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PURE DISCRETE SPECTRUM ON TOEPLITZ ARRAYS

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Abstract. We look at Toeplitz arrays on $\mathbb{Z}^d$ and study a characterizing property for pure discrete spectrum in terms of the periodic structures of the Toeplitz arrays.

1. Introduction

There have been a lot of investigations about the property of pure discrete spectrum on the various settings like Delone sets, tilings, and Toeplitz arrays. Especially on substitution Delone sets or tilings, the pure discrete spectrum is characterized by simple properties like algebraic coincidence and overlap coincidence [9, 13], so one can easily check the pure discrete spectrum using algorithms deriven from the coincidences. Toeplitz arrays have been good examples for studying pure discrete spectrum [1, 7]. It can be observed that Toeplitz arrays are regular if and only if the point sets coming from the associated cut-and-project scheme are regular from [1, 10]. Since every regular cut-and-project point set has pure discrete spectrum, every regular Toeplitz array has pure discrete spectrum. However characterizing property for pure discrete spectrum in Toeplitz arrays is not yet known. In this paper, we provide a simple equivalent property for pure discrete spectrum on Toeplitz arrays in terms of the periodic structures of the Toeplitz arrays.

2. Preliminary

Let $\Sigma = \{1, 2, \ldots, m\}$ be a finite set of colours and $Z \subseteq \mathbb{Z}^d$ a subgroup isomorphic to $\mathbb{Z}^d$. For $x = \{x(v)\}_{v \in \mathbb{Z}^d} \in \Sigma^{\mathbb{Z}^d}$ we define

$$\mathrm{Per}(x, Z, i) = \{w \in \mathbb{Z}^d : x(w + z) = \sigma \text{ for all } z \in Z\},\ i \in \Sigma,$$

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$$\mathrm{Per}(x, Z) = \bigcup_{\sigma \in \Sigma} \mathrm{Per}(x, Z, i).$$

When $\mathrm{Per}(x, Z) \neq \emptyset$, we say that Z is a *group of periods of* x. We say that x is a $\mathbb{Z}^d$-Toeplitz array (or a Toeplitz array) if for any $1 \leq i \leq m$ and $v \in \mathbb{Z}^d$, there exists $Z \subseteq \mathbb{Z}^d$ subgroup isomorphic to $\mathbb{Z}^d$ such that $v + Z \subseteq \mathrm{Per}(x, Z, i)$.

Let $x \in \Sigma^{\mathbb{Z}^d}$. A group $Z \subset \mathbb{Z}^d$ of periods of x is called a *group generated by essential periods of* x if $\mathrm{Per}(x, Z) \subseteq \mathrm{Per}(x, Z')$ implies $Z' \subseteq Z$. It is shown in [5] that for any group $Z \subseteq \mathbb{Z}^d$ of periods of x, there exists $K \subseteq \mathbb{Z}^d$ a group generated by essential periods of x such that $\mathrm{Per}(x, Z) \subseteq \mathrm{Per}(x, K)$. Furthermore there exists a sequence $\{Z_n\}_{n \geq 0}$ of groups generated by essential periods of x such that $Z_{n+1} \subseteq Z_n$ and $\bigcup_{n \geq 0} \mathrm{Per}(x, Z_n) = \mathbb{Z}^d$. We call the sequence of groups $\{Z_n\}_{n \geq 0}$ a *period structure* of x.

Let $\{Z_i\}_{i \geq 0}$ be a decreasing sequence of subgroups isomorphic to $\mathbb{Z}^d$ and let $\phi_i : \mathbb{Z}^d/Z_{i+1} \to \mathbb{Z}^d/Z_i$ be the function induced by the inclusion $Z_{i+1} \subseteq Z_i$, $i \geq 0$. Consider the inverse limit

$$G = \lim_{\leftarrow i} (\mathbb{Z}^d/Z_i, \phi_i).$$

Consider the homomorphism $\tau : \mathbb{Z}^d \to \Pi_{i \geq 0} \mathbb{Z}^d/Z_i$ defined for $v \in \mathbb{Z}^d$ by

$$\tau(v) = \{\tau_i(v)\}_{i \geq 0},$$

where $\tau_i : \mathbb{Z}^d \to \mathbb{Z}^d/Z_i$ is the canonical projection. The image of $\mathbb{Z}^d$ by τ is dense in G, which implies that the $\mathbb{Z}^d$-action $v(\mathbf{g}) = \tau(v) + \mathbf{g}$, $v \in \mathbb{Z}^d$, $\mathbf{g} \in G$, is well defined and $(G, \mathbb{Z}^d)$ is a minimal equicontinuous system. We call $(G, \mathbb{Z}^d)$ an *odometer system* or simply an *odometer*.

A *van Hove sequence* for $\mathbb{R}^d$ is a sequence $\mathcal{F} = \{F_n\}_{n \geq 1}$ of bounded measurable subsets of $\mathbb{R}^d$ satisfying

$$\lim_{n \to \infty} \mathrm{Vol}((\partial F_n)^{+r})/\mathrm{Vol}(F_n) = 0, \text{ for all } r > 0. \tag{2.1}$$

A multi-colour set or m-multi-colour set in $\mathbb{Z}^d$ is a subset $\mathbf{\Lambda} = \cup_{i \leq m} (\Lambda_i, i)$, where $\Lambda_i \subseteq \mathbb{Z}^d$ and $1 \leq i \leq m$. It is convenient to think of it as a set whose points come in various colours, i being the colour of the points in Λ_i. Then $\mathbf{\Lambda}$ is a $\mathbb{Z}^d$-Toeplitz array (or a Toeplitz array) if for any $1 \leq i \leq m$ and $v \in \Lambda_i$, there exists $Z \subseteq \mathbb{Z}^d$ subgroup isomorphic to $\mathbb{Z}^d$ such that $v + Z \subseteq \Lambda_i$.

Let $\mathbf{\Lambda}$ be a Toeplitz array in $\mathbb{Z}^d$. Let us define a dynamical hull defined by a local topology. Let $X_{\mathbf{\Lambda}} = \overline{\{x + \mathbf{\Lambda} : x \in \mathbb{Z}^d\}}^{\rho}$, where ρ is the metric on the set of multi-colour points in $\mathbb{Z}^d$ for which two point sets are close if they agree on a large region around the origin with small shift(see the detail of this topology in [3, 12]). Then we get a topological dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$ with a translation action of $\mathbb{Z}^d$.

On the other hand, we can consider a slightly different pseudo metric e on the set of $\mathbb{Z}^d$-Toeplitz arrays. For any two $\mathbb{Z}^d$-Toeplitz arrays $\mathbf{\Lambda}', \mathbf{\Lambda}''$, we define

$$e(\mathbf{\Lambda}', \mathbf{\Lambda}'') = \limsup_{n\to\infty} \frac{\sum_{i=1}^{m} \sharp((x+\Lambda_i')\triangle\Lambda_i'') \cap F_n)}{\sharp(\mathbb{Z}^d \cap F_n)},$$

where $\{F_n\}$ is a van Hove sequence on $\mathbb{R}^2$. We let $A(\mathbf{\Lambda}) = \overline{\{x+\mathbf{\Lambda} : x \in \mathbb{Z}^d\}^e}$ be the orbit closure of $\mathbf{\Lambda}$ under the topology defined by the pseudo metric e(see [3, 11]for the detail about this topology). For each $\epsilon > 0$, define

$$U(\epsilon) := \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : e(x+\mathbf{\Lambda}, y+\mathbf{\Lambda}) < \epsilon\}.$$

Let $\mathcal{U} = \{U(\epsilon) \subset \mathbb{Z}^d \times \mathbb{Z}^d : \epsilon > 0\}$. Then $\mathcal{U}$ forms a fundamental set of entourages for a uniformity on $\mathbb{Z}^d$. Since each $U(\epsilon)$ is $\mathbb{Z}^d$-invariant, we obtain a topological group structure on $\mathbb{Z}^d$. Let $A(\mathbf{\Lambda})$ be the completion of $\mathbb{Z}^d$ in this topology, which is a new topological group. For $y \in \mathbb{Z}^d$ and $U \in \mathcal{U}$, define $U[y] = \{x \in \mathbb{Z}^d : (x, y) \in U\}$. Let $P_\epsilon = \{x \in \mathbb{Z}^d : e(x+\mathbf{\Lambda}, \mathbf{\Lambda}) < \epsilon\}$ for each $\epsilon > 0$. Then $U(\epsilon)[0] = P_\epsilon$. Note that for any $\epsilon > 2e(\mathbf{\Lambda}, \emptyset)$, $P_\epsilon = \mathbb{Z}^d$. Although $A(\mathbf{\Lambda})$ is the completion of $\mathbb{Z}^d$ under this topology, one can also think of it as the hull (completion) of $\mathbf{\Lambda}$ under translation action by $\mathbb{Z}^d$ when the topology is supplied by the pseudo-metric e.

Let $\{F_n\}_{n\geq 1}$ be a van Hove sequence. A cluster of $\mathbf{\Lambda}$ is a family $\mathbf{P} = (P_i)_{i\leq m}$ where $P_i \subset \Lambda_i$ is finite for all $i \leq m$. The Toeplitz array $\mathbf{\Lambda}$ has *uniform cluster frequencies* (UCF) (relative to $\{F_n\}_{n\geq 1}$) if for any cluster $\mathbf{P}$, the limit

$$\mathrm{freq}(\mathbf{P}, \mathbf{\Lambda}) = \lim_{n\to\infty} \frac{L_{\mathbf{P}}(x+F_n)}{\mathrm{Vol}(F_n)} \geq 0\,,$$

where $L_{\mathbf{P}}(x+F_n) = \sharp\{v \in \mathbb{R}^d : v+\mathbf{P} \subset (x+F_n)\cap\mathbf{\Lambda}\}$, exists uniformly in $x \in \mathbb{R}^d$. It is known that for a $\mathbb{Z}^d$-Toeplitz array $\mathbf{\Lambda}$, the dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$ is uniquely ergodic if and only if $\mathbf{\Lambda}$ has UCF (see [12, Theorem 2.7]).

2.1. Maximal equicontinuous factor The topological dynamical system $(X, \mathbb{Z}^d)$ is said to be *equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{\Gamma}, \mathbf{\Gamma}' \in X$ satisfy $d(\mathbf{\Gamma}, \mathbf{\Gamma}') < \delta$, then $d(\mathbf{\Gamma}+v, \mathbf{\Gamma}'+v) < \delta$ for all $v \in \mathbb{Z}^d$. We say that $(X, \mathbb{Z}^d)$ is an *extension* of $(Y, \mathbb{Z}^d)$, or $(Y, \mathbb{Z}^d)$ is a factor of $(X, \mathbb{Z}^d)$, if there exists a continuous surjection $\phi : X \to Y$ such that ϕ preserves the $\mathbb{Z}^d$-action. We call ϕ a factor map. We call $(Y, \mathbb{Z}^d)$ the *maximal equicontinuous factor* of $(X, \mathbb{Z}^d)$ if it is an equicontinuous factor of $(X, \mathbb{Z}^d)$ such that for any other equicontinuous factor $(Y', \mathbb{Z}^d)$ of $(X, \mathbb{Z}^d)$ there exists a factor map $\phi : Y \to Y'$ that satisfies $\phi \circ f = f'$, with $f : X \to Y$ and $f' : X \to Y'$ factor maps.

Proposition 2.1 ([5, Proposition 5])**.** *If $\{Z_n\}_{n\geq 0}$ is a period structure of $\mathbf{\Lambda}$, then the odometer $G = \lim_{\leftarrow}(\mathbb{Z}^d/Z_n, \phi_n)$ is the maximal equicontinuous factor of $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$.*

2.2. Dynamical spectrum and diffraction spectrum Suppose that $\mathbf{\Lambda}$ is a $\mathbb{Z}^d$-Toeplitz array with UCF. In this case, we observe that the autocorrelation is unique for any measure of the form

$$\nu = \sum_{i\leq m} a_i \delta_{\Lambda_i}, \quad \text{where} \ \ \delta_{\Lambda_i} = \sum_{x\in\Lambda_i} \delta_x \text{ and } a_i \in \mathbb{C}\,. \tag{2.2}$$

Let $\gamma(\nu)$ denote its autocorrelation, that is, the vague limit

$$\gamma(\nu) = \lim_{n\to\infty} \frac{1}{\mathrm{Vol}(F_n)} \left(\nu|_{F_n} * \widetilde{\nu}|_{F_n}\right), \tag{2.3}$$

where $*$ is a convolution of the two measures, $\{F_n\}_{n\geq 1}$ is a van Hove sequence and $\tilde{\nu} = \sum_{i\leq m} \overline{a_i}\delta_{-\Lambda_i}$. Simple computation shows

$$\gamma(\nu) = \sum_{i,j=1}^{m} a_i \overline{a}_j \sum_{y\in\Lambda_i, z\in\Lambda_j} \mathrm{freq}((y,z), \mathbf{\Lambda})\delta_{y-z}. \tag{2.4}$$

Here (y, z) stands for a cluster consisting of two points $y \in \Lambda_i$, $z \in \Lambda_j$. The measure $\gamma(\nu)$ is positive definite, so by Bochner's Theorem the Fourier transform $\widehat{\gamma(\nu)}$ is a positive measure on $\mathbb{R}^d$, called the *diffraction measure* for ν. We say that the measure ν(or $\mathbf{\Lambda}$) has *pure point diffraction spectrum* if $\widehat{\gamma(\nu)}$ is a pure point or discrete measure.

On the other hand, we also have the measure-preserving system $(X_{\mathbf{\Lambda}}, \mu, \mathbb{Z}^d)$ associated with $\mathbf{\Lambda}$. Consider the associated group of unitary operators $\{U_x\}_{x\in\mathbb{Z}^d}$ on $L^2(X_{\mathbf{\Lambda}}, \mu)$:

$$U_x f(\mathbf{\Lambda}') = f(-x + \mathbf{\Lambda}').$$

Every $f \in L^2(X_{\mathbf{\Lambda}}, \mu)$ defines a function on $\mathbb{Z}^d$ by $x \mapsto (U_x f, f)$. This function is positive definite on $\mathbb{Z}^d$, so its Fourier transform is a positive measure σ_f on $\mathbb{R}^d$ called the *spectral measure* corresponding to f. We say that the Toeplitz array $\mathbf{\Lambda}$ has *pure discrete dynamical spectrum* if σ_f is pure discrete for every $f \in L^2(X_{\mathbf{\Lambda}}, \mu)$. We recall that $f \in L^2(X_{\mathbf{\Lambda}}, \mu)$ is an eigenfunction for the $\mathbb{Z}^d$-action if for some $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$,

$$U_x f = e^{2\pi i x\cdot\alpha} f, \quad \text{for all} \ \ x \in \mathbb{Z}^d,$$

where $\cdot$ is the standard inner product on $\mathbb{Z}^d$.

We recall the following series of theorems in the literature. We consider a $\mathbb{Z}^d$-Toeplitz array $\mathbf{\Lambda}$ with UCF.

Theorem 2.2 ([2, 8, 12]). *$\mathbf{\Lambda}$ has pure point diffraction spectrum if and only if $\mathbf{\Lambda}$ has pure discrete dynamical spectrum.*

Theorem 2.3 ([4, Theorem 5]). *$\mathbf{\Lambda}$ has pure point diffraction spectrum if and only if for all ϵ, P_ϵ is relatively dense.*

Theorem 2.4 ([4]). *$\mathbf{\Lambda}$ has pure point diffraction spectrum if and only if $A(\mathbf{\Lambda})$ is compact.*

Theorem 2.5 ([3, Theorem 7]). *The dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$ has pure discrete dynamical spectrum with continuous eigenfunctions if and only if there exists a continuous surjective $\mathbb{Z}^d$-map $\beta : X_{\mathbf{\Lambda}} \to A(\mathbf{\Lambda})$.*

3. Characterization of Pure Discrete Dynamical Spectrum on $\mathbb{Z}^d$-Toeplitz Arrays

For two $\mathbb{Z}^d$-Toeplitz arrays $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$, define

$$\text{dens}(\mathbf{\Lambda} \cap \mathbf{\Gamma}) = \lim_{n\to\infty} \frac{\sharp(\mathbf{\Lambda} \cap \mathbf{\Gamma} \cap F_n)}{\sharp(\mathbb{Z}^d \cap F_n)},$$

where $\{F_n\}_{n\in\mathbb{N}}$ is a van Hove sequence on $\mathbb{R}^d$.

Theorem 3.1. *Let $\mathbf{\Lambda}$ be a Toeplitz array in $\mathbb{Z}^d$ and $\{p_n\mathbb{Z}^d\}_{n\in\mathbb{N}}$ is a periodic structure of $\mathbf{\Lambda}$. Let $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$ be a minimal topological dynamical system with unique ergodicity. Suppose that every measurable eigenfunctions for $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$ can be considered as continuous eigenfunctions. Then $\mathbf{\Lambda}$ has pure discrete dynamical spectrum if and only if for any $\epsilon > 0$, there exists $p_m\mathbb{Z}^d \in \{p_n\mathbb{Z}^d\}_{n\in\mathbb{N}}$ such that*

$$dens((t + \mathbf{\Lambda}) \cap \mathbf{\Lambda}) > 1 - \epsilon, \ \forall \ t \in p_m\mathbb{Z}^d. \tag{3.1}$$

Proof. Since $\mathbf{\Lambda}$ is a $\mathbb{Z}^d$-Toeplitz array, the odometer $G = \lim_{\leftarrow}(\mathbb{Z}^d/p_n\mathbb{Z}^d, \phi_n)$ is the maximal equicontinuous factor of $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$ from Proposition 2.1. Let $\psi : X_{\mathbf{\Lambda}} \to G$ be an equicontinuous factor map. Then ψ is a continuous and surjective map.

From Theorem 2.2 and Theorem 2.5, if $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$ has pure discrete dynamical spectrum with continuous eigenfunctions, there exists a continuous onto map $\phi : G \to A(\mathbf{\Lambda})$, since G is the maximal equicontinuous factor. It is known from

Theorem 2.3 that $\mathbf{\Lambda}$ has pure point diffraction spectrum if and only if P_ϵ is relatively dense. Note that P_ϵ is an open set in $A(\mathbf{\Lambda})$. Since $\phi^{-1}(P_\epsilon)$ is an open set by continuous map ϕ, $\exists\ m \in \mathbb{N}$ s.t $p_m\mathbb{Z}^d \subset P_\epsilon$ where $p_m\mathbb{Z}^d \in \{p_n\mathbb{Z}^d\}_{n\in\mathbb{N}}$. Thus if $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$ is pure discrete dynamical spectrum with continuous eigenfunctions and has unique ergodicity, then the formula (3.1) follows. On the other hand, if the formula (3.1) holds, then P_ϵ becomes relatively dense. From Theorem 2.3 and Theorem 2.2, $(X_{\mathbf{\Lambda}}, \mathbb{Z}^d)$ is pure discrete dynamical spectrum. □

Remark 3.2. The paper [6] provides a sufficient condition for $L_2(X_{\mathbf{\Lambda}}, \mu)$ to have continuous eigenfunctions in linear recurrent dynamical Cantor system, although it is not a necessary condition. It would be interesting to know that under what conditions, every measurable eigenfunctions in $L_2(X_{\mathbf{\Lambda}}, \mu)$ can be considered as continuous eigenfunctions.

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