J. Appl. Math. & Informatics Vol. **36***(2018), No. 3 - 4, pp. 301 - 306*
https://doi.org/10.14317/jami.2018.301

PORTFOLIO AND CONSUMPTION OPTIMIZATION PROBLEM WITH COBB-DOUGLAS UTILITY AND NEGATIVE WEALTH CONSTRAINTS†

KUM-HWAN ROH

ABSTRACT. I obtain the optimal portfolio and consumption strategies of an investor who have a Cobb-Douglas utility function. And I assume that there is negative wealth constraints. This constraints mean that the investor can borrow partially against her future labor income.

AMS Mathematics Subject Classification : 91G10, 49L20.
Key words and phrases : Negative wealth constraints, Cobb-Douglas utility, utility maximization, martingale methods.

1. Introduction

The continuous time portfolio optimization problem have been widely studied after the works of Merton ([4], [5]). And many researchers consider various realistic constraints such as borrowing constraints (see [1], [3], [2]). A negative wealth constraint is a general version of borrowing constraints. This constraints means that an agent can borrow partially against her future labor income (see [6], [7]).

In the portfolio selection problem the Cobb-Douglas utility function is used for considering the leisure choice and the optimal consumption ([7]). To focus on the impact of negative constraints, I assume that the leisure rate process is constant in the rest of the paper.

In this paper I investigate an optimal portfolio and consumption selection problem of an investor who have a Cobb-Douglas utility function with negative wealth constraints. And I use the martingale method for deriving the optimal solution in the closed-form.

Received February 2, 2018. Revised March 20, 2018. Accepted April 10, 2018.
†This work was supported by the 2015 Hannam University Research Fund.

2. The Financial Market Model

I consider a continuous-time financial market in which an investor can trade two assets. There are one risk-free asset with constant interest rate $r > 0$ and one risky asset S_t which follows the geometric Brownian motion $dS_t/S_t = \mu dt + \sigma dB_t$, where $\mu > r$ and $\sigma > 0$ are constants, and B_t is a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the augmentation of the filtration generated by the standard Brownian motion $\{B_t\}_{t\geq 0}$ under $\mathbb{P}$. The market price of risk is defined by $\theta \triangleq (\mu - r)/\sigma$.

Let π_t be the the $\mathcal{F}_t$-progressively measurable portfolio process and c_t be the nonnegative $\mathcal{F}_t$-progressively measurable consumption process at time t. Furthermore I assume that they satisfy the following integrability conditions:

$$\int_0^t \pi_s^2 ds < \infty, \text{ and } \int_0^t c_s ds < \infty, \text{ for all } t \geq 0, \text{ almost surely (a.s.)}.$$

I also consider the leisure rate process $l_t < \bar{L}$ where a constant $\bar{L}$ is the sum of rates of labor and leisure at time $t \geq 0$. So $\bar{L} - l_t$ means the rate of work at time t. Let $\epsilon_t > 0$ be the agent's labor income rate. Then the agent's wealth process x_t follows

$$dx_t = \left[rx_t + \pi_t(\mu - r) - c_t + \epsilon_t(\bar{L} - l_t)\right] dt + \sigma\pi_t dB_t, \tag{1}$$

with initial wealth $x_0 = x$.

I assume that the agent has a Cobb-Douglas utility function of consumption and leisure as follows:

$$u(c_t, l_t) \triangleq \frac{1}{\alpha}\frac{(l_t^{1-\alpha}c_t^{\alpha})^{1-\gamma^*}}{1-\gamma^*}, \ 0 < \alpha < 1 \text{ and } \gamma^* > 0(\gamma^* \neq 1), \tag{2}$$

where γ^* is the agent's coefficient of relative risk aversion and α is a a constant weight for consumption. If I assume that the leisure rate process is constant i.e. $l_t = L$ and I define $\gamma = 1 - \alpha(1 - \gamma^*)$, then the utility function (2) becomes

$$u_L(c_t) = L^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma} \text{ for } \ t \geq 0. \tag{3}$$

And I define the Merton's constant $K > 0$ such that

$$K \triangleq r + \frac{\rho - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 > 0.$$

I also assume that the labor income rate ϵ is constant and there is a negative wealth constraint given by

$$x_t \geq -\nu\frac{\epsilon}{r}, \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad \nu \in [0, 1]. \tag{4}$$

3. The Optimization Problem

The state price density $H(t)$ is defined as

$$H(t) \triangleq \exp\left\{-\left(r + \frac{1}{2}\theta^2\right)t - \theta B(t)\right\}.$$

From the wealth process (1), I can derive the budget constraint as follows:

$$\mathbb{E}\left[\int_0^\infty H_t(c_t - \epsilon)dt\right] \le x. \tag{5}$$

And I define the agent's optimization problem as follows:

$$V(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u_L(c_t)dt\right]$$

with the negative wealth constraint (4). Here $\rho > 0$ is a subjective discount rate, $\mathcal{A}(x)$ is an admissible set of pairs (c, π) at x, and the utility $u_L(c_t)$ is a Cobb-Douglas utility function which is defined in (3).

Using a Lagrange multiplier $\lambda > 0$, I define a dual value function as follow:

$$\widetilde{V}(\lambda) = \sup_c \mathbb{E}\left[\int_0^\infty e^{-\rho t} u_L(c_t)dt - \lambda\int_0^\infty H_t(c_t - \epsilon)dt\right] \tag{6}$$

$$= \sup_c \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{u_L(c_t)dt - \lambda e^{\rho t} H_t(c_t - \epsilon)\right\}dt\right] \tag{7}$$

$$= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\widetilde{u}_L(y_t)dt\right], \tag{8}$$

where $\widetilde{u}_L(\cdot)$ is the dual utility function of the Cobb-Douglas utility and $y_t \triangleq \lambda e^{\beta t} H_t$. The dual utility $\widetilde{u}_L(y)$ is defined by as follows:

$$\begin{aligned}\widetilde{u}_L(y) &= \sup_c \{u_L(c) - y(c - \epsilon)\}\\ &= \frac{\gamma}{1-\gamma}\left(L^{\gamma-\gamma^*}\right)^{\frac{1}{\gamma}} y^{-\frac{1-\gamma}{\gamma}} + y\epsilon\\ &= \frac{\gamma}{1-\gamma} L^* y^{-\frac{1-\gamma}{\gamma}} + y\epsilon,\end{aligned}$$

where $L^* = L^{1-\frac{\gamma^*}{\gamma}}$.

Now I define a function

$$\phi(t, y) \triangleq \mathbb{E}\left[\int_t^\infty e^{-\rho t}\widetilde{u}_L(y_s)ds \middle| y_t = y\right].$$

Using the Feymann-Kac formula, I can derive the partial differential equation as follows:

$$\mathcal{L}\phi(t, y) + e^{-\rho t}\widetilde{u}_L(y) = 0,$$

where the partial differential operator is given by

$$\mathcal{L} := \frac{\partial}{\partial t} + (\rho - r)y\frac{\partial}{\partial y} + \frac{1}{2}\theta^2 y^2 \frac{\partial^2}{\partial y^2}.$$

If I conjecture that $\phi(t, y) = e^{-\rho t} v(y)$, then I am able to derive the following equation,

$$\frac{1}{2}\theta^2 y^2 v''(y) + (\rho - r)yv'(y) - \beta v(y) + \widetilde{u}_L(y) = 0. \tag{9}$$

Theorem 3.1. *The value function $V(x)$ is obtained by*

$$V(x) = C_1(1-m_+)\xi^{m_+} + \frac{L^*}{(1-\gamma)K}\xi^{-\frac{1-\gamma}{\gamma}},$$

where

$$C_1 = \left((\nu-1)\frac{\epsilon}{r} + \frac{L^*}{K}\hat{y}^{-\frac{1}{\gamma}}\right)\frac{\hat{y}^{1-m_+}}{m_+},$$

$$\hat{y} = \left(\frac{\gamma(1-\nu)(m_+-1)}{\gamma(m_+-1)+1}\frac{\epsilon}{r}\frac{K}{L^*}\right)^{-\gamma},$$

and ξ is determined from the algebraic equation

$$x = -m_+C_1\xi^{m_+-1} + \frac{L^*}{K}\xi^{-\frac{1}{\gamma}} - \frac{\epsilon}{r}.$$

Proof. A general solution of the equation (9) is given as

$$v(y) = C_1 y^{m_+} + \frac{\gamma}{1-\gamma}\frac{L^*}{K}y^{-\frac{1-\gamma}{\gamma}} + \frac{\epsilon}{r}y,$$

where C_1 is constant to be determined, and $m_+ > 1$ is a root of the quadratic equation $f(m) = \frac{1}{2}\theta^2 m^2 + \left(\rho - r - \frac{1}{2}\theta^2\right)m - \beta = 0$. The negative wealth constraint (4) implies two free boundary conditions,

$$v'(\hat{y}) = \nu\frac{\epsilon}{r}, \quad v''(\hat{y}) = 0, \tag{10}$$

where $\hat{y} > 0$ is the dual value corresponding to the wealth level $-\nu\frac{\epsilon}{r}$. From the conditions (10), I obtain that

$$\hat{y} = \left(\frac{\gamma(1-\nu)(m_+-1)}{\gamma(m_+-1)+1}\frac{\epsilon}{r}\frac{K}{L^*}\right)^{-\gamma}$$

and

$$C_1 = \left((\nu-1)\frac{\epsilon}{r} + \frac{L^*}{K}\hat{y}^{-\frac{1}{\gamma}}\right)\frac{\hat{y}^{1-m_+}}{m_+}.$$

Using the Legendre inverse transform formula,

$$V(x) = \inf_{y>0}\{v(y) + yx\},$$

I obtain the following value function

$$V(x) = C_1(1-m_+)\xi^{m_+} + \frac{L^*}{(1-\gamma)K}\xi^{-\frac{1-\gamma}{\gamma}},$$

where ξ is determined from the algebraic equation

$$x = -m_+C_1\xi^{m_+-1} + \frac{L^*}{K}\xi^{-\frac{1}{\gamma}} - \frac{\epsilon}{r}.$$

□

Theorem 3.2. *The optimal consumption and portfolio pair* (c_t^*, π_t^*) *is given by*

$$c_t^* = L^* \xi_t^{-\frac{1}{\gamma}}$$

and

$$\pi_t^* = \frac{\theta}{\sigma}\left(C_1 m_+(m_+ - 1)\xi_t^{m_+-1} + \frac{L^*}{\gamma K}\xi_t^{-\frac{1}{\gamma}}\right),$$

where ξ_t *is determined from the following equation,*

$$x = -m_+ C_1 \xi^{m_+-1} + \frac{L^*}{K}\xi^{-\frac{1}{\gamma}} - \frac{\epsilon}{r}.$$

Proof. From the proof of Theorem 3.1 we know that the optimal strategies are occurred when the equation, $x = -v'(y^*)$ is satisfied. By applying Itô formula to this equation and comparing with equation (1), I obtain the optimal consumption and portfolio pair (c_t^*, π_t^*) as follows:

$$c_t^* = -rv'(y^*) + (\rho - r + \theta^2)y^* v''(y^*) + \frac{1}{2}\theta^2 {y^*}^2 v'''(y^*) + I,$$

$$\pi_t^* = \frac{\theta}{\sigma} y^* v''(y^*).$$

Since $v(y) = C_1 y^{m_+} + \dfrac{\gamma}{1-\gamma}\dfrac{L^*}{K} y^{-\frac{1-\gamma}{\gamma}} + \dfrac{\epsilon}{r} y$, I can derive (c_t^*, π_t^*) as follows:

$$c_t^* = L^* \xi_t^{-\frac{1}{\gamma}}$$

and

$$\pi_t^* = \frac{\theta}{\sigma}\left(C_1 m_+(m_+ - 1)\xi_t^{m_+-1} + \frac{L^*}{\gamma K}\xi_t^{-\frac{1}{\gamma}}\right),$$

where ξ_t is determined from the following equation,

$$x = -m_+ C_1 \xi^{m_+-1} + \frac{L^*}{K}\xi^{-\frac{1}{\gamma}} - \frac{\epsilon}{r}.$$

□

References

1. P.H. Dybvig and H. Liu, *Lifetime Consumption and Investment: Retirement and Constrained Borrowing*, J. of Eco. Theory **145**(3) (2010), 885-907.
2. H.-S Lee, B.L. Koo and Y.H Shin, *A dynamic programming approach to a consumption/investment and retirement choice problem under borrowing constraints*, Japan J. Indust. Appl;. Math. **34** (2017), 793-809.
3. B.H. Lim and Y.H. Shin, *Optimal Investment, Consumption and Retirement Decision with Disutility and Borrowing Constraints*, Quant. Fin. **11**(10) (2011), 1581-1592.
4. R.C. Merton, *Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case*, Rev. of Eco. and Stat. **51**(3) (1969), 247-257.
5. R.C. Merton, *Optimum Consumption and Portfolio Rules in a Continuous-Time Model*, J. of Eco. Theory **3**(4) (1971), 373-413.

6. K. Park, M. Kang and Y.H. Shin, *An Optimal Consumption, Leisure, and Investment Problem with an Option to Retire and Negative Wealth Constraints*, Chaos, Solitons and Fractals **103** (2017), 374-381.
7. S. Park and B.-G. Jang, *Optimal retirement strategy with a negative wealth constraint*, Oper. Res. Let. **42** (2014), 208-212.

Kum-Hwan Roh received M.Sc. from Sogang University and Ph.D at KAIST. Since 2012 he has been at Hannam University. His research interests include mathematical finance.

Department of Mathematics, Hannam University, Daejeon 34430, Korea.
e-mail: khroh@hnu.kr