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ON SOME PROPERTIES OF BARRIERS AT INFINITY FOR SECOND ORDER UNIFORMLY ELLIPTIC OPERATORS

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Abstract. We consider the boundary value problem with a Dirichlet condition for a second order linear uniformly elliptic operator in a non-divergence form. We study some properties of a barrier at infinity which was introduced by Meyers and Serrin to investigate a solution in an exterior domains. Also, we construct a modified barrier for more general domain than an exterior domain.

1. Introduction

We consider the following second order linear uniformly elliptic partial differential operator in a non-divergence form:

$$L = \sum_{i,j=1}^{n} a_{ij}(x)D_{ij} + \sum_{i=1}^{n} b_i(x)D_i + c(x), \quad c(x) \le 0, \tag{L}$$

where D_i represents a partial derivative with x_i direction, namely

$$D_i = \frac{\partial}{\partial x_i}, \quad \text{and} \quad D_{ij} = D_j D_i = \frac{\partial^2}{\partial x_i \partial x_j}.$$

A uniformly ellipticity means that, for some strictly positive constant λ_1, λ_2,

$$\lambda_1 |x|^2 \le \sum_{i,j=1}^{n} a_{ij}(x) x_i x_j \le \lambda_2 |x|^2, \quad x = (x_1, x_2, \ldots, x_n), \tag{UE}$$

for any $x \in \mathbb{R}^n$. When $b_i(x) = c(x) = 0$, the operator turns into

$$\Lambda = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}, \tag{Λ}$$

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which will be frequently treated. A prototype of these operators is the well known Laplace operator:

$$(\Delta) \qquad \Delta = \sum_{i=1}^{n} D_{ii} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial x_i}.$$

In fact, the operator (Λ) turns into the Laplace operator (Δ) when $\lambda_1 = \lambda_2 = 1$.

With these operators, we consider the following boundary value problem of the first kind, so called a Dirichlet problem:

$$(1.1) \qquad \begin{cases} Lu = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is a domain, open and connected set in $\mathbb{R}^n$, $\partial\Omega$ denotes its topological boundary, and g is a given function defined on $\partial\Omega$.

Throughout the paper, we assume that the given data, such as $g, \partial\Omega, a_{ij}, b_i, c$ are smooth. Thus we deal a solution in a classical sense, namely, a solution u is differentiable up to second order, $u \in C^2$, provided that if it exists.

When Ω is bounded, to obtain the uniqueness of a solution, it is common to use the maximum principle. In fact, the sign condition $c(x) \leq 0$ from (L) is needed to hold the principle. The following version of the maximum principle appears in [4, Corollary 3.2]. See also [10] for more about maximum principles.

Theorem 1.1 (Maximum principle)**.** *Let L be elliptic in a bounded domain Ω. Suppose that in Ω,*

$$Lu \geq 0 (\leq 0), \quad c \leq 0,$$

with $u \in C^0(\overline{\Omega})$. Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ \quad (\inf_{\Omega} u \geq \inf_{\partial\Omega} u^-).$$

If $Lu = 0$ in Ω, then

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|.$$

If we have two solutions u_1, u_2, then $u_1 - u_2$ is also a solution with a zero condition on its boundary. Thus, $u_1 = u_2$ by the maximum principle, Theorem 1.1.

For further results of bounded domains, such as a existence, uniqueness, regularity, one may refer to [4, 2, 6, 7].

But for unbounded domains, the uniqueness is not immediate. For example, one consider for $n \geq 3$,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus B_1(0), \\ u = 0 & \text{on } \partial B_1(0). \end{cases}$$

Here, $B_1(0)$ denotes a open ball of radius 1 centered at the origin. For the problem, $u(x) = c(1 - \frac{1}{|x|^{n-2}})$ for any constant c are solutions. Thus, the uniqueness fails in general, for an unbounded domain. The domain $\mathbb{R}^n \setminus B_1(0)$ is called an exterior domain, which means the complement of a bounded domain.

Among other things, Meyers and Serrin [9] considered the existence and uniqueness of a Dirichlet problem with an additional condition at infinity in an exterior domain. For the purpose, they introduced a barrier at infinity (see Definition 2.2), which has an important role for the behavior of a solution at infinity.

In this paper, we study some properties of a barrier at infinity.

Briefly, we illustrate the contents of the paper. In Section 2, we prepare some known results, and in Section 3, we present main results about some properties of barrier functions including a non-uniqueness, decay at infinity, explicit barrier functions for some uniformly elliptic operators, and some examples related.

Throughout the paper, the summation convention over repeated indices is assumed.

2. Preliminaries

In this section, we gather some known results to present the main results in Section 3, and introduce some definitions.

The following version of a maximum principle is called the strong maximum principle due to E. Hopf [5]. One may compare Theorem 1.1 which is also called a weak maximum principle.

Theorem 2.1 (Strong maximum principle). *Let L be uniformly elliptic, $c = 0$ and $Lu \geq 0(\leq)$ in a domain Ω (not necessarily bounded). Then if u achieves its maximum (minimum) in the interior of Ω, u is constant.*

Proof. For the proof, one may refer to [4, Theorem 3.5]. □

For the next we present a Harnack inequality:

Theorem 2.2 (Harnack Inequality)**.** *Let $u \in W^{2,n}(\Omega)$ satisfy $Lu = 0$, $u \geq 0$ in Ω. Here, $W^{2,n}$ is a Sobolev space. Then for any ball $B_{2r}(y) \subset \Omega$, we have*

$$\sup_{B_R(y)} u \leq C \inf_{B_R(y)} u,$$

where $C = C(n, \lambda_1, \lambda_2)$

Proof. For the proof, see [4, Corollary 9.22], for example. □

In [9], Meyers and Serrin treated a Dirichlet problem for second order elliptic partial differential equations in an exterior domain:

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

As mentioned in Sec. 1. Introduction, this problem is not well set, namely we do not have the existence and uniqueness of the solution, since the behavior near infinity is not fixed. In this respect, they treat two types of Dirichlet problems:

Definition 2.1. The exterior Dirichlet problem I: The function u tend to an assigned limit l as x tends to infinity.

The exterior Dirichlet problem II: The function u be bounded in Ω.

We say that the given Dirichlet problem is well set if there exists a solution and the solution is unique. For the existence of the exterior Dirichlet problem I, the following barrier function is useful.

Definition 2.2. Barrier at infinity:

(i) v is defined and positive in some neighborhood of infinity.

(ii) v tend to 0 as $r \to \infty$.

(iii) $v \in C^2$ and $\Lambda v \leq 0$.

or

(iii*) $v \in C$; if u satisfies $\Lambda u \leq 0$ in some region R which is contained the domain, and if $u \leq v$ at every point of FR, then $u \leq v$ in R.

The role of barrier function become clear from the next theorem. From [9, Theorem 1], we have that there exists a barrier at infinity if and only if the exterior Dirichlet problem I is well set. We state it as a following theorem:

Theorem 2.3. *Problem I is well set for the equation $\Lambda u = 0$ if and only if this equation has a barrier at infinity.*

Thus the existence of a barrier is necessary and sufficient condition for the well set of the Dirichlet problem type 1. In the next section, we study the various properties of the barrier at infinity.

3. Properties of a Barrier at Infinity

In this main section, we study some various properties of a barrier at infinity. We begin this section with the study of a uniqueness about a barrier function at infinity (see Definition 2.2).

3.1. A uniqueness of a barrier at infinity

Example 3.1. Consider the Laplace operator $\Delta = \delta_{ij} D_{ij} = D_{ii}$ in $\mathbb{R}^n$, $n \geq 3$.

Choose v to be $v = |x|^{-\mu}$, $\mu > 0$. Then it is obvious to see that it satisfies (i), (ii) of Definition 2.2. For (iii), note that

$$D_i v = -\mu |x|^{-\mu-2} x_i,$$

$$D_{ii} v = -\mu |x|^{-\mu-2} + \mu(\mu+2)|x|^{-\mu-4} x_i^2,$$

$$\Delta v = \mu |x|^{-\mu-4}(-\delta_{ij}|x|^2 + \delta_{ij}(\mu+2)x_i^2) = \mu |x|^{-\mu-4}(-n|x|^2 + (\mu+2)|x|^2).$$

Thus we have $\Delta v \leq 0$ for $\mu \leq n-2$ for $n \geq 3$. In all, $v = |x|^{-\mu}$ are barriers at infinity for $\mu \leq n-2$, $n \geq 3$. Note also that

$$\begin{cases} 0 < \mu < n-2 & \to \Delta v < 0, \\ \mu = n-2 & \to \Delta v = 0, \\ \mu > n-2 & \to \Delta v > 0. \end{cases}$$

The above example shows us that a barrier function is not unique. We state the observation as a theorem without a proof.

Theorem 3.1. *A barrier at infinity is not unique, in general.*

3.2. Decay of a barrier at infinity For the next, we consider the decay of a barrier at infinity.

Let v be a barrier at infinity corresponding to Λ defined in $\mathbb{R}^n \setminus B_R$ for some $R > 0$. By Theorem 2.3, we obtain the solution of the exterior Dirichlet problem I, namely,

$$\Lambda u = 0 \text{ in } \mathbb{R}^n \setminus B_R, \quad u = v \text{ in } \partial B_R, \quad \lim_{x \to \infty} u = 0.$$

Note that the solution u is also a barrier at infinity. For this, it is enough to check that $u > 0$. If $u(x_0) < 0$ for some x_0, then for sufficiently small $\epsilon > 0$, $u + \epsilon < 0$

for some compact domain D containing x_0, $u = 0$ on ∂D. But it contradict to the maximum principle, Theorem 1.1.

If $u(x_0) = 0$ for some x_0 as a local minimum, it also contradicts the strong maximum principle, Theorem 2.1.

Thus $u > 0$ in $\mathbb{R}^n \setminus B_R$. Thus in all, u is a barrier at infinity.

We claim that a solution u has a maximal growth (the fastest decaying) barrier at infinity.

If not, there exists a barrier w such that $w << u$ in some neighborhood of infinity. Without loss of generality we may assume that u, w are defined in $\mathbb{R}^n \setminus B_R$, and $w > cu$ on ∂B_R for some positive constant c.

Since $w << u$, there exist $R' > 0$ and a open connected set D such that $(\mathbb{R}^n \setminus B_{R'}) \subset D$, $w = cu$ on ∂D, and $w \leq cu$ in D.

We claim $w = cu$ in D. If not, there exists a point x_0 such that $w(x_0) < cu(x_0)$. We can choose $\epsilon > 0$ such that $w(x_0) + \epsilon < cu(x_0)$. Let $D' := \{x \in D | w(x) + \epsilon < cu(x)\}$. Since $\lim_{x\to\infty} w + \epsilon > 0$ and $\lim_{x\to\infty} cu = 0$, D' is bounded and $w + \epsilon = cu$ on $\partial D'$. Note that $\Lambda(w + \epsilon) \leq \Lambda cu = 0$, by a maximum principle, $w + \epsilon \geq cu$ in D', which leads to a contradiction.

Thus in all, $w << u$ is not a possible case. In conclusion we have that the solution u is a maximal growth (the fastest decaying) barrier at infinity.

But for the slowest decaying barrier, it is not clear. We may impose the following question:

Problem: Is there a minimal (slowest decaying) growth barrier at infinity for a fixed linear operator L?

For the Laplace operator, we observe the following:

Let v be a barrier at infinity for the Laplace operator, and $V := (-\ln v(x))^{-m}$. Then

$$D_i V = m(-\ln v)^{-m-1} v^{-1} D_i v,$$

and

$$\begin{aligned} D_{ii} V =& m(m+1)(-\ln v)^{-m-2} v^{-2} (D_i v)^2 - m(-\ln v)^{-m-1} v^{-2} (D_i v)^2 \\ &+ m(-\ln v)^{-m-1} v^{-1} D_{ii} v. \end{aligned}$$

Thus

$$\Delta V = [m(m+1) - m(-\ln v)]\,(-\ln v)^{-m-2} v^{-2} |Dv|^2 + m(-\ln v)^{-m-1} v^{-1} \Delta v.$$

Here, $-\ln v \to +\infty$ since $v \to 0$ as $x \to \infty$. Thus in all, $\Delta V \leq 0$ for $\Delta v \leq 0$.

More generally, let v be a barrier at infinity for Λ, and $V := (-\ln v(x))^{-m}$. Then, by direct computations,

$$D_iV = m(-\ln v)^{-m-1}v^{-1}D_iv,$$

and

$$\begin{aligned} D_{ij}V =& m(m+1)(-\ln v)^{-m-2}v^{-2}D_ivD_jv - m(-\ln v)^{-m-1}v^{-2}D_ivD_jv \\ &+ m(-\ln v)^{-m-1}v^{-1}D_{ij}v. \end{aligned}$$

Thus,

$$\begin{aligned} \Lambda V \leq& m(m+1)(-\ln v)^{-m-2}v^{-2}\lambda_2|Dv|^2 - m(-\ln v)^{-m-1}v^{-2}\lambda_1|Dv|^2 \\ &+ m(-\ln v)^{-m-1}v^{-1}\Lambda v. \end{aligned}$$

Here, $-\ln v \to +\infty$ since $v \to 0$ as $x \to \infty$. Thus in all, $\Lambda V \leq 0$ for $\Lambda v \leq 0$.

Note that

$$\lim_{x\to\infty} \frac{v(x)}{(-\ln v(x))^{-m}} = \lim_{y\to 0} \frac{y}{(-\ln y)^{-m}} = 0.$$

Hence for Δ or Λ, if we have a barrier v at infinity we always obtain another barrier V, which vanishes to 0 more slowly.

For the next, we consider the behavior of a given barrier at infinity with respect to directions.

Using a Harnack inequality, a solution has the following property: it has the comparable growth near infinity.

Theorem 3.2. *Let $\Lambda u = 0$ in $\mathbb{R}^n \setminus B_R(0)$ and positive. Then, for any $x, y \in \mathbb{R}^n$ such that $|x| = |y| \geq 2R$, the following holds:*

$$N_1 \leq \frac{u(x)}{u(y)} \leq N_2 \tag{3.1}$$

for some positive fixed constant N_1, N_2 depending on n, λ_1, λ_2.

Proof. We use a Harnack inequality. First consider the following covering:

$$T := \{B_1(z) |\, |z| = 2\}.$$

Since T covers the set $\{1 < |z| < 3\}$ and the set $\{\frac{3}{2} \leq |z| \leq \frac{5}{2}\}$ is compact, there exists a finite covering

$$T_n := \{B_1(z_i) |\, |z_i| = 2, i = 1, 2, ..., t_n\},$$

where t_n depends on the dimensional constant n.

Note that

$$T_n^r := \{B_r(rz_i), i = 1, 2, ..., t_n\}, \quad r > 2R,$$

covers $\{\frac{3r}{2} \le |z| \le \frac{5r}{2}\}$. Thus for any $|x| = |y| \ge 2R$, using Harnack inequality t_n-times at most,

$$u(x) \le N^{t_n} u(y), \quad u(y) \le N^{t_n} u(x),$$

where N depends on n, λ_1, λ_2. This leads to

$$N_1 \le \frac{u(x)}{u(y)} \le N_2, \tag{3.2}$$

where $N_1^{-1} = N_2 = N^{t_n}$. □

From Theorem 3.2, for a solution u at infinity, it follows that

$$0 < N_1 \le \liminf_{r\to\infty} \frac{u(rx)}{u(ry)} \le \limsup_{r\to\infty} \frac{u(rx)}{u(ry)} \le N_2 < +\infty,$$

for any $|x| = |y| = 1$. Thus a solution has a uniform decay at infinity in every direction.

Remark 3.2. In general we do not have a Harnack inequality for the super solution. Thus for a general barrier at infinity satisfying $\Lambda v \le 0$, we do not have the above result.

3.3. Barriers at infinity for some uniformly elliptic operators For some uniformly elliptic operators, explicitly, we can construct barriers at infinity.

Theorem 3.3. *For a uniform elliptic operator Λ, there exists a barrier at infinity of the form $|x|^{-\mu}$ provided that $\frac{n\lambda_1}{\lambda_2} - 2 > 0$, where λ_1, λ_2 are ellipticity constants from* (UE).

Proof. Let $v = |x|^{-\mu}$ for $\mu > 0$. Then it is obvious to see that v satisfies (i), (ii) of Definition 2.2. For (iii), note that

$$D_i v = -\mu|x|^{-\mu-2}x_i, \qquad D_{ij}v = -\mu|x|^{-\mu-2}\delta_{ij} + \mu(\mu+2)|x|^{-\mu-4}x_i x_j,$$

$$\Lambda v \le -\mu|x|^{-\mu-2}n\lambda_1 + \mu(\mu+2)|x|^{-\mu-4}\lambda_2|x|^2$$

$$= \mu|x|^{-\mu-2}(-n\lambda_1 + (\mu+2)\lambda_2) \le 0$$

for $\mu \in (0, \frac{n\lambda_1}{\lambda_2} - 2]$. □

In fact, there is other type of barriers.

Example 3.3. Choose $v = (\log |x|)^{-m}$. Then
$D_i v = -m(\log |x|)^{-m-1}|x|^{-2}x_i$,
$D_{ij}v = m(m+1)(\log |x|)^{-m-2}|x|^{-4}x_ix_j - m(\log |x|)^{-m-1}(|x|^{-2}\delta_{ij} - 2|x|^{-4}x_ix_j)$
$= m(\log |x|)^{-m-2}\left((m+1)|x|^{-4}x_ix_j - (\log |x|)(|x|^{-2}\delta_{ij} - 2|x|^{-4}x_ix_j\right)$,
$\Lambda v \le m(\log |x|)^{-m-2}\left((m+1)\lambda_2|x|^{-2} - (\log |x|)(n\lambda_1|x|^{-2} - 2\lambda_2|x|^{-2}\right)$
$= m(\log |x|)^{-m-2}|x|^{-2}\left((m+1)\lambda_2 + (\log |x|)(-n\lambda_1 + 2\lambda_2)\right) \le 0 \quad \text{for } \frac{n\lambda_1}{\lambda_2} - 2 > 0.$

Remark 3.4. For $n \ge 3$, the fundamental solution $\Gamma(x) = |x|^{2-n}$ is a barrier at infinity for the Laplace operator Δ. Then by direct computations $\Gamma(x_1, x_2, ..., \frac{x_n}{\sqrt{\lambda}})$ is a barrier for the operator $\Lambda = \Delta + (\lambda - 1)D_{nn}$ for any $\lambda > 0$. Note that the operator Λ has sufficiently small or large ellipticity constant λ_1 and 1, 1 and λ_2 depending on λ. Thus it is possible Theorem 3.3 does hold even when $\frac{n\lambda_1}{\lambda_2} - 2 > 0$ does not hold.

Furthermore, with the same uniform elliptic constants λ_1, λ_2, there are two operator Λ_1, Λ_2 such that Λ_1 has a barrier and Λ_2 has an anti barrier (see Definition 3.7). For this, see [9, Section 9].

Theorem 3.4. *For a uniform elliptic operator Λ with $a_{ij} = a_{ji}$, $n \ge 3$, there exists a barrier at infinity if the coefficients $a_{ij}(x)$ converges to some limits as $x \to \infty$, namely, $a_{ij}(x) \to a^0_{ij}$ as $|x| \to \infty$.*

Proof. Since a_{ij} is symmetric, by a orthogonal transformation, we may assume that $a^0_{ij} = \delta_{ij}$. For any $\epsilon > 0$, there exists a large $M > 0$, such that $|a_{ij}(x) - \delta_{ij}| < \epsilon$ for $|x| > M$. Thus, for $v = |x|^{-\mu}$, $\mu > 0$,

$$\Lambda v = (\Lambda - \Delta)v + \Delta v \le |a_{ij}(x) - \delta_{ij}||D_{ij}v| + \Delta v.$$

Using $|a_{ij}(x) - \delta_{ij}| < \epsilon$,

$$D_{ij}v = -\mu|x|^{-\mu-2}\delta_{ij} + \mu(\mu+2)|x|^{-\mu-4}x_ix_j,$$
$$\Delta v = \mu|x|^{-\mu-4}(-n|x|^2 + (\mu+2)|x|^2),$$
$$\Lambda v \le \epsilon\mu(n + (\mu+2))|x|^{-\mu-2} + \mu(-n+\mu+2)|x|^{-\mu-2} \le 0$$

for $\epsilon \le \frac{n-\mu-2}{n+\mu+2}$, $\mu < n-2$. □

Remark 3.5. From [9, Theorem 3], there is a so called three eigenvalue criterion: For a elliptic case, if $(a_{ik}) \to (a^0_{ik})$, $r \to \infty$, and if (a^0_{ik}) has at least three positive eigenvalues (equivalently if $Rank(a^0_{ik}) \ge 3$), then problem I is well-set for the equation $\Lambda u = 0$, which imply the theorem above.

Also, we remark here that using Theorem 3.3, we can construct a operator Λ, where a_{ij} do not converge at infinity but having a barrier at infinity.

For the next, we apply the method to a general elliptic operator L.

Theorem 3.5. *For a general uniform elliptic operator L, there exists a barrier at infinity provided that $(-a_{ij}(x)\delta_{ij} + (\mu+2)a_{ij}(x)x_ix_j|x|^{-2} - b_i(x)x_i) \leq 0$ for any $x = (x_1, x_2, ..., x_n) \in \Omega$.*

Proof. Let $v = |x|^{-\mu}$ for $\mu > 0$.

Then it is obvious to see that v satisfies (i), (ii) of Definition 2.2. For (iii), note that

$$D_iv = -\mu|x|^{-\mu-2}x_i, \quad D_{ij}v = -\mu|x|^{-\mu-2}\delta_{ij} + \mu(\mu+2)|x|^{-\mu-4}x_ix_j,$$

$$Lv = -\mu|x|^{-\mu-2}a_{ij}\delta_{ij} + \mu(\mu+2)|x|^{-\mu-4}a_{ij}x_ix_j - \mu|x|^{-\mu-2}b_ix_i$$

$$= \mu|x|^{-\mu-2}(-a_{ij}\delta_{ij} + (\mu+2)a_{ij}x_ix_j|x|^{-2} - b_ix_i) \leq 0$$

for $(-a_{ij}(x)\delta_{ij} + (\mu+2)a_{ij}(x)x_ix_j|x|^{-2} - b_i(x)x_i) \leq 0$. □

Corollary 3.6. *For a uniform elliptic operator L, there exists a barrier at infinity provided that $\frac{n\lambda_1 - B}{\lambda_2} - 2 > 0$, where $B := \sup_{\mathbb{R}^n \setminus B_R} b_i(x)x_i$ for some large R.*

Proof. Let $v = |x|^{-\mu}$ for $\mu > 0$.

Similar to the previous proof, check the following:

$$Lv \leq -\mu|x|^{-\mu-2}n\lambda_1 + \mu(\mu+2)|x|^{-\mu-4}\lambda_2|x|^2 - \mu|x|^{-\mu-2}b_ix_i$$

$$= \mu|x|^{-\mu-2}(-n\lambda_1 + (\mu+2)\lambda_2 - b_ix_i) = \mu|x|^{-\mu-2}(-n\lambda_1 + (\mu+2)\lambda_2 + B) \leq 0$$

for $\mu \in (0, \frac{n\lambda_1 - B}{\lambda_2} - 2]$. □

Similar to a barrier, we may apply the above method to an anti-barrier, which also appear in [9, p. 522]. Here anti-barrier means the following:

Definition 3.7. A function v is an anti-barrier at infinity for the equation $\Lambda u = 0$ if

(i) v tends to $+\infty$ as $r \to \infty$ and
(ii) v is twice continuously differentiable and $\Lambda v \leq 0$.

Similar to above, let $v = |x|^\mu$ for $\mu \in (0, 2)$. Then it is obvious to see that v satisfies (i) of Definition 3.7. For (ii), note that

$D_i v = \mu|x|^{\mu-2}x_i, \quad D_{ij}v = \mu|x|^{\mu-2}\delta_{ij} + \mu(\mu-2)|x|^{\mu-4}x_i x_j,$

$\Lambda v \le \mu|x|^{\mu-2}n\lambda_2 + \mu(\mu-2)|x|^{\mu-4}\lambda_1|x|^2 = \mu|x|^{\mu-2}(n\lambda_2 + (\mu-2)\lambda_1) \le 0$

for $0 < \mu \le \frac{-n\lambda_2}{\lambda_1} + 2$. But $\frac{-n\lambda_2}{\lambda_1} + 2 \le 0$.

For the next, one try with $\mu \ge 2$ when $v = |x|^\mu$. For (ii) of Definition 3.7, note that

$\Lambda v \le \mu|x|^{\mu-2}n\lambda_2 + \mu(\mu-2)|x|^{\mu-4}\lambda_2|x|^2 = \mu|x|^{\mu-2}(n\lambda_2 + (\mu-2)\lambda_2).$

The last term is strictly positive for $\mu \ge 2$.

Thus, unlike a barrier at infinity, we do not have a condition to guarantee the existence of anti barrier of the form $|x|^\mu$. But $n = 2$ for the Laplace operator, the fundamental solution $\log|x|$ is an anti barrier at infinity.

Remark 3.8. In fact, from [9, Theorem 2 and Theorem 6], we can conclude that whether $\Delta(t)$ function is a Dini function or not, the barrier or anti barrier does exists, respectively, where

$$\mathcal{A} = \frac{\sum a_{ii}}{\sum a_{ik}\frac{x_i x_k}{|x|^2}}, \quad \mathcal{A} - 2 = \epsilon, \quad \Delta(t) \equiv exp\left\{-\int^t \epsilon(s)\frac{ds}{s}\right\}.$$

Thus, Theorem 3.3 become clear from $\mathcal{A} \ge \frac{n\lambda_1}{\lambda_2}$.

3.4. Some examples There are other types of an unbounded domain than an exterior domain. One may consider the upper half space, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n_+ | x_n > 0, x = (x_1, ..., x_n)\}$, and a Dirichlet problem:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ u = 0 & \text{on } \partial\mathbb{R}^n_+ = \{x_n = 0\}. \end{cases}$$

As one can easily see, $u = cx_n$ for any constant c are solutions. Thus the uniqueness is not guaranteed for the general domain, like an exterior domain.

To treat the general unbounded domain, we may introduce the following definition which is generalized from Definition 2.2.

Definition 3.9. Barrier at infinity:

(i) v is defined and positive in the intersection of some neighborhood of infinity and the given general unbounded domain.

(ii) v tend to 0 as $r \to \infty$ in the domain.

(iii) $v \in C^2$ and $\Lambda v \le 0$ in the domain.

or

(iii*) $v \in C$; if u satisfies $\Lambda u \leq 0$ in some region R, and if $u \leq v$ at every point of FR, then $u \leq v$ in R.

Let $\Omega \subset \overline{\Omega} \subset \mathbb{R}^n_+$, $n \geq 2$, and $v := \frac{x_n}{|x|^\alpha} = x_n|x|^{-\alpha}$ for $\alpha > 1$. Then v satisfies Definition 3.9 for the Laplace operator Δ. Note that

$$D_i v = \delta_{in}|x|^{-\alpha} - \alpha x_n x_i |x|^{-\alpha-2},$$

$$D_{ij} v = -\alpha\delta_{in}x_j|x|^{-\alpha-2} - \alpha\delta_{jn}x_i|x|^{-\alpha-2} - \alpha x_n\delta_{ij}|x|^{-\alpha-2} + \alpha(\alpha+2)x_n x_i x_j|x|^{-\alpha-4},$$

$$\Delta v = -\alpha|x|^{-\alpha-2}\left(x_n + x_n + n x_n\right) + \alpha(\alpha+2)x_n|x|^{-\alpha-2} \leq 0$$

if $\alpha \leq n$. More generally, for a barrier at infinity v satisfying Definition 2.2, one consider the following:

$$D_i(x_n v) = \delta_{in}v + x_n D_i v, \quad D_{ij}(x_n v) = \delta_{in}D_j v + \delta_{nj}D_i v + x_n D_{ij} v,$$

$$\Lambda(x_n v) = a_{nj}D_j v + a_{in}D_i v + x_n\Lambda v \leq 0$$

provided that $a_{ni} = a_{in} = \delta_{ni}$, $D_n v < 0$.

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