J. Appl. Math. & Informatics Vol. **36***(2018), No. 3 - 4, pp. 285 - 299*
https://doi.org/10.14317/jami.2018.285

ON STANCU TYPE GENERALIZATION OF (p, q)-SZÁSZ-MIRAKYAN KANTOROVICH TYPE OPERATORS

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ABSTRACT. In this article, we present the Stancu generalization of (p, q)-Szász-Mirakyan Kantorovich type linear positive operators. Using Korovkin's result, approximation properties are investigated. First, we evaluate moments and direct results. By choosing p and q, the convergence rate have been estimated for better approximation. For the particular case $\alpha = 0, \beta = 0$ we obtain results for (p, q)-Szász-Mirakyan Kantorovich type operators.

AMS Mathematics Subject Classification : 41A25, 41A35.
Key words and phrases : (p, q)-Calculus, (p, q)-Szász-Mirakyan Kantorovich operators, Korovkin's result, Rate of convergence, Voronovskaya results.

1. Introduction

Approximation theory is an important and useful tool in mathematics. There are so many research going on on approximating the continuous functions with the help of linear positive operators [5, 6, 7, 8, 18]. In approximation theory, the use of Bernstein polynomial in q-calculus was first introduced by Lupas [2]. The constant development in q-calculus has led us towards the new generalized approximating operators depending on q-integers [13, 15, 16, 17]. In recent years, Mursaleen et al. [9] introduced new way of approximating linear positive operators in $(p, q)-$ calculus. The more research is going on in this area [10, 14, 19, 21].
We initiate by recollecting standard definitions from (p, q)-calculus (Ref. [9, 10, 14, 20]). Let $q < p; q, p \in (0, 1]$.

$$[\xi]_{p,q} = \frac{p^\xi - q^\xi}{p - q}, \quad \xi = 0, 1, 2, \cdots, \ [0]_{p,q} = 0$$

and

Received September 13, 2017. Revised December 13, 2017. Accepted March 26, 2018.
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$$[\xi]_{p,q}! = [\xi]_{p,q}[\xi-1]_{p,q}\cdots 1 \quad , \quad \xi \geq 1 \text{ and } [0]_{p,q}! = 1.$$

The (p,q)-binomial expansion:

$$(y+z)^{\xi}_{p,q} = (y+z)(py+qz)(p^2y+q^2z)\cdots(p^{\xi-1}y+q^{\xi-1}z),$$

Also, the (p,q)-binomial coefficients:

$$\begin{bmatrix}\xi\\ r\end{bmatrix}_{p,q} = \frac{[\xi]_{p,q}!}{[r]_{p,q}![\xi-r]_{p,q}!}, \qquad 0 \leq r \leq \xi.$$

Now, g is mapping on complex numbers $\mathbb{C}$. The (p,q)-differentiability of g is given by:

$$D_{p,q}g(x) = \frac{g(px)-g(qx)}{(p-q)x}, \quad x \neq 0,$$

and $(D_{p,q}g)(0) = g'(0)$, on the condition that g is differentiable at 0.
Now, g is an arbitrary mapping and c is any real number, we have

$$\int_0^c g(x)d_{p,q}x = (q-p)c\sum_{j=0}^{\infty}\frac{p^j}{q^{j+1}}g\left(\frac{p^j}{q^{j+1}}c\right) \quad if \quad \left|\frac{p}{q}\right| < 1.$$

$$\int_0^c g(x)d_{p,q}x = (p-q)c\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}g\left(\frac{q^j}{p^{j+1}}c\right) \quad if \quad \left|\frac{p}{q}\right| > 1.$$

The $(p,q)-$analogue of exponential function e^x [3] is:

$$e_{p,q}(y) = \sum_{\xi=0}^{\infty}\frac{p^{\frac{\xi(\xi-1)}{2}}y^{\xi}}{[\xi]_{p,q}!},$$

$$E_{p,q}(y) = \sum_{\xi=0}^{\infty}\frac{q^{\frac{\xi(\xi-1)}{2}}y^{\xi}}{[\xi]_{p,q}!}.$$

The $(p,q)-$exponential function satisfy following property:

$$e_{p,q}(y)E_{p,q}(-y) = E_{p,q}(y)e_{p,q}(-y) = 1.$$

In recent times, Acar [12] proposed a Szász-Mirakyan operators in $(p,q)-$calculus as :

$$S_{\xi,p,q}(\psi;x) = \sum_{j=0}^{\infty}s_{\xi}(p,q;x)\psi\left(\frac{[j]_{p,q}}{q^{j-2}[\xi]_{p,q}}\right),$$

where

$$s_{\xi}(p,q;x) = \frac{1}{E_{p,q}([\xi]_{p,q}x)}q^{\frac{j(j-1)}{2}}\frac{[\xi]^j_{p,q}x^j}{[j]_{p,q}!}; j = 0,1,2,...$$

Lemma 1.1. *([12]) Let* $p,q \in (0,1]; q < p$ *and* $\xi \in \mathbb{N}$ *&* $e_i(t) = t^i, i = 0,1,2,$

$$S_{\xi,p,q}(e_0(t);x) = 1,$$
$$S_{\xi,p,q}(e_1(t);x) = qx,$$
$$S_{\xi,p,q}(e_2(t);x) = pqx^2 + \frac{q^2x}{[\xi]_{p,q}}.$$

Encouraged by Acar, recently, Sharma and Gupta [11] introduced Kantorovich type generalization of (p,q)-Szász-Mirakyan operator; $p,q\in(0,1]; q<p, \xi\in\mathbb{N}$ for $\psi:\mathbb{R}^+\to\mathbb{R}$ as:

$$K_{\xi}^{(p,q)}(\psi;x)=[\xi]_{p,q}\sum_{j=0}^{\infty}s_{\xi}(p,q;x)p^{-j}q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}\psi(t)d_{p,q}t.$$

Lemma 1.2. *([11]) Suppose $p,q\in(0,1]; q<p, \xi\in\mathbb{N}$,*

$$q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}d_{p,q}t=\frac{p^j}{[\xi]_{p,q}},$$

$$q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}td_{p,q}t=\frac{p^jq^{-j+2}([j+1]_{p,q}+q[j]_{p,q})}{(p+q)[\xi]_{p,q}^2},$$

$$q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}t^2d_{p,q}t=\frac{p^jq^{-2j+4}([j+1]_{p,q}^2+q[j]_{p,q}[j+1]_{p,q}+q^2[j]_{p,q}^2)}{(p^2+pq+q^2)[\xi]_{p,q}^3}.$$

Lemma 1.3. *([11])Suppose $p,q\in(0,1]; q<p, \xi\in\mathbb{N}$, $e_i(t)=t^i, i=0,1,2:$*

$$K_{\xi}^{(p,q)}(e_0;x)=1,$$

$$K_{\xi}^{(p,q)}(e_1;x)=qx+\frac{q^2}{[\xi]_{p,q}(p+q)},$$

$$K_{\xi}^{(p,q)}(e_2;x)=pqx^2+\frac{(2q^4+3pq^3+p^2q^2)x}{(p^2+pq+q^2)[\xi]_{p,q}}+\frac{q^4}{[\xi]_{p,q}^2(p^2+pq+q^2)}.$$

2. Construction of the operators

In this paper, inspired by Acar [12] and Sharma & Gupta [11], we propose Stancu on Kantorovich type generalization of (p,q)-Szász-Mirakyan operator; $p,q\in(0,1]; q<p, \xi\in\mathbb{N}$ for $\psi:\mathbb{R}^+\to\mathbb{R}$ as:

$$K_{\xi,\alpha,\beta}^{(p,q)}(\psi;x)=[\xi]_{p,q}\sum_{j=0}^{\infty}s_{\xi}(p,q;x)p^{-j}q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}\psi\left(\frac{[\xi]_{p,q}t+\alpha}{[\xi]_{p,q}+\beta}\right)d_{p,q}t. \tag{1}$$

Lemma 2.1. *For $p,q\in(0,1]; q<p, \xi\in\mathbb{N}$,*

$$q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}d_{p,q}t=\frac{p^j}{[\xi]_{p,q}},$$

$$q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}\frac{[\xi]_{p,q}t+\alpha}{[\xi]_{p,q}+\beta}d_{p,q}t=\frac{p^jq^{-j+2}([j+1]_{p,q}+q[j]_{p,q})}{([\xi]_{p,q}+\beta)(p+q)[\xi]_{p,q}}+\frac{\alpha p^j}{([\xi]_{p,q}+\beta)[\xi]_{p,q}},$$

$$q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}\left(\frac{[\xi]_{p,q}t+\alpha}{[\xi]_{p,q}+\beta}\right)^2 d_{p,q}t$$
$$=\frac{1}{([\xi]_{p,q}+\beta)^2}$$
$$\Bigg[\frac{p^jq^{-2j+4}([j+1]_{p,q}^2+q[j]_{p,q}[j+1]_{p,q}+q^2[j]_{p,q}^2)}{(p^2+pq+q^2)[\xi]_{p,q}}$$
$$+\frac{2\alpha p^jq^{-j+2}([j+1]_{p,q}+q[j]_{p,q})}{(p+q)[\xi]_{p,q}}+\frac{\alpha^2p^j}{[\xi]_{p,q}}\Bigg].$$

Proof. With the help of Lemma 1.2, we get the result. □

Lemma 2.2. *For $p,q\in(0,1]; q<p, \xi\in\mathbb{N}, e_i(t)=t^i, i=0,1,2,$*

$$K_{\xi,\alpha,\beta}^{(p,q)}(e_0(t);x)=1, \tag{2}$$

$$K_{\xi,\alpha,\beta}^{(p,q)}(e_1(t);x)=\frac{q^2}{([\xi]_{p,q}+\beta)(p+q)}+\frac{[\xi]_{p,q}qx+\alpha}{[\xi]_{p,q}+\beta}, \tag{3}$$

$$K_{\xi,\alpha,\beta}^{(p,q)}(e_2(t);x)=\frac{1}{([\xi]_{p,q}+\beta)^2}\Bigg[[\xi]_{p,q}^2pqx^2$$
$$+\frac{(2p+q)q^3+(2\alpha+q)(p^2+pq+q^2)q}{(p^2+pq+q^2)}[\xi]_{p,q}x$$
$$+\frac{q^4}{(p^2+pq+q^2)}+\frac{2\alpha q^2}{(p+q)}+\alpha^2\Bigg]. \tag{4}$$

Proof. With the help of (1), lemma 2.1, lemma 1.1 we obtain moments as follow:

$$K_{\xi,\alpha,\beta}^{(p,q)}(e_0(t);x)=[\xi]_{p,q}\sum_{j=0}^{\infty}s_\xi(p,q;x)p^{-j}q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}d_{p,q}t$$
$$=\sum_{j=0}^{\infty}s_\xi(p,q;x)$$
$$=1.$$

And applying $[j+1]_{p,q}=q^j+p[j]_{p,q}$, we obtain

$$K_{\xi,\alpha,\beta}^{(p,q)}(e_1(t);x)=[\xi]_{p,q}\sum_{j=0}^{\infty}s_\xi(p,q;x)p^{-j}q^{j-2}\int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}}\frac{[\xi]_{p,q}t+\alpha}{[\xi]_{p,q}+\beta}d_{p,q}t$$
$$=[\xi]_{p,q}\sum_{j=0}^{\infty}s_\xi(p,q;x)p^{-j}\Bigg[\frac{p^jq^{-j+2}([j+1]_{p,q}+q[j]_{p,q})}{([\xi]_{p,q}+\beta)(p+q)[\xi]_{p,q}}$$
$$+\frac{\alpha p^j}{[\xi]_{p,q}([\xi]_{p,q}+\beta)}\Bigg]$$

$$= \frac{1}{([\xi]_{p,q}+\beta)(p+q)} \sum_{j=0}^{\infty} s_\xi(p,q;x) q^{-j+2}(q^j + (p+q)[j]_{p,q})$$

$$+ \frac{\alpha}{[\xi]_{p,q}+\beta} \sum_{j=0}^{\infty} s_\xi(p,q;x)$$

$$= \frac{q^2}{([\xi]_{p,q}+\beta)(p+q)} S_{\xi,p,q}(1;x) + \frac{[\xi]_{p,q}}{[\xi]_{p,q}+\beta} S_{\xi,p,q}(t;x) + \frac{\alpha}{[\xi]_{p,q}+\beta} S_{\xi,p,q}(1;x)$$

$$= \frac{q^2}{([\xi]_{p,q}+\beta)(p+q)} + \frac{[\xi]_{p,q} qx + \alpha}{[\xi]_{p,q}+\beta}.$$

And now;

$$K_{\xi,\alpha,\beta}^{(p,q)}(e_2(t);x) =$$

$$[\xi]_{p,q} \sum_{j=0}^{\infty} s_\xi(p,q;x) p^{-j} q^{j-2} \int_{\frac{q^{-j+3}[j]_{p,q}}{[\xi]_{p,q}}}^{\frac{q^{-j+2}[j+1]_{p,q}}{[\xi]_{p,q}}} \left(\frac{[\xi]_{p,q} t + \alpha}{[\xi]_{p,q}+\beta} \right)^2 d_{p,q}t$$

$$= \frac{[\xi]_{p,q}}{([\xi]_{p,q}+\beta)^2} \sum_{j=0}^{\infty} s_\xi(p,q;x) p^{-j}$$

$$\left[\frac{p^j q^{-2j+4}([j+1]_{p,q}^2 + q[j]_{p,q}[j+1]_{p,q} + q^2[j]_{p,q}^2)}{(p^2+pq+q^2)[\xi]_{p,q}} + \frac{2\alpha p^j q^{-j+2}([j+1]_{p,q} + q[j]_{p,q})}{(p+q)[\xi]_{p,q}} + \frac{\alpha^2 p^j}{[\xi]_{p,q}} \right]$$

$$= \frac{1}{([\xi]_{p,q}+\beta)^2} \left[[\xi]_{p,q}^2 \sum_{j=0}^{\infty} \frac{s_\xi(p,q;x)[j]_{p,q}^2}{q^{2j-4}[\xi]_{p,q}^2} + \frac{(2p+q)q^2[\xi]_{p,q}}{(p^2+pq+q^2)} \sum_{j=0}^{\infty} \frac{s_\xi(p,q;x)[j]_{p,q}}{q^{j-2}[\xi]_{p,q}} \right.$$

$$+ \frac{q^4}{(p^2+pq+q^2)} \sum_{j=0}^{\infty} s_\xi(p,q;x) + \frac{2\alpha q^2}{(p+q)} \sum_{j=0}^{\infty} s_\xi(p,q;x)$$

$$\left. + 2\alpha[\xi]_{p,q} \sum_{j=0}^{\infty} \frac{s_\xi(p,q;x)[j]_{p,q}}{q^{j-2}[\xi]_{p,q}} + \alpha^2 \sum_{j=0}^{\infty} s_\xi(p,q;x) \right]$$

$$= \frac{1}{([\xi]_{p,q}+\beta)^2} \left[[\xi]_{p,q}^2 S_{\xi,p,q}(t^2;x) + \frac{(2p+q)q^2[\xi]_{p,q}}{(p^2+pq+q^2)} S_{\xi,p,q}(t;x) \right.$$

$$+ \frac{q^4}{(p^2+pq+q^2)} S_{\xi,p,q}(1;x) + \frac{2\alpha q^2}{(p+q)} S_{\xi,p,q}(1;x)$$

$$
\begin{aligned}
&+ 2\alpha[\xi]_{p,q}S_{\xi,p,q}(t;x) + \alpha^2 S_{\xi,p,q}(1;x)\Big] \\
&= \frac{1}{([\xi]_{p,q}+\beta)^2}\Big[[\xi]_{p,q}^2 pqx^2 + [\xi]_{p,q}q^2x \\
&\quad + 2\alpha q[\xi]_{p,q}x + \frac{(2p+q)q^3[\xi]_{p,q}}{(p^2+pq+q^2)}x + \frac{q^4}{(p^2+pq+q^2)} \\
&\quad + \frac{2\alpha q^2}{(p+q)} + \alpha^2\Big] \\
&= \frac{1}{([\xi]_{p,q}+\beta)^2}\Big[[\xi]_{p,q}^2 pqx^2 \\
&\quad + \frac{(2p+q)q^3 + (2\alpha+q)(p^2+pq+q^2)q}{(p^2+pq+q^2)} \\
&\quad [\xi]_{p,q}x + \frac{q^4}{(p^2+pq+q^2)}\frac{2\alpha q^2}{(p+q)} + \alpha^2\Big].
\end{aligned}
$$

□

Corollary 2.3. *Central moments* $\Phi^{(p,q)}_{\xi,\alpha,\beta}(x) = K^{(p,q)}_{\xi,\alpha,\beta}((t-x)^\xi;x),\ \xi = 1,2$*:*

$$
\begin{aligned}
\Phi^{(p,q)}_{1,\alpha,\beta}(x) &= \frac{q^2}{([\xi]_{p,q}+\beta)(p+q)} + \frac{[\xi]_{p,q}qx+\alpha}{[\xi]_{p,q}+\beta} - x \\
\Phi^{(p,q)}_{2,\alpha,\beta}(x) &= \left(\frac{[\xi]_{p,q}pq}{([\xi]_{p,q}+\beta)^2} - \frac{2[\xi]_{p,q}q}{([\xi]_{p,q}+\beta)} + 1\right)x^2 \\
&\quad + \left(\frac{(2p+q)q^3 + (2\alpha+q)q(p^2+pq+q^2)[\xi]_{p,q}}{(p^2+pq+q^2)([\xi]_{p,q}+\beta)^2}\right. \\
&\quad \left. - \frac{2q^2}{(p+q)([\xi]_{p,q}+\beta)} - \frac{2\alpha}{([\xi]_{p,q}+\beta)}\right)x \\
&\quad + \frac{1}{([\xi]_{p,q}+\beta)^2}\left(\frac{q^4}{(p^2+pq+q^2)} + \frac{2\alpha q^2}{p+q} + \alpha^2\right).
\end{aligned}
$$

Remark 2.1. For $q < p; q,p \in (0,1]$, we get that $\lim\limits_{\xi\to\infty}[\xi]_{p,q} = \dfrac{1}{p-q}$. To get the approximation results of the operators, we consider a sequences $0 < q_\xi < p_\xi \leq 1$ in such a way that $p_\xi \to 1, q_\xi \to 1$; $p_\xi^\xi \to N, q_\xi^\xi \to N'$ as $\xi \to \infty$. Hence, $\frac{1}{[\xi]_{p_\xi,q_\xi}} \to 0$ as $\xi \to \infty$.
We can construct above mentioned sequences. For example, let $p_\xi = 1 + \frac{1}{2\xi}$ and $q_\xi = 1 + \frac{1}{3\xi}$. Then $p_\xi \to 1, q_\xi \to 1$; $p_\xi^\xi \to e^{1/2}, q_\xi^\xi \to e^{1/3}$ as $\xi \to \infty$, and $\lim\limits_{\xi\to\infty} 1/[\xi]_{p_\xi,q_\xi} = 0$.

Theorem 2.4. *Suppose* $(p_\xi)_\xi$ & $(q_\xi)_\xi$ *are the sequences* $\ni p_\xi \to 1, q_\xi \to 1$; $p_\xi^\xi \to N, q_\xi^\xi \to N'$ *and* $\frac{1}{[\xi]_{p_\xi,q_\xi}} \to 0$ *as* $\xi \to \infty$*, then for every* $\psi \in C[0,\infty)$, $K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(\psi;x)$ *converging to* ψ *uniformly.*

Proof. With the help of Korovkin theorem, we'll only prove that $\|K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(t^\xi;x) - x^\xi\|_{C[0,\infty)}; \xi = 0,1,2.$
Result is trivial for $\xi = 0$; with the help of eq:(2). Using eq:(3) we'll get the result for $\xi = 1$, as follow:

$$\begin{aligned}\lim_{\xi\to\infty}\|K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(t;x) - x\|_{C[0,\infty)} &= \lim_{\xi\to\infty}\left|\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}\right.\\ &\quad \left. + \frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} - x\right|\\ &\le \lim_{\xi\to\infty}\left|\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}\right|\\ &\quad + \lim_{\xi\to\infty}\left|\frac{\alpha}{[\xi]_{p_\xi,q_\xi}+\beta}\right|\\ &\quad + \lim_{\xi\to\infty}\left|\frac{[\xi]_{p_\xi,q_\xi}q_\xi}{[\xi]_{p_\xi,q_\xi}+\beta} - 1\right|x\\ &= 0.\end{aligned}$$

Now, with the help of eq:(4), we obtain

$$\begin{aligned}&\lim_{\xi\to\infty}\|K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(t^2;x) - x^2\|_{C[0,\infty)}\\ &= \lim_{\xi\to\infty}\left|\frac{[\xi]_{p_\xi,q_\xi}^2 p_\xi q_\xi x^2}{([\xi]_{p_\xi,q_\xi}+\beta)^2}\right.\\ &+ \left(\frac{(2p_\xi+q_\xi)q_\xi^3 + (2\alpha+q_\xi)(p_\xi^2+p_\xi q_\xi+q_\xi^2)q_\xi}{(p_\xi^2+p_\xi q_\xi+q_\xi^2)([\xi]_{p_\xi,q_\xi}+\beta)^2}\right)\\ &[\xi]_{p_\xi,q_\xi}x + \frac{q_\xi^4}{(p_\xi^2+p_\xi q_\xi+q_\xi^2)([\xi]_{p_\xi,q_\xi}+\beta)^2}\\ &+ \frac{2\alpha q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)^2}\\ &\left. + \frac{\alpha^2}{([\xi]_{p_\xi,q_\xi}+\beta)^2} - x^2\right|\\ &\le \lim_{\xi\to\infty}\\ &\left|\frac{(2p_\xi+q_\xi)q_\xi^3 + (2\alpha+q_\xi)(p_\xi^2+p_\xi q_\xi+q_\xi^2)q_\xi}{(p_\xi^2+p_\xi q_\xi+q_\xi^2)([\xi]_{p_\xi,q_\xi}+\beta)^2}[\xi]_{p_\xi,q_\xi}x\right|\end{aligned}$$

$$
\begin{aligned}
&+\lim_{\xi\to\infty}\left|\frac{q_\xi^4}{(p_\xi^2+p_\xi q_\xi+q_\xi^2)([\xi]_{p_\xi,q_\xi}+\beta)^2}\right|\\
&+\lim_{\xi\to\infty}\left|\frac{2\alpha q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)^2}\right|\\
&+\lim_{\xi\to\infty}\left|\frac{\alpha^2}{([\xi]_{p_\xi,q_\xi}+\beta)^2}\right|\\
&+\lim_{\xi\to\infty}\left|\frac{[\xi]_{p_\xi,q_\xi}^2 p_\xi q_\xi}{([\xi]_{p_\xi,q_\xi}+\beta)^2}-1\right|x^2\\
&=0.
\end{aligned}
$$

□

3. Main result

In this segment, we show result on local approximation for our operators. Here, $C_B[0,\infty)$ is the space of real valued bounded and continuous functions ψ on $[0,\infty)$. The sup-norm on $C_B[0,\infty)$ is $\|\psi\|=\sup_{x\in[0,\infty)}|\psi(x)|$.

Peetre's K-functional is given by

$$K_2(\psi,\delta)=\inf_{g\in W^2}\left\{\|\psi-g\|+\delta\|g''\|\right\},$$

here $W^2=\{g\in C_B[0,\infty): g',g''\in C_B[0,\infty)\}$. From ([1], p.177), $\exists$ a constant $C>0$ such that $K_2(\psi,\delta)\le C\omega_2(\psi,\delta^{1/2}),\delta>0$, where

$$\omega_2(\psi,\delta^{1/2})=\sup_{0<\eta<\delta^{1/2},x\in[0,\infty)}|\psi(x+2\eta)-2\psi(x+\eta)+\psi(x)|$$

be the modulus of continuity of second order of the functions ψ in $C_B[0,\infty)$. The first order modulus of continuity of function $\psi\in C_B[0,\infty)$ is defined as

$$\omega(\psi,\delta^{1/2})=\sup_{0<\eta<\delta^{1/2},x\in[0,\infty)}|\psi(x+\eta)-\psi(x)|.$$

Theorem 3.1. *Suppose $(p_\xi)_\xi$ & $(q_\xi)_\xi$ are the sequences $\ni p_\xi\to 1, q_\xi\to 1$; $p_\xi^\xi\to N, q_\xi^\xi\to N'$ and $\frac{1}{[\xi]_{p,q}}\to 0$ as $\xi\to\infty$. For $\psi\in C_B[0,\infty)$, and for all $\xi\in\mathbb{N}$, there exists an absolute constant $C>0$ such that*

$$|K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(\psi;x)-\psi(x)|\le C\omega_2(\psi,\delta_\xi(x))+\omega(\psi,\gamma_\xi(x)),$$

where

$$\delta_\xi^2(x)=\left(\Phi_{2,\alpha,\beta}^{(p_\xi,q_\xi)}(x)+(\Phi_{1,\alpha,\beta}^{(p_\xi,q_\xi)}(x))^2\right),$$

and

$$\gamma_\xi(x)=\left|\frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta}+\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}-x\right|.$$

Proof. Let $x\in[0,\infty)$, we are taking the auxiliary operators $K_{\xi,\alpha,\beta}^*(\psi;x)$ as;

$$K^{*}_{\xi,\alpha,\beta}(\psi;x)=$$
$$K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(\psi;x)+\psi(x)-\psi\bigg(\frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta}+\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}\bigg).$$

With the help of eq:(3) and $K^{*}_{\xi,\alpha,\beta}(\psi;x)$, we obtain

$$\begin{aligned}K^{*}_{\xi,\alpha,\beta}(t-x;x)&=K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(t-x;x)\\&\quad-\bigg(\frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta}+\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}-x\bigg)\\&=K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(t;x)-xK^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(1;x)\\&\quad-\bigg(\frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta}+\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}-x\bigg)\\&=0.\end{aligned}$$

Now, $x\in[0,\infty)$ and $\eta\in W^2$. With the help of Taylor's formula, we obtain

$$\eta(t)=\eta(x)+\eta'(x)(t-x)+\int_x^t(t-v)\eta''(v)dv.$$

Using $K^{*}_{\xi,\alpha,\beta}$ on both sides, we obtain

$$\begin{aligned}K^{*}_{\xi,\alpha,\beta}(\eta;x)-\eta(x)&=K^{*}_{\xi,\alpha,\beta}(\eta'(x)(t-x);x)+K^{*}_{\xi,\alpha,\beta}\bigg(\int_x^t(t-v)\eta''(v)dv;x\bigg)\\&=\eta'(x)K^{*}_{\xi,\alpha,\beta}(t-x;x)+K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}\bigg(\int_x^t(t-v)\eta''(v)dv;x\bigg)\\&\quad-\int_x^{\frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta}+\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}}\\&\quad\bigg(\frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta}+\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}-v\bigg)\eta''(v)dv\\&=K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}\bigg(\int_x^t(t-v)\eta''(v)dv;x\bigg)\\&\quad-\int_x^{\frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta}+\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}}\\&\quad\bigg(\frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta}+\frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}-v\bigg)\eta''(v)dv\end{aligned}$$

Also,

$$\bigg|\int_x^t(t-v)\eta''(v)dv|\bigg)\leq\int_x^t|t-v||\eta''(v)|dv$$

$$\leq \|\eta''(v)\| \int_x^t |t-v| dv$$
$$\leq (t-x)^2 \|\eta''\|,$$

and

$$\left| \int_x^{\frac{q_\xi [\xi]_{p_\xi,q_\xi} x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} + \frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}} \left(\frac{q_\xi[\xi]_{p_\xi,q_\xi} x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} + \frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)} - v \right) \eta''(v) dv \right|$$
$$\leq \left(\frac{q_\xi[\xi]_{p_\xi,q_\xi} x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} + \frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)} - x \right)^2 \|\eta''\|.$$

Hence, we get

$$|K^*_{\xi,\alpha,\beta}(\eta;x) - \eta(x)| = \left| K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta} \left(\int_x^t (t-v)\eta''(v) dv; x \right) - \int_x^{\frac{q_\xi [\xi]_{p_\xi,q_\xi} x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} + \frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)}} \left(\frac{q_\xi[\xi]_{p_\xi,q_\xi} x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} + \frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)} - v \right) \eta''(v) dv \right|$$
$$\leq \|\eta''\| K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}((t-x)^2; x)$$
$$+ \left(\frac{q_\xi[\xi]_{p_\xi,q_\xi} x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} + \frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)} - x \right)^2 \|\eta''\|$$
$$= \delta_\xi^2(x) \|\eta''\|.$$

Also, we get

$$|K^*_{\xi,\alpha,\beta}(\psi;x) \leq |K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(\psi;x)| + 2\|\psi\| \leq 3\|\psi\|.$$

Hence,

$$|K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(\psi;x) - \psi(x)| \leq |K^*_{\xi,\alpha,\beta}(\psi-\eta;x) - (\psi-\eta)(x)|$$
$$+ \left| \psi\left(\frac{q_\xi[\xi]_{p_\xi,q_\xi} x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} + \frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)} \right) - \psi(x) \right| + |K^*_{\xi,\alpha,\beta}(\eta;x) - \eta(x)|$$
$$\leq |K^*_{\xi,\alpha,\beta}(\psi-\eta;x)| + |(\psi-\eta)(x)|$$
$$+ \left| \psi\left(\frac{q_\xi[\xi]_{p_\xi,q_\xi} x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} + \frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)} \right) - \psi(x) \right| + |K^*_{\xi,\alpha,\beta}(\eta;x) - \eta(x)|$$

$$\leq 4\|\psi-\eta\| + \delta_\xi^2(x)\|\eta''\| + \omega\left(\psi : \left|\frac{q_\xi[\xi]_{p_\xi,q_\xi}x+\alpha}{[\xi]_{p_\xi,q_\xi}+\beta} + \frac{q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)} - x\right|\right).$$

Now, we are applying infimum on the RHS over all $\eta \in W^2$, we have

$$|K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(\psi;x)-\psi(x)| \leq 4K_2(\psi,\delta_\xi^2(x)) + \omega(\psi,\gamma_\xi(x)).$$

With the help of K-functional property, we have

$$|K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(\psi;x)-\psi(x)| \leq C\omega_2(\psi,\delta_\xi(x)) + \omega(\psi,\gamma_\xi(x)).$$

Hence, we get the proof. □

Now, we assume the following space of functions. $B_{x^2}[0,\infty)$ is the class of functions ψ on the interval $[0,\infty)$ with condition $|\psi(x)| \leq M_\psi^+(1+x^2)$, M_ψ^+ is an absolute constant depends on ψ. $C_{B_{x^2}}[0,\infty)$ is the space contained in $B_{x^2}[0,\infty)$ where all the functions ψ are continuous. $C^*_{B_{x^2}}[0,\infty)$ be a subspace of functions ψ in $C_{B_{x^2}}[0,\infty)$ where $\frac{\psi(x)}{1+x^2}$ tends to finite limit as x tends to ∞. $B_{x^2}[0,\infty)$ be a normed linear space having sup-norm:

$$\|\psi\|_{x^2} = \sup_{x\geq 0}\frac{|\psi(x)|}{1+x^2}.$$

We define modulus of continuity on $[0,a]$ by,

$$\omega_a(\psi,\delta) = \sup_{|t-x|\leq\delta;x,t\in[0,a]}|\psi(t)-\psi(x)|.$$

Theorem 3.2. *Suppose $0<q_\xi<p_\xi\leq 1$ such that $p_\xi\to 1, q_\xi\to 1, p_\xi^\xi\to b$ and $q_\xi^\xi\to c$ as $\xi\to\infty$. For $\psi\in C_{B_{x^2}}[0,\infty), \omega_{a+1}(\psi;\delta)$ be its modulus of continuity on the interval $[0,a+1]\subset[0,\infty), a>0$ and for each $\xi>1$,*

$$\|K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(\psi;x)-\psi\|_{C[0,a]} \leq 6M_\psi(1+a^2)\lambda_\xi + 2\omega_{a+1}(\psi;\sqrt{\lambda_\xi})$$

here

$$\lambda_\xi = \left(1-\frac{p_\xi q_\xi[\xi]_{p_\xi,q_\xi}}{([\xi]_{p_\xi,q_\xi}+\beta)^2}\right)a^2 + \frac{6a(1-\beta-2\alpha\beta)+2(1+3\alpha+3\alpha^2)}{([\xi]_{p_\xi,q_\xi}+\beta)^2(p_\xi^2+p_\xi q_\xi+q_\xi^2)(p_\xi+q_\xi)}.$$

Proof. If $x\in[0,a]$; $t\geq 0$, we get (see [4])

$$|\psi(t)-\psi(x)| \leq 6M_\psi(1+a^2)(t-x)^2 + \omega_{a+1}(\psi;\delta_\xi)\left(\frac{|t-x|}{\delta_\xi}+1\right).$$

With the help Cauchy-Schwarz inequality and above inequality, we obtain

$$\begin{aligned}\|K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(\psi;x)-\psi(x)\|_{C[0,a]} &\leq K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(|\psi(t)-\psi(x)|);x)\\ &\leq 6M_\psi(1+a^2)K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}((t-x)^2;x) + \omega_{a+1}(\psi;\delta_\xi)\\ &\left(\left(\frac{K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}((t-x)^2;x)}{\delta_\xi^2}\right)+1\right)^{1/2}.\end{aligned}$$

We use above corollary and for $x \in [0,a]$, we get

$$
\begin{aligned}
K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}((t-x)^2;x) &= \left(\frac{[\xi]_{p_\xi,q_\xi} p_\xi q_\xi}{([\xi]_{p_\xi,q_\xi}+\beta)^2} - \frac{2[\xi]_{p_\xi,q_\xi} q_\xi}{([\xi]_{p_\xi,q_\xi}+\beta)} + 1\right)x^2 \\
&+ \left(\frac{(2p_\xi+q_\xi)q_\xi^3 + (2\alpha+q_\xi)q_\xi(p_\xi^2+p_\xi q_\xi+q_\xi^2)[\xi]_{p_\xi,q_\xi}}{(p_\xi^2+p_\xi q_\xi+q_\xi^2)([\xi]_{p_\xi,q_\xi}+\beta)^2}\right. \\
&\left. - \frac{2q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)} - \frac{2\alpha}{([\xi]_{p_\xi,q_\xi}+\beta)}\right)x \\
&+ \frac{1}{([\xi]_{p_\xi,q_\xi}+\beta)^2}\left(\frac{q_\xi^4}{(p_\xi^2+p_\xi q_\xi+q_\xi^2)} + \frac{2\alpha q_\xi^2}{p_\xi+q_\xi} + \alpha^2\right) \\
&\leq \left(\frac{[\xi]_{p_\xi,q_\xi} p_\xi q_\xi}{([\xi]_{p_\xi,q_\xi}+\beta)^2} - \frac{2[\xi]_{p_\xi,q_\xi} q_\xi}{([\xi]_{p_\xi,q_\xi}+\beta)} + 1\right)a^2 \\
&+ \left(\frac{(2p_\xi+q_\xi)q_\xi^3 + (2\alpha+q_\xi)q_\xi(p_\xi^2+p_\xi q_\xi+q_\xi^2)[\xi]_{p_\xi,q_\xi}}{(p_\xi^2+p_\xi q_\xi+q_\xi^2)([\xi]_{p_\xi,q_\xi}+\beta)^2}\right. \\
&\left. - \frac{2q_\xi^2}{(p_\xi+q_\xi)([\xi]_{p_\xi,q_\xi}+\beta)} - \frac{2\alpha}{([\xi]_{p_\xi,q_\xi}+\beta)}\right)a \\
&+ \frac{1}{([\xi]_{p_\xi,q_\xi}+\beta)^2}\left(\frac{q_\xi^4}{(p_\xi^2+p_\xi q_\xi+q_\xi^2)} + \frac{2\alpha q_\xi^2}{p_\xi+q_\xi} + \alpha^2\right) \\
&\leq \left(\frac{[\xi]_{p_\xi,q_\xi} q_\xi}{([\xi]_{p_\xi,q_\xi}+\beta)}\left(1 - \frac{p_\xi}{([\xi]_{p_\xi,q_\xi}+\beta)}\right)\right. \\
&\left. - \frac{[\xi]_{p_\xi,q_\xi} q_\xi}{[\xi]_{p_\xi,q_\xi}+\beta} + 1\right)a^2 \\
&+ \frac{(6-6\beta-12\alpha\beta)a + 2 + 6\alpha + 6\alpha^2}{([\xi]_{p_\xi,q_\xi}+\beta)^2(p_\xi^2+p_\xi q_\xi+q_\xi^2)(p_\xi+q_\xi)} \\
&= \left(1 - \frac{p_\xi q_\xi [\xi]_{p_\xi,q_\xi}}{([\xi]_{p_\xi,q_\xi}+\beta)^2}\right)a^2 + \\
&\frac{6a(1-\beta-2\alpha\beta) + 2(1+3\alpha+3\alpha^2)}{([\xi]_{p_\xi,q_\xi}+\beta)^2(p_\xi^2+p_\xi q_\xi+q_\xi^2)(p_\xi+q_\xi)} = \lambda_\xi.
\end{aligned}
$$

Here we take $\delta_\xi = \sqrt{\lambda_\xi}$, we obtain the theorem. □

4. Voronovskaya theorem

Theorem 4.1. *Suppose $0 < q_\xi < p_\xi \leq 1$ such that $p_\xi \to 1, q_\xi \to 1, p_\xi^\xi \to a$ and $q_\xi^\xi \to b$ as $\xi \to \infty$. If $\psi \in C_{x^2}[0,\infty)$, such that $\psi', \psi'' \in C_{x^2}[0,\infty)$, we obtain*

$$\lim_{\xi\to\infty} [\xi]_{p_\xi,q_\xi} |K_{\xi,\alpha,\beta}^{(p_\xi,q_\xi)}(\psi;x) - \psi(x)| = (\zeta x + \alpha + 1/2)\psi'(x) + x(\gamma x + 1)\frac{\psi''(x)}{2}$$

uniformly on $[0, A]$ *for any* $A > 0$*, where*

$$\zeta = \lim_{\xi\to\infty} [\xi]_{p_\xi,q_\xi}(q_\xi - 1)$$
$$\gamma = \lim_{\xi\to\infty} [\xi]_{p_\xi,q_\xi}(p_\xi q_\xi - 2q_\xi + 1).$$

Proof. Using Taylor's formula, we get

$$\psi(t) = \psi(x) + (t-x)\psi'(x) + \tfrac{1}{2}\psi''(x)(t-x)^2 + r(t,x)(t-x)^2,$$

where $r(t,x)$ is remainder and $r(t,x)$ tends to zero as $t \to x$.
Hence,

$$\begin{aligned}[\xi]_{p_\xi,q_\xi}(K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(\psi;x) - \psi(x)) &= [\xi]_{p_\xi,q_\xi}\psi'(x)K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}((t-x);x)\\ &+ [\xi]_{p_\xi,q_\xi}\frac{\psi''(x)}{2}K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}((t-x)^2;x)\\ &+ [\xi]_{p_\xi,q_\xi}K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(r(t,x)(t-x)^2;x).\end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(r(t,x)(t-x)^2;x) \leq \sqrt{K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(r^2(t,x);x)}\sqrt{K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}((t-x)^4;x)}.$$

Since, $r(t,x) \in C^*_{x^2}[0,\infty)$, hence using Theorem 2.4, also considering that $\lim_{t\to x} r(t,x) = 0$, we get

$$\lim_{\xi\to\infty} K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(r^2(t,x);x) = 0, \tag{5}$$

converges uniformly, $x \in [0, A]$. Hence, with the help of above equation (5) and using the fact that above linear operator is positive, we obtain

$$\lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(r(t,x)(t-x)^2;x) = 0.$$

Hence,

$$\begin{aligned}\lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}(K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}(\psi;x) - \psi(x)) &= \lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}\psi'(x)K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}((t-x);x)\\ &+ \lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}\frac{\psi''(x)}{2}K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}((t-x)^2;x).\end{aligned}$$

Here,

$$\begin{aligned}\lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}((t-x);x) &= \lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}\Phi^{(p_\xi,q_\xi)}_{1,\alpha,\beta}(x)\\ &= \lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}(q_\xi - 1)x + \alpha + 1/2\\ &= \zeta x + \alpha + 1/2, \qquad (6)\end{aligned}$$

and

$$\begin{aligned}\lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}K^{(p_\xi,q_\xi)}_{\xi,\alpha,\beta}((t-x)^2;x) &= \lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}\Phi^{(p_\xi,q_\xi)}_{2,\alpha,\beta}(x)\\ &= \lim_{\xi\to\infty}[\xi]_{p_\xi,q_\xi}(p_\xi q_\xi - 2q_\xi + 1)x^2 + x\end{aligned}$$

$$= x(\gamma x + 1). \tag{7}$$

Therefore, by using equations (6), (7); we obtain the result. □

5. Discussion

The solutions obtained in this research article are relatively more generalized and prcised as compare to other papers in operator theory, which improves the literature of applications of (p,q)-calculus. This paper will be beneficial to the researchers and experts studying or aim to study in the domain of functional analysis and applications of functional analysis. Moreover, the solutions can be beneficial in various areas of Mathematics and physics, i.e. Mathematical physics, Applied math. and Math. analysis.

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