$$\begin{cases} x_0 = P_{C_0}x, \\ v_{n-1} \in C \ \text{ such that } \ \phi(v_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - v_{n-1}, v_{n-1} - x_{n-1}\rangle \ \text{ for } \ y \in C, \\ y_{n-1} = a_{n-1}v_{n-1} + (1 - a_{n-1})T_{i(n)}^{h(n)}v_{n-1}, \\ C_n = \{v \in C_{n-1} : \|y_{n-1} - v\|^2 \leq \|x_{n-1} - v\|^2 + \theta_{n-1}\}, \\ x_n = P_{C_n}x \ \text{ for } \ n \in \mathbb{N}, \end{cases}$$

where P_C is a metric projection of H onto C, $\{\theta_n\}$ is a sequence in $(0, \infty)$ and $i(n) \equiv n(mod\ N)$.

Very recently, Kim et al. [2] introduced the following iterative scheme with errors for generalized equilibrium problems and common fixed point problems;

Algorithm 1.1 ([2])**.** *Let C be a nonempty closed convex subset of a Hilbert space H, $T_i, \psi : C \to C (i = 1, 2, \cdots, N)$ be mappings and $\phi : C \times C \to \mathbb{R}$ be a bifunction. For any $x_0 \in C$, let $\{x_n\}$ and $\{v_n\}$ be sequences generated by*

$$\begin{cases} \phi(v_{n-1}, y) + \langle \psi v_{n-1}, y - v_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - v_{n-1}, v_{n-1} - x_{n-1}\rangle \geq 0 \ \textit{ for } \ y \in C \ , \\ x_n = a_{n-1}v_{n-1} + b_{n-1}T_{i(n)}^{h(n)}v_{n-1} + c_{n-1}u_{n-1} \ \textit{ for } \ n \in \mathbb{N} \ , \end{cases}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0, 1)$ such that $a_n + b_n + c_n = 1$, $a_n \geq k + \varepsilon$, $b_n \geq \varepsilon$ for some $\varepsilon \in (0, 1)$, $\sum_{n=1}^{\infty} c_n < \infty$, $\{u_n\}$ is a bounded sequence in C, $\{r_n\}$ is a sequence in $(0, \infty)$ such that $\lim_{n\to\infty} \inf r_n > 0$ and $i(n) \equiv n(mod\ N)$, $h(n) = \lceil \frac{n}{N} \rceil$ with a ceiling function $\lceil \cdot \rceil$.

And, the authors proved the weak limits of sequences $\{x_n\}$ and $\{v_n\}$, obtained under the given scheme for N finite asymptotically k_i-strictly pseudo-contractive mappings $\{T_i\}_{i=1}^N$ and a firmly nonexpansive mapping S_r, are the same and hence the point is a common fixed point of $\{T_i\}_{i=1}^N$ and S_r.

Inspired by those results, in this paper, we prove a strong convergence result under Algorithm 1.1. Then, we modify Algorithm 1.1 to obtain a strong convergence result by hybrid methods. Our results extend and unify the corresponding ones in [1, 2, 3, 8]. In particular, some necessary and sufficient conditions for strong convergence under Algorithm 1.1 are obtained.

First of all, we recall some definitions and results which will be needed in the main results.

Definition 1.1. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction and $\psi : C \to C$ a nonlinear mapping.
(a) ϕ is said to be *monotone* if $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$.
(b) ψ is said to be *monotone* if $\langle \psi x - \psi y, x - y\rangle \geq 0$ for all $x, y \in C$.

Definition 1.2. A mapping $T : C \to C$ is asymptotically k-strictly pseudo-contractive if there exist a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ and $k \in [0, 1)$ such that

$$\|T^n x - T^n y\|^2 \leq k_n^2 \|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2 \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

Lemma 1.3 ([4, 7])**.** *Let H be a real Hilbert space, then we have the following identities;*
(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ for $x, y \in H$,
(ii) $\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2$ for $x, y \in H$ and $a, b, c \in [0, 1]$ with $a + b + c = 1$,
(iii) if $\{x_n\}$ is a sequence in H weakly converging to z, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2 \text{ for } y \in H.$$

Lemma 1.4 ([7])**.** *Let $\{a_n\}$, $\{c_n\}$ and $\{\delta_n\}$ be nonnegative real sequences satisfying the condition: $a_{n+1} \leq (1 + \delta_n)a_n + c_n$ for $n \in \mathbb{N}$.*

If $\sum\limits_{n=1}^{\infty} \delta_n < \infty$ and $\sum\limits_{n=1}^{\infty} c_n < \infty$, then $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 1.5 ([2])**.** *Let C be a closed convex subset of a Hilbert space H. Assume that a function $\phi : C \times C \to \mathbb{R}$ satisfies the following conditions;*
(i) $\phi(x, x) = 0$ for all $x \in C$;
(ii) ϕ is monotone;
(iii) ϕ is upper hemi-continuous in the first variable;
(iv) ϕ is convex and lower semi-continuous in the second variable.
Let $\psi : C \to C$ be a monotone nonlinear mapping. For $r > 0$ and $x \in H$, define a mapping $S_r : H \to 2^C$ by

$$S_r x = \{z \in C : \phi(z, y) + \langle \psi z, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C\}.$$

Then the following holds;
(a) for each $x \in H$, $S_r x$ is a singleton;
(b) S_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle;$$

(c) The set $F(S_r)$ of all fixed points of S_r is a closed and convex subset of C as a solution set of the following equilibrium problem considered in [6]: finding $x \in C$ such that $\phi(x, y) + \langle \psi x, y - x\rangle \geq 0$ for all $y \in C$.

Lemma 1.6 ([8]). *Let H be a real Hilbert space, C a nonempty subset of H and $T : C \to C$ be an asymptotically k-strictly pseudo-contractive mapping. Then the fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well-defined.*

Lemma 1.7 ([5]). *Let C be a closed convex subset of a real Hilbert space H, $\{x_n\}$ be a sequence in H and $q = P_C u$. If $W(x_n) := \bigcup\{A(\{x_{n_k}\}) : \{x_{n_k}\}$ is a subsequence of $\{x_n\}\} \subset C$ and $\|x_n - u\| \leq \|u - q\|$ for $n \in \mathbb{N}$, then $x_n \to q$ as $n \to \infty$.*

2. Main Result

Now, we consider two strong convergences, one is founded on Algorithm 1.1 and the other is on hybrid algorithm.

Theorem 2.1. *Assume the following conditions;*
(i) C is a closed convex subset of a Hilbert space H;
(ii) $T_i : C \to C$ is asymptotically k_i-strictly pseudo-contractive with $k_i \in [0, 1)$ and sequences $\{k_{n,i}\}_{n=1}^{\infty}$ in $[1, \infty)$ satisfying $\sum\limits_{n=1}^{\infty}(k_{n,i}^2 - 1) < \infty$ for $i = 1, 2, \cdots, N$;
(iii) a bifunction $\phi : C \times C \to \mathbb{R}$ satisfies the conditions (i)-(iv) in Lemma 1.5 and $\psi : C \to C$ is a monotone nonlinear mapping with a nonempty set $F := (\bigcap\limits_{i=1}^{N} F(T_i)) \bigcap F(S_r)$.

Then, for any fixed $x_0 \in C$, two sequences $\{x_n\}$ and $\{v_n\}$ generated by Algorithm 1.1 converge strongly to the unique same element of F if and only if $\liminf\limits_{n\to\infty} d(x_n, F) = 0$, where $d(x, F)$ denotes the metric distance from the point x to F.

Proof. Let $p \in F$. By Algorithm 1.1 and Lemma 1.5 (a), $v_{n-1} = S_{r_{n-1}} x_{n-1}$ and $\|v_{n-1} - p\| = \|S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} p\| \leq \|x_{n-1} - p\|$ for $n \in \mathbb{N}$. From Algorithm 1.1

and Lemma 1.3 (ii), we have

$$\begin{aligned}
\|x_n - p\|^2 &= \|a_{n-1}(v_{n-1} - p) + b_{n-1}(T_{i(n)}^{h(n)} v_{n-1} - p) + c_{n-1}(u_{n-1} - p)\|^2 \\
&\le a_{n-1}\|v_{n-1} - p\|^2 + b_{n-1}\|T_{i(n)}^{h(n)} v_{n-1} - T_{i(n)}^{h(n)} p\|^2 + c_{n-1}\|u_{n-1} - p\|^2 \\
&\quad - a_{n-1}b_{n-1}\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\|^2 \\
&\le a_{n-1}\|v_{n-1} - p\|^2 + b_{n-1}\{(k_{h(n)}^{'})^2\|v_{n-1} - p\|^2 + k\|(I - T_{i(n)}^{h(n)})v_{n-1} \\
&\quad - (I - T_{i(n)}^{h(n)})p\|^2\} + c_{n-1}\|u_{n-1} - p\|^2 - a_{n-1}b_{n-1}\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\|^2 \\
&\le (k_{h(n)}^{'})^2\|v_{n-1} - p\|^2 - b_{n-1}(a_{n-1} - k)\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\|^2 \\
&\quad + c_{n-1}\|u_{n-1} - p\|^2 \qquad (2.1) \\
&\le [1 + ((k_{h(n)}^{'})^2 - 1)]\|x_{n-1} - p\|^2 + c_{n-1}\|u_{n-1} - p\|^2, \qquad (2.2)
\end{aligned}$$

where $k = \max\{k_i : 1 \le i \le N\}$ and $k_n^{'} = \max\{k_{n,i} : 1 \le i \le N\}$ for each $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty}(k_{n,i}^2 - 1) < \infty$, by Lemma 1.4, $\lim_{n\to\infty}\|x_n - p\|$ exists. On the other hand, since $a_n \ge k + \varepsilon$, $b_n \ge \varepsilon$ for some $\varepsilon \in (0,1)(n \in \mathbb{N})$, we have

$$\begin{aligned}
&(k_{h(n)}^{'})^2\|x_{n-1} - p\|^2 - \|x_n - p\|^2 + c_{n-1}\|u_{n-1} - p\|^2 \\
\ge\ & b_{n-1}(a_{n-1} - k)\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\|^2 \\
\ge\ & \varepsilon^2\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty} k_{h(n)}^{'} = 1$ and $\lim_{n\to\infty} c_n = 0$, taking the limits as $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty}\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\|^2 = 0. \qquad (2.3)$$

Observe that

$$\begin{aligned}
\|x_n - v_{n-1}\| &= \|a_{n-1}v_{n-1} + b_{n-1}T_{i(n)}^{h(n)} v_{n-1} + c_{n-1}u_{n-1} - v_{n-1}\| \\
&= \|-(1 - a_{n-1})(v_{n-1} - T_{i(n)}^{h(n)} v_{n-1}) + c_{n-1}(u_{n-1} - T_{i(n)}^{h(n)} v_{n-1})\| \\
&\le (1 - a_{n-1})\|v_{n-1} - T_{i(n)}^{h(n)} v_{n-1}\| + c_{n-1}\|u_{n-1} - T_{i(n)}^{h(n)} v_{n-1}\|.
\end{aligned}$$

From (2.3), it follows that

$$\lim_{n\to\infty}\|x_n - v_{n-1}\| = 0.$$

By the firm nonexpansiveness of $S_{r_{n-1}}$ and Lemma 1.3 (i), we have

$$\begin{aligned}\|v_{n-1}-p\|^2 &= \|S_{r_{n-1}}x_{n-1}-S_{r_{n-1}}p\|^2 \le \langle S_{r_{n-1}}x_{n-1}-S_{r_{n-1}}p, x_{n-1}-p\rangle \\ &= \langle v_{n-1}-p, x_{n-1}-p\rangle = -\langle -(x_{n-1}-v_{n-1})-(x_{n-1}-p), x_{n-1}-p\rangle \\ &= -\frac{1}{2}(\|x_{n-1}-v_{n-1}\|^2-\|x_{n-1}-p\|^2-\|v_{n-1}-p\|^2),\end{aligned}$$

and hence $\|v_{n-1}-p\|^2 \le \|x_{n-1}-p\|^2 - \|x_{n-1}-v_{n-1}\|^2$. Applying this inequality to (2.1), we have

$$\|x_n-p\|^2 \le (k'_{h(n)})^2(\|x_{n-1}-p\|^2-\|x_{n-1}-v_{n-1}\|^2)+c_{n-1}\|u_{n-1}-p\|^2.$$

Since $\lim_{n\to\infty}\|x_n-p\|$ exists and $\lim_{n\to\infty}k'_{h(n)}=1$,

$$\lim_{n\to\infty}\|x_{n-1}-v_{n-1}\|=0. \tag{2.4}$$

Since $0\le d(x_n,F)\le \|x_n-v_n\|+\|v_n-p\|$, from (2.4), we have $\liminf_{n\to\infty} d(x_n,F)=0$. Conversely, we asuume that $\liminf_{n\to\infty} d(x_n,F)=0$. From the fact that $1+x\le e^x$ for $x\ge 0$ and (2.2), we have

$$\begin{aligned}\|x_{n+m}-p\|^2 &\le [1+\{(k'_{h(n+m)})^2-1\}]\|x_{(n+m)-1}-p\|^2+c_{(n+m)-1}\|u_{(n+m)-1}-p\|^2 \\ &\le e^{(k'_{h(n+m)})^2-1}\|x_{(n+m)-1}-p\|^2+c_{(n+m)-1}\|u_{(n+m)-1}-p\|^2 \\ &\le e^{(k'_{h(n+m)})^2-1}\{[1+\{(k'_{h(n+m-1)})^2-1\}]\|x_{(n+m)-2}-p\|^2 \\ &\quad + c_{(n+m)-2}\|u_{(n+m)-2}-p\|^2\}+c_{(n+m)-1}\|u_{(n+m)-1}-p\|^2 \\ &\le e^{(k'_{h(n+m)})^2-1}e^{(k'_{h(n+m-1)})^2-1}\|x_{(n+m)-2}-p\|^2 \\ &\quad + e^{(k'_{h(n+m)})^2-1}c_{(n+m)-2}\|u_{(n+m)-2}-p\|^2+c_{(n+m)-1}\|u_{(n+m)-1}-p\|^2 \\ &\le \cdots \\ &\le e^{\sum_{j=n}^{n+m}\{(k'_{h(j)})^2-1\}}\|x_n-p\|^2+e^{\sum_{j=n}^{n+m}\{(k'_{h(j)})^2-1\}}\sum_{j=n}^{n+m}c_j\|u_j-p\|^2 \\ &\le M\|x_n-p\|^2+MN\sum_{j=n}^{n+m}c_j,\end{aligned}$$

where $e^{\sum_{j=n}^{n+m}\{(k'_{h(j)})^2-1\}}=M$ and $\|u_j-p\|^2=N$ for some $M,N>0$. Since $\sum_{n=1}^{\infty}c_n<\infty$, for any $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that $\sum_{j=n}^{n+m}c_j<\varepsilon$ for $m,n\ge K$,

$$\|x_{n+m}-p\|^2\le M\|x_n-p\|^2+MN\varepsilon \quad \text{for } m,n\ge K.$$

Letting $\varepsilon \to 0$, we have $\|x_{n+m} - p\|^2 \le M\|x_n - p\|^2$ for $m, n \ge K$. By Lemma 1.4, we get $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. For given $\varepsilon > 0$, there exists $K_1 \in \mathbb{N}$ such that $d(x_n, F) \le \frac{\varepsilon}{2\sqrt{M}}$ for $n \ge K_1$. Put $K_2 = \max\{K, K_1\}$, then for $n \ge K_2$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\le \|x_{n+m} - p\| + \|x_n - p\| \\ &\le \sqrt{M}\|x_{K_1} - p\| + \sqrt{M}\|x_{K_1} - p\| \\ &\le \sqrt{M}\frac{\varepsilon}{2\sqrt{M}} + \sqrt{M}\frac{\varepsilon}{2\sqrt{M}} = \varepsilon. \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence. Assume that $x_n \to x^*(\in H)$, then $d(x^*, F) = \lim_{n\to\infty} d(x_n, F) = 0$. Since T_i is an asymptotically k_i-strictly pseudo-contractive mapping, from Lemma 1.6 and Lemma 1.5 (c), F is also closed and convex. Consequently, $x^* \in F$. Since $\|v_n - x_n\| \to 0$ as $n \to \infty$, we conclude that both $\{x_n\}$ and $\{v_n\}$ converges strongly to an element of F. $\square$

The following strong convergence result founded on hybrid method is our main result.

Theorem 2.2. *Let C, H, ϕ, ψ, $\{T_i\}_{i=1}^N$, $\{k_{n,i}\}$, k, k'_n and F be the same as Theorem 2.1. For $x \in H$ and $C_0 = C$, let $\{x_n\}$ and $\{v_n\}$ be two sequences generated by the following algorithm;*

$$\begin{cases} x_0 = P_{C_0}x, \\ v_{n-1} \in C \text{ such that } \phi(v_{n-1}, y) + \langle \psi v_{n-1}, y - v_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - v_{n-1}, v_{n-1} - x_{n-1}\rangle \\ \text{for } y \in C, \\ y_{n-1} = a_{n-1}v_{n-1} + b_{n-1}T_{i(n)}^{h(n)}v_{n-1} + c_{n-1}u_{n-1}, \\ C_n = \{v \in C_{n-1} : \|y_{n-1} - v\|^2 \le \|x_{n-1} - v\|^2 + \theta_{n-1}\}, \\ x_n = P_{C_n}x \text{ for } n \in \mathbb{N}, \end{cases} \tag{2.5}$$

where $\theta_{n-1} = \{(k'_{h(n)})^2 - 1\}(1 - a_{n-1} - c_{n-1})\rho_{n-1}^2 + c_{n-1}\gamma_{n-1}^2 \to \infty$ as $n \to \infty$, $\rho_{n-1} = \sup_{v\in F}\{\|x_{n-1} - v\|\} < \infty$, $\gamma_{n-1} = \sup_{v\in F}\{\|u_{n-1} - v\|\} < \infty$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0, 1)$ such that $a_n + b_n + c_n = 1$, $a_n \ge k + \varepsilon$, $b_n \ge \varepsilon$ for some $\varepsilon \in (0, 1)$, $\sum_{n=1}^{\infty} c_n < \infty$, $\{u_n\}$ is a bounded sequence in C, $\{r_n\}$ is a sequence in $(0, \infty)$ such that $\lim_{n\to\infty} \inf r_n > 0$ and $i(n) \equiv n(\text{mod } N)$, $h(n) = \lceil \frac{n}{N} \rceil$ with a ceiling function $\lceil\cdot\rceil$.

Then, $\{x_n\}$ and $\{v_n\}$ converege strongly to $P_F x$.

Proof. First of all, we show that $F \subset C_n$ for $n \geq 0$ by the induction. It is obvious that $F \subset C = C_0$. Assume that $F \subset C_{j-1}$ for some $j \in \mathbb{N}$, then for $p \in F \subset C_{j-1}$, from the fact that $v_{j-1} = S_{r_{j-1}} x_{j-1}$, we get

$$\|v_{j-1} - p\| = \|S_{r_{j-1}} x_{j-1} - S_{r_{j-1}} p\| \leq \|x_{j-1} - p\|.$$

By Lemma 1.3 (ii), we have

$$\begin{aligned}
\|y_{j-1} - p\|^2 &= \|a_{j-1} v_{j-1} + b_{j-1} T_{i(n)}^{h(n)} v_{j-1} + c_{j-1} u_{j-1} - (a_{j-1} + b_{j-1} + c_{j-1}) p\|^2 \\
&= \|a_{j-1}(v_{j-1} - p) + b_{j-1}(T_{i(n)}^{h(n)} v_{j-1} - p) + c_{j-1}(u_{j-1} - p)\|^2 \\
&= a_{j-1}\|v_{j-1} - p\|^2 + b_{j-1}\|T_{i(n)}^{h(n)} v_{j-1} - p\|^2 + c_{j-1}\|u_{j-1} - p\|^2 \\
&\quad - a_{j-1} b_{j-1}\|v_{j-1} - T_{i(n)}^{h(n)} v_{j-1}\|^2 - b_{j-1} c_{j-1}\|T_{i(n)}^{h(n)} v_{j-1} - u_{j-1}\|^2 \\
&\quad - c_{j-1} a_{j-1}\|v_{j-1} - u_{j-1}\|^2 \\
&\leq a_{j-1}\|v_{j-1} - p\|^2 + b_{j-1}\{(k'_{h(n)})^2\|v_{j-1} - p\|^2 + k\|(I - T_{i(n)}^{h(n)}) v_{j-1} \\
&\quad - (I - T_{i(n)}^{h(n)}) p\|^2\} + c_{j-1}\|u_{j-1} - p\|^2 - a_{j-1} b_{j-1}\|v_{j-1} - T_{i(n)}^{h(n)} v_{j-1}\|^2 \\
&= a_{j-1}\|v_{j-1} - p\|^2 + b_{j-1}(k'_{h(n)})^2\|v_{j-1} - p\|^2 + b_{j-1} k\|T_{i(n)}^{h(n)} v_{j-1} - v_{j-1}\|^2 \\
&\quad + c_{j-1}\|u_{j-1} - p\|^2 - a_{j-1} b_{j-1}\|v_{j-1} - T_{i(n)}^{h(n)} v_{j-1}\|^2 \\
&\leq (1 + a_{j-1} + c_{j-1} - 1)\|v_{j-1} - p\|^2 + (1 - a_{j-1} - c_{j-1})(k'_{h(n)})^2\|v_{j-1} - p\|^2 \\
&\quad - b_{j-1}(a_{j-1} - k)\|v_{j-1} - T_{i(n)}^{h(n)} v_{j-1}\|^2 + c_{j-1}\|u_{j-1} - p\|^2 \\
&= \|v_{j-1} - p\|^2 + (1 - a_{j-1} - c_{j-1})\{(k'_{h(n)})^2 - 1\}\|v_{j-1} - p\|^2 \\
&\quad - b_{j-1}(a_{j-1} - k)\|v_{j-1} - T_{i(n)}^{h(n)} v_{j-1}\|^2 + c_{j-1}\|u_{j-1} - p\|^2 \qquad (2.6) \\
&\leq \|x_{j-1} - p\|^2 + (1 - a_{j-1} - c_{j-1})\{(k'_{h(n)})^2 - 1\}\|x_{j-1} - p\|^2 \\
&\quad + c_{j-1}\|u_{j-1} - p\|^2 \\
&\leq \|x_{j-1} - p\|^2 + \theta_{j-1}.
\end{aligned}$$

Therefore, $p \in C_j$, which implies that $F \subset C_n$ $(n \geq 0)$.

Next, we prove that C_n $(n \geq 0)$ is closed and convex. It is obvious that $C_0 = C$ is closed and convex. Suppose that C_{k-1} is closed and convex for some $k \in \mathbb{N}$. For $v \in C_{k-1}$, we know that $\|y_{k-1} - v\|^2 \leq \|x_{k-1} - v\|^2 + \theta_{k-1}$ is equivalent to

$$2\langle x_{k-1} - y_{k-1}, v\rangle \leq \|x_{k-1}\|^2 - \|y_{k-1}\|^2 + \theta_{k-1}.$$

So, C_k is closed and convex which shows that C_n is closed and convex for $n \geq 0$. This implies that $\{x_n\}$ is well-defined. From Lemma 1.5, the sequence $\{v_n\}$ is also well-defined. From (2.5), $x_n = P_{C_n}x$ $(n \geq 0)$, we have

$$\langle x - x_n, x_n - y\rangle \geq 0 \text{ for } y \in C_n.$$

Since $F \subset C_n$, we get $\langle x - x_n, x_n - p\rangle \geq 0$ for $p \in F$ and $n \geq 0$. Hence, we have

$$\begin{aligned} 0 &\leq \langle x - x_n, x_n - p\rangle = \langle x - x_n, x_n - x + x - p\rangle \\ &= -\langle x_n - x, x_n - x\rangle + \langle x - x_n, x - p\rangle \\ &\leq -\|x_n - x\|^2 + \|x - x_n\| \cdot \|x - p\|, \end{aligned}$$

which implies that

$$\|x_n - x\| \leq \|x - p\| \text{ for } p \in \mathbb{N} \text{ and } n \geq 0. \tag{2.7}$$

Thus, $\{x_n\}$ is bounded. It follows that $\{y_n\}$ is also bounded. From the fact that $x_{n-1} = P_{C_{n-1}}x$ and $x_n = P_{C_n}x$ $(\in C_n \subset C_{n-1})$, we have

$$\langle x - x_{n-1}, x_{n-1} - x_n\rangle \geq 0, \tag{2.8}$$

which implies that

$$\begin{aligned} 0 &\leq \langle x - x_{n-1}, x_{n-1} - x_n\rangle \\ &= \langle x - x_{n-1}, x_{n-1} - x + x - x_n\rangle \\ &\leq -\|x - x_{n-1}\|^2 + \|x - x_{n-1}\| \cdot \|x - x_n\| \end{aligned}$$

and hence $\|x - x_{n-1}\| \leq \|x - x_n\|$. Thus, $\{\|x_n - x\|\}$ is non-decreasing, so $\lim_{n\to\infty} \|x_n - x\|$ exsits.

Next, we show that $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$. From (2.8), we have

$$\begin{aligned} \|x_{n-1} - x_n\|^2 &= \|x_{n-1} - x + x - x_n\|^2 \\ &= \|x_{n-1} - x\|^2 + 2\langle x_{n-1} - x, x - x_n\rangle + \|x - x_n\|^2 \\ &= -\|x_{n-1} - x\|^2 + 2\langle x_{n-1} - x, x_{n-1} - x_n\rangle + \|x - x_n\|^2 \\ &\leq -\|x_{n-1} - x\|^2 + \|x - x_n\|^2. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x\|$ exists, we get

$$\lim_{n\to\infty} \|x_{n-1} - x_n\| = 0. \tag{2.9}$$

On the other hand, $x_n \in C_n$, we obtain

$$\|y_{n-1} - x_n\|^2 \leq \|x_{n-1} - x_n\|^2 + \theta_{n-1}.$$

So, we have $\lim_{n\to\infty} \|y_{n-1} - x_n\| = 0$. It follows that

$$\|y_{n-1} - x_{n-1}\| \leq \|y_{n-1} - x_n\| + \|x_n - x_{n-1}\| \to 0 \text{ for } n \to \infty. \tag{2.10}$$

Next, we claim that $\lim_{n\to\infty} \|x_n - v_n\| = 0$. Let $p \in F$. Since $v_{n-1} = S_{r_{n-1}} x_{n-1}$, we have

$$\begin{aligned}
\|v_{n-1} - p\|^2 &= \|S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} p\|^2 \\
&\leq \langle S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} p, x_{n-1} - p\rangle \\
&= \langle v_{n-1} - p, x_{n-1} - p\rangle \\
&= \langle v_{n-1} - x_{n-1} + x_{n-1} - p, x_{n-1} - p\rangle \\
&= \langle -(x_{n-1} - v_{n-1}) + (x_{n-1} - p), x_{n-1} - p\rangle \\
&= -\langle (x_{n-1} - v_{n-1}) - (x_{n-1} - p), x_{n-1} - p\rangle \\
&= \frac{1}{2}(\|v_{n-1} - p\|^2 - \|x_{n-1} - v_{n-1}\|^2 + \|x_{n-1} - p\|^2)
\end{aligned}$$

and hence

$$\|v_{n-1} - p\|^2 \leq \|x_{n-1} - p\|^2 - \|x_{n-1} - v_{n-1}\|^2. \tag{2.11}$$

From (2.6) and (2.11), we have

$$\begin{aligned}
\|y_{n-1} - p\|^2 &\leq \|v_{n-1} - p\|^2 + \theta_{n-1} \\
&\leq \|x_{n-1} - p\|^2 - \|x_{n-1} - v_{n-1}\|^2 + \theta_{n-1},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n-1} - v_{n-1}\|^2 &\leq \|x_{n-1} - p\|^2 - \|y_{n-1} - p\|^2 + \theta_{n-1} \\
&\leq \|x_{n-1} - v_{n-1}\|(\|x_{n-1} - p\| - \|y_{n-1} - p\|) + \theta_{n-1}.
\end{aligned}$$

From (2.10) and the boundedness of $\{x_n\}$ and $\{y_n\}$, we have

$$\lim_{n\to\infty} \|x_{n-1} - v_{n-1}\| = 0. \tag{2.12}$$

From (2.9) and (2.12), we obtain

$$\|v_n - v_{n-1}\| \leq \|v_n - x_n\| + \|x_n - x_{n-1}\| + \|x_{n-1} - v_{n-1}\| \to 0 \ \text{ as } \ n \to \infty,$$

which implies that

$$\lim_{n\to\infty} \|v_n - v_{n+j}\| = 0 \ \text{ for } \ j \in \{1, 2, \cdots, N\}.$$

From the fact that $y_{n-1} = a_{n-1} v_{n-1} + b_{n-1} T_{i(n)}^{h(n)} v_{n-1} + c_{n-1} u_{n-1} (n \in \mathbb{N})$,
we have

$$\begin{aligned}
b_{n-1}\|T_{i(n)}^{h(n)} v_{n-1} - x_{n-1}\| &= \|y_{n-1} - a_{n-1} v_{n-1} - c_{n-1} u_{n-1} - (1 - a_{n-1} - c_{n-1}) x_{n-1}\| \\
&\leq \|y_{n-1} - x_{n-1}\| + a_{n-1}\|v_{n-1} - x_{n-1}\| \\
&\quad + c_{n-1}\|u_{n-1} - x_{n-1}\|.
\end{aligned}$$

Applying (2.10) and (2.12), we have

$$\lim_{n\to\infty} \|T_{i(n)}^{h(n)} v_{n-1} - x_{n-1}\| = 0. \tag{2.13}$$

From (2.12) and (2.13), we obtain

$$\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\| \le \|T_{i(n)}^{h(n)} v_{n-1} - x_{n-1}\| + \|v_{n-1} - x_{n-1}\| \to 0 \text{ as } n \to \infty.$$

By using the same method as in the proof of Theorem 3.2 in [2], we can obtain

$$\lim_{n\to\infty} \|T_l v_n - v_n\| = 0$$

and

$$W(x_n) \subset F.$$

Hence, the strong convergence of $\{x_n\}$ to $p = P_F x$ is guaranteed by Lemma 1.7 under (2.7). From (2.12), we also have the strong convergence of $\{v_n\}$ to $p = P_F x$. □

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