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A RESERCH ON NONLINEAR (p, q)-DIFFERENCE EQUATION TRANSFORMABLE TO LINEAR EQUATIONS USING (p, q)-DERIVATIVE†

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Abstract. In this paper, we introduce various first order (p, q)-difference equations. We investigate solutions to equations which are linear (p, q)-difference equations and nonlinear (p, q)-difference equations. We also find some properties of (p, q)-calculus, exponential functions, and inverse function.

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1. Introduction

For a long time, studies on q-difference equations appeared in intensive works especially by F. H. Jackson[9], R. D. Carmichael[4], T. E. Mason[7], and other authors[11]. q-calculus is considered as one of the most useful concepts to study with special numbers and polynomials. This subject appears in many areas of mathematics, physics, engineering, and applications including q-combinatorics, q-arithmetics, q-integrable system, variational q-calculus, and so on(see [1,3,4,7,11]).

For any $n \in \mathbb{C}$, the q-number is defined by

$$[n]_q = \frac{1 - q^n}{1 - q}, \qquad |q| < 1.$$

In 1991, R. Chakrabarti and R. Jagannathan[5] introduced the (p, q)-number in order to unify varied forms of q-oscilator algebras in physics literature. Around

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the same time, independently, G. Brodimas, et al. and M. Arik, et al. discovered the (p,q)-number(see [1,2]). Also around the same time, Wachs and White[12] introduced the (p,q)-number in mathematics literature by certain combinatorial problems without any connection to the quantum group related to mathematics and physics literature.

For any $n \in \mathbb{C}$, the (p,q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \qquad \left|\frac{q}{p}\right| < 1.$$

Here, we can observe a difference that is q-number don't have the symmetric property. It is clear that (p,q)-number possesses the symmetric property, and this number is q-number when $p=1$. In particular, we can see $\lim_{q\to 1}[n]_{p,q} = n$ with $p=1$.

Heretofore, many mathematicians have studied (p,q)-calculus including (p,q)-exponential, integration, series and differentiation from (p,q)-number. (p,q)-extension of q-number has taken many new conceptions and has advanced since much properties of (p,q)-number is different from properties of q-number. For example, R. Jagannathan and K. S. Rao[8] created the (p,q)-extensions of q-identites in 2006. In [6], R. B. Corcino created the theorem of (p,q)-extension of binomials coefficients and found various properties which are related to horizontal function, triangular function, and vertica functionl. P. N. Sadjang[10] represented two appropriate polynomials of the (p,q)-derivative and investigated some properties of these polynomials. In addition, he discovered two (p,q)-Talyor formulas of polynomials and dotained the formula of (p,q)-integration by part. We define some basic notations about (p,q)-calculus which are found in [2,5,6,8,10,12].

Definition 1.1. We define the (p,q)-derivative operator of any function f, also referred to as the Jackson derivative, as follows:

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,$$

and $D_{p,q}f(0) = f'(0)$.

Since $D_{p,q}z^n = [n]_{p,q}z^{n-1}$, if $t(x) = \sum_{k=0}^{n} a_k x^k$ then

$$D_{p,q}t(x) = \sum_{k=0}^{n-1} a_{k+1}[k+1]_{p,q}x^k.$$

This equation is equivalent to the (p,q)-difference equation in q with known f

$$D_{p,q}g(x) = f(x).$$

From Definition 1.1, one has

$$\frac{1-T_{p,q}}{\left(1-\frac{q}{p}\right)x}g(x)=f\left(\frac{1}{p}x\right),\qquad T_{p,q}g(x)=g\left(\frac{q}{p}x\right).$$

Thus, we can see that

$$\begin{aligned} g(x)&=\left(1-\frac{q}{p}\right)\sum_{i=0}^{\infty}T_{p,q}^{i}\left\{xf\left(\frac{1}{p}x\right)\right\}\\ &=\left(1-\frac{q}{p}\right)x\sum_{i=0}^{\infty}\left(\frac{q}{p}\right)^{i}f\left(\frac{q^{i}}{p^{i+1}}x\right). \end{aligned}$$

If the series in the right hand side of the above is convergent, then we can find the previous calculus is obviously valid. Let f be an arbitrary function. In [10], we note that the definition of (p,q)- integral is

$$\int f(x)d_{p,q}x=(p-q)x\sum_{k=0}^{\infty}\frac{q^{k}}{p^{k+1}}f\left(\frac{q^{k}}{p^{k+1}}x\right).$$

Theorem 1.1. This operator,$D_{p,q}$, has the following basic properties:

(i) Derivative of a product
$$\begin{aligned} D_{p,q}(f(x)g(x))&=f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x)\\ &=g(px)D_{p,q}f(x)+f(qx)D_{p,q}g(x). \end{aligned}$$

(ii) Derivative of a ratio
$$\begin{aligned} D_{p,q}\left(\frac{f(x)}{g(x)}\right)&=\frac{g(qx)D_{p,q}f(x)-f(qx)D_{p,q}g(x)}{g(px)g(qx)}\\ &=\frac{g(px)D_{p,q}f(x)-f(px)D_{p,q}g(x)}{g(px)g(qx)}. \end{aligned}$$

Definition 1.2. The (p,q)-analogue of $(x+a)^n$ is defined by

(i) $(x+a)_{p,q}^{n}=\begin{cases} 1 & \text{if } n=0\\ (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}) & \text{if } n\neq 0\end{cases}$,

(ii) $(x+a)_{p,q}^{n}=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}p^{\binom{k}{2}}q^{\binom{n-k}{2}}x^{k}a^{n-k}$,

where $\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}$ is (p,q)-Gauss Binomial coefficient. In addition, we can see the notation, $((x,-a);(p,q))_n$, in other papers. This means $((x,-a);(p,q))_n=(x+a)_{p,q}^{n}$.

Definition 1.3. Let z be any complex numbers with $|z| < 1$. The two forms of (p, q)-exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!},$$
$$E_{p,q}(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!}.$$

The useful relation of two forms of (p, q)-exponential functions is taken by

$$e_{p,q}(z)E_{p,q}(-z) = 1, \qquad E_{p,q}(z) = e_{p^{-1},q^{-1}}(z).$$

Definition 1.4. For $n \neq 0$, we define

$$\mathcal{E}_{p,q}\left(\frac{z}{a+b}\right) = \sum_{n=0}^{\infty} \frac{1}{(a+b)_{p,q}^n} \frac{z^n}{[n]_{p,q}!}.$$

We can note that $\lim_{p,q\to 1} \mathcal{E}_{p,q}\left(\frac{z}{a+b}\right) = e^{\frac{z}{a+b}}$.

The most important aim of this paper is to find solutions of various first-order linear or nonlinear differential equations. The paper is organised as follows. In Section 2, we investigate various cases of first-order linear (p, q)-differential equations. In Section 3, we derive and illustrate with examples solutions to some first-order nonlinear (p, q)-differential equations.

2. First order linear (p, q)-difference equations

As in the case of differential or difference equations, first order linear (p, q)-difference equations are of particular interest in the theory and applications of (p, q)-difference equations. In this section, we investigate the solution for each basic type of equations.

We can write a general first order linear (p, q)-difference equation in the form:

$$D_{p,q}y(x) = a(x)y(qx) + b(x). \tag{2.1}$$

This equation is a non homogenous first order equation while the correspoding homogenous one has

$$D_{p,q}y(x) = a(x)y(qx). \tag{2.2}$$

Theorem 2.1. Consider the form $D_{p,q}(y) = a(x)y(qx)$. Then, we find

$$\begin{aligned} y(x) &= y\left(\left(\frac{q}{p}\right)^N x\right) \prod_{i=0}^{N-1} \left\{1 + \left(1 - \frac{q}{p}\right)\left(\frac{q}{p}\right)^i xa\left(\frac{q^i}{p^{i+1}}x\right)\right\} \\ &= y(x_0) \prod_{k=pq^{-1}x_0}^{x} \left\{1 + \left(1 - \frac{q}{p}\right) ka\left(\frac{k}{p}\right)\right\}. \end{aligned}$$

Proof. Applying the definition of $D_{p,q}$ in the homogenous equation, (2.2), we have

$$\begin{aligned} y(px) &= y(qx) + (p-q)xa(x)y(qx) \\ &= \{1 + (p-q)xa(x)\}\, y(qx). \end{aligned}$$

By replacing px by x in the above equation, one has

$$y(x) = \left\{1 + \left(1 - \frac{q}{p}\right) xa\left(\frac{1}{p}x\right)\right\} y\left(\frac{q}{p}x\right). \tag{2.3}$$

From (2.3), we get the result below by using the recurrence relation, and the theorem is proved.

$$\begin{aligned} y(x) &= y\left(\left(\frac{q}{p}\right)^N x\right) \prod_{i=0}^{N-1} \left\{1 + \left(1 - \frac{q}{p}\right)\left(\frac{q}{p}\right)^i xa\left(\frac{q^i}{p^{i+1}}x\right)\right\} \\ &= y(x_0) \prod_{k=pq^{-1}x_0}^{x} \left\{1 + \left(1 - \frac{q}{p}\right) ka\left(\frac{k}{p}\right)\right\}. \end{aligned}$$

□

If $N \to \infty$ with $0 < \frac{q}{p} < 1$, then we can see $\frac{q}{p} \to 0$ and also find

$$y(x) = y(0) \prod_{i=0}^{\infty} \left\{1 + \left(1 - \frac{q}{p}\right)\left(\frac{q}{p}\right)^i xa\left(\frac{q^i}{p^{i+1}}x\right)\right\}.$$

Corollary 2.1. Consider the equation $D_{p,q}(y) = a(px)y(qx)$. In this case, from Theorem 2.1, we can find the solutions

(i) $$y(x) = y\left(\left(\frac{q}{p}\right)^N x\right) \prod_{i=0}^{N-1} \left\{1 + \left(1 - \frac{q}{p}\right)\left(\frac{q}{p}\right)^i xa\left(\left(\frac{q}{p}\right)^i x\right)\right\}.$$

(ii) $$y(x) = y(0) \prod_{i=0}^{\infty} \left\{1 + \left(1 - \frac{q}{p}\right)\left(\frac{q}{p}\right)^i xa\left(\left(\frac{q}{p}\right)^i x\right)\right\}.$$

Example 2.1. Suppose $a(x) = \frac{p(q^k - p^k)}{(q-p)(q^k x - p^k)}$. Then we can find the following result,

$$y(x) = \frac{(-1)^k y(0)}{p^{\binom{k}{2}}} \prod_{i=0}^{k-1} (p^i - q^i x).$$

Solution. First, we can transform $a(x)$ as the follows:

$$a(x) = \frac{\left(\frac{q}{p}\right)^k - 1}{\left(\frac{q}{p} - 1\right)\left(\left(\frac{q}{p}\right)^k x - 1\right)}.$$

From Corollary 2.1 (ii), we obtain the following result:

$$y(x) = y(0)\prod_{i=0}^{\infty}\left\{1+\left(1-\frac{q}{p}\right)\left(\frac{q}{p}\right)^{i} x\frac{\left(\frac{q}{p}\right)^{k}-1}{\left(\frac{q}{p}-1\right)\left(\left(\frac{q}{p}\right)^{k+i}x-1\right)}\right\}$$

$$= y(0)\prod_{i=0}^{\infty}\left(\frac{\left(\frac{q}{p}\right)^{i}x-1}{\left(\frac{q}{p}\right)^{k+i}x-1}\right) = \frac{(-1)^k y(0)}{p^{\binom{k}{2}}}\prod_{i=0}^{k-1}(p^i-q^i x).$$

□

Theorem 2.2. Equation of the form $D_{p,q}y(x) = a(x)y(qx) + b(x)$ has the general solution,

$$y(x) = \int_{x_0}^{x} y_0(x)y_0^{-1}(pt)b(t)d_{p,q}t + y_0(x)c,$$

where $c = y_0^{-1}(x_0)y(x_0)$.

Proof. From variation of constants in equation 2.1, we can get

$$y(x) = c(x)y_0(x),$$

where $c(x)$ is an unknown function to be determined and $y_0(x)$ is a homogenous solution. By using (p,q)-derivative formula in the above equation, we get

$$\begin{aligned} D_{p,q}y(x) &= D_{p,q}c(x)y_0(x) \\ &= y_0(px)D_{p,q}c(x) + c(qx)D_{p,q}y_0(x). \end{aligned}$$

We can also transform the above equation from the given equation.

$$b(x) = c(qx)\left\{D_{p,q}y_0(x) - a(x)y_0(qx)\right\} + y_0(px)D_{p,q}c(x) = y_0(px)D_{p,q}c(x).$$

Thus, this equation can be written as

$$D_{p,q}c(x) = y_0^{-1}(px)b(x).$$

Using the integral formula on both sides, we get

$$c(x) = \int_{x_0}^{x} y_0^{-1}(pt)b(t)d_{p,q}t + c(x_0) = \int_{x_0}^{x} y_0^{-1}(pt)b(t)d_{p,q}t + c,$$

where $c = c(x_0) = y_0^{-1}(x_0)y(x_0)$. Therefore, we find the solution, and the theorem is completed.

$$\begin{aligned} y(x) &= \left(\int_{x_0}^{x} y_0^{-1}(pt)b(t)d_{p,q}t + c\right)y_0(x) \\ &= \int_{x_0}^{x} y_0(x)y_0^{-1}(pt)b(t)d_{p,q}t + y_0(x)c. \end{aligned}$$

□

Corollary 2.2. Let $D_{p,q}y(x) = a(px)y(qx) + b(px)$. Then we get

$$y(x) = \int_{x_0}^{x} y_0(x)y_0^{-1}(pt)b(pt)d_{p,q}t + y_0(x)c.$$

Theorem 2.3. Let a be some constant. Then the equation $D_{p,q}y(x) = ay(px)$ becomes

$$y(x) = \sum_{n=0}^{\infty} C_0 p^{\binom{n}{2}} \frac{(ax)^n}{[n]_{p,q}!} = C_0 e_{p,q}(ax).$$

Proof. From the definition of (p,q)-difference, the given equation can be written as

$$y(qx) = \{1 + (q-p)xa\}\, y(px).$$

In order to obtain the solution, we put

$$y(x) = \sum_{n=0}^{\infty} C_n x^n.$$

Then. we have

$$\begin{aligned} y(qx) = \sum_{n=0}^{\infty} C_n(qx)^n &= \{1 + (q-p)xa\} \sum_{n=0}^{\infty} C_n(px)^n \\ &= \sum_{n=0}^{\infty} C_n(px)^n + (q-p)a \sum_{n=0}^{\infty} C_n p^n x^{n+1}. \end{aligned}$$

From the above equation, we can write the k-th term as the follows.

$$C_k = ap^{k-1} \frac{p-q}{p^k - q^k} C_{k-1}.$$

By using recursive calculation, we get

$$C_n = C_0 p^{\binom{n}{2}} a^n \left(\prod_{k=1}^{n} \frac{p-q}{p^k - q^k} \right).$$

From the definition of $[n]_{p,q}$ and $[n]_{p,q}!$, we can change this to

$$C_n = C_0 p^{\binom{n}{2}} a^n \frac{1}{[n]_{p,q}!}.$$

Therefore, the solution is a (p,q)-exponential function,

$$y(x) = \sum_{n=0}^{\infty} C_0 p^{\binom{n}{2}} \frac{(ax)^n}{[n]_{p,q}!} = C_0 e_{p,q}(ax).$$

□

Theorem 2.4. An equation of $D_{p,q}y(x) = ay(qx)$ gives a result of the form

$$y(x) = \sum_{n=0}^{\infty} C_0 q^{\binom{n}{2}} \frac{(ax)^n}{[n]_{p,q}!} = C_0 e_{p^{-1},q^{-1}}(ax) = C_0 E_{p,q}(ax).$$

Proof. In the given equation, it is clear that

$$y(px) = \{1 + (p - q)xa\}\, y(qx).$$

This proof is very similar to the proof of Theorem 2.3, but the result is different. In other words, the result of Theorem 2.4 is the inverse function of $e_{p,q}(x)$. Hence, we omit the detailed proof of Theorem 2.4.

□

From Theorem 2.3 and Theorem 2.4, we note that

$$\int e_{p,q}(apx) d_{p,q}x = \frac{1}{a} e_{p,q}(ax), \qquad \int e_{p^{-1},q^{-1}}(aqx) d_{p,q}x = \frac{1}{a} e_{p^{-1},q^{-1}}(ax).$$

Theorem 2.5. Let $D_{p,q}y(x) = ay(px)$, $D_{p,q}z(x) = -a(x)z(qx)$ and $y(x_0)z(x_0) = 1$. Then we have $z(x)y(x) = 1$.

Proof. Using the differential formula, we have

$$D_{p,q}z(x)y(x) = y(px)D_{p,q}z(x) + z(qx)D_{p,q}y(x) = 0.$$

Therefore, the proof of Theorem 2.5 is complete.

□

Theorem 2.6. The equation of the form $D_{p,q}y(x) = \alpha x y(x)$ can seek a solution under the form $y(x) = \sum_{n=0}^{\infty} C_n x^n$. Thus, we find the solution

$$y(x) = C_0 \sum_{n=0}^{\infty} \frac{(\alpha x^2)^n}{(2)_{p,q}^n [n]_{p,q}!} = C_0 \mathcal{E}_{p,q}\left(\frac{\alpha x^2}{2}\right).$$

Proof. In order to find the solution of equation, we write

$$D_{p,q}y(x) = \sum_{n=1}^{\infty} C_n [n]_{p,q} x^{n-1} = \alpha \sum_{n=0}^{\infty} C_n x^{n+1}.$$

By using the coefficients of both sides in the above equation, we observe

$$C_{2n} = \alpha^n \frac{C_0}{[2n]_{p,q}[2(n-1)]_{p,q} \cdots [2]_{p,q}[1]_{p,q}} \quad \text{and} \quad C_{2n-1} = 0, \quad \text{for} \quad n \geq 1.$$

Here, we can apply a property of $[n]_{p,q}$(see [3]).

$$[2n]_{p,q}[2(n-1)]_{p,q} \cdots [2]_{p,q}[1]_{p,q} = [n]_{p,q}!(2)_{p,q}^n.$$

Therefore, we have the solution

$$y(x) = C_0 \sum_{n=0}^{\infty} \frac{(\alpha x^2)^n}{(2)_{p,q}^n [n]_{p,q}!} = C_0 \mathcal{E}_{p,q}\left(\frac{\alpha x^2}{2}\right),$$

where $\mathcal{E}_{p,q}(\frac{\alpha x^2}{2})$ is (p,q)-version of $e^{\frac{\alpha x^2}{2}}$.

□

Theorem 2.7. For the equation of the form $D_{p,q}y(x) = ay(px) + b$ with $x_0 = 0$, the solution is

$$y(x) = \left(y(0) + \frac{b}{a}\right) e_{p,q}(ax) - \frac{b}{a}.$$

Proof. Letting $y(x) = c(x)y_0(x)$, we get

$$D_{p,q}c(x)y_0(x) = c(px)D_{p,q}y_0(x) + y_0(qx)D_{p,q}c(x) = ay(px) + b.$$

From Theorem 2.3, one can write

$$D_{p,q}c(x) = y_0^{-1}(qx)b.$$

To search for a solution we can write

$$c(x) = c(0) + \frac{b}{a} - \frac{b}{a}e_{p^{-1},q^{-1}}(-ax).$$

Therefore, the result is

$$\begin{aligned} y(x) &= e_{p,q}(ax)\left\{y(0) + \frac{b}{a} - \frac{b}{a}e_{p^{-1},q^{-1}}(-ax)\right\} \\ &= \left(y(0) + \frac{b}{a}\right) e_{p,q}(ax) - \frac{b}{a}, \end{aligned}$$

and the theorem is completely proved.

□

Theorem 2.8. Consider the equation of the form $D_{p,q}y(x) = ay(qx)+b$ with $x_0 = 0$. Its solution is

$$y(x) = \left(y(0) + \frac{b}{a}\right) e_{p^{-1},q^{-1}}(ax) - \frac{b}{a}.$$

Proof. To solve the equation we set

$$y(x) = c(x)y_0(x).$$

By using the result of Theorem 2.2, we can derive

$$\begin{aligned} y(x) &= \int_{x_0}^{x} y_0(x)y_0^{-1}(pt)b(t)d_{p,q}t + y_0(x)c \\ &= y_0(x)\left\{b\int_0^x y_0^{-1}(pt)d_{p,q}t + c\right\} \\ &= e_{p^{-1},q^{-1}}(ax)\left\{b\int_0^x e_{p,q}(-apt)d_{p,q}t + y(0)\right\} \\ &= \left(y(0) + \frac{b}{a}\right) e_{p^{-1},q^{-1}}(ax) - \frac{b}{a}. \end{aligned}$$

Thus, the theorem is proved.

□

3. Nonlinear (p,q)-difference equations transformable to linear equations

In this section, we are concerned with first order nonlinear (p,q)-difference equations. The method of solving these equations is using first order linear equations. We also consider (p,q)-Riccati type equations.

Remark 3.1. Consider equations of the following form:

$$f\left(\frac{D_{p,q}y(x)}{y(px)}, x\right) = 0.$$

This equation can be transformed into a linear equation in $z(x)$ where $z(x) = \frac{D_{p,q}y(x)}{y(x)}$.

Example 3.1. Solve the equation.

$$\{D_{p,q}y(x)\}^2 - y(px)D_{p,q}y(x) - 6\{y(px)\}^2 = 0$$

Solution. Clearly, it gives $z^2(x) - z(x) - 6 = 0$ where $z(x) = \frac{D_{p,q}y(x)}{y(px)}$. Thus, $z(x) = 3$ and $z(x) = -2$ or $y(x) = c_1 e_{p,q}(3x)$ and $y(x) = c_2 e_{p,q}(-2x)$, respectively.

□

Example 3.2. Solve the equation.

$$\{D_{p,q}y(x)\}^2 - y(qx)D_{p,q}y(x) - 6\{y(qx)\}^2 = 0$$

Solution. Letting $z(x) = \frac{D_{p,q}y(x)}{y(qx)}$, one has

$$z^2(x) - z(x) - 6 = 0.$$

Thus, $z(x) = 3$ and $z(x) = -2$ or $y(x) = ce_{p^{-1},q^{-1}}(3x)$ and $y(x) = ce_{p^{-1},q^{-1}}(-2x)$, respectively.

□

Generally, we can derive the result where the solution for the equation of the form $\{D_{p,q}y(x)\}^2 - (a+b)y(px)D_{p,q}y(x) + ab\{y(px)\}^2 = 0$ is $y(x) = c_1 e_{p,q}(ax)$, $y(x) = c_2 e_{p,q}(bx)$. We can also find the solution of the form $\{D_{p,q}y(x)\}^2 - (a+b)y(qx)D_{p,q}y(x) + ab\{y(qx)\}^2 = 0$ is the inverse function of the (p,q)-exponential function.

Remark 3.2. Suppose (p,q)-Riccati type equation is as the follows:

$$D_{p,q}y(x) = a(x)y(qx) + b(x)y(px)y(qx).$$

If we set $y(x) = \frac{1}{z(x)}$ in order to solve the equation, we can find this following result:

$$D_{p,q}z(x) = -\{a(x)z(px) + b(x)\}.$$

Example 3.3. Calculate the following equation.

$$y(qx)y(px)\ln pq^{-1} - y(qx) + y(px) = 0$$

Solution. We can make $z(px) - z(qx) = \ln p - \ln q$ from $y(x) = \frac{1}{z(x)}$. According to the (p, q)-differential definition, one has

$$D_{p,q}z(x) = \frac{\ln p - \ln q}{(p-q)x}.$$

We can see $z(x) = \ln x$ by using the integral. Hence, the solution is $y(x) = \frac{1}{\ln x}$.

□

Remark 3.3. Let c_1, c_2 be some constants. Equation of the form $g(x) = \{y(px)\}^{c_1} \{y(qx)\}^{c_2}$ becomes the following equation by using ln function.

$$c_1 \ln y(px) + c_2 \ln y(qx) = \ln g(x).$$

Setting $z(x) = \ln y(x)$ we obtain

$$c_1 z(px) + c_2 z(qx) = \ln g(x).$$

Example 3.4. Contemplate the equation.

$$y^3(px) = e^{x^5} y(qx)$$

Solution. By using ln function in both sides, one has

$$3\ln y(px) - \ln y(qx) = x^5.$$

We also put $\ln y(x) = z(x)$ and represent

$$3z(px) - z(qx) = x^5. \tag{3.1}$$

Thus, we obtain the homogenous solution,

$$z(x) = cx^{\frac{\ln 3}{\ln p^{-1} q}}.$$

Now we will find particular solutions from Equation (3.1). Using the operator $T_{p,q}$, this equation can transform as follows:

$$3\left(1 - \frac{1}{3}T_{p,q}\right) z(x) = x^5$$

Calculating the equation, one has

$$z(x) = \frac{1}{3}\left(1 - \frac{1}{3}T_{p,q}\right)^{-1} x^5 = \frac{1}{3}\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i \left(\frac{q}{p}x\right)^{5i} = \frac{p^5 x^5}{3p^5 - q^5}.$$

We can find

$$z(x) = cx^{\frac{\ln 3}{\ln p^{-1} q}} + \frac{(px)^5}{3p^5 - q^5}.$$

Hence, we find to solution

$$y(x) = \exp\left(= cx^{\frac{\ln 3}{\ln p^{-1} q}} + \frac{(px)^5}{3p^5 - q^5}\right).$$

□

Remark 3.4. Consider the equation of the form $D_{p,q}y(x) = f(x)$. From the definition of (p,q)-differential equation we have

$$y(x) - y\left(\frac{q}{p}x\right) = (p-q)\frac{1}{p}xf\left(\frac{1}{p}x\right).$$

Thus, we have the general solution,

$$y(x) = (p-q)\frac{1}{p}(1-T_{p,q})^{-1}xf\left(\frac{1}{p}x\right) = \left(1-\frac{q}{p}\right)x\sum_{i=0}^{\infty}\left(\frac{q}{p}\right)^{i} f\left(\frac{q^i}{p^{i+1}}x\right).$$

Example 3.5. Solve the equation $y(qx) - ay(px) = h(x)$.
Solution. From Remark 3.4, we can find

$$y(x) = -\frac{1}{a}\left(1-\frac{1}{a}T_{p,q}\right)^{-1} h\left(\frac{1}{p}x\right) = -\sum_{i=0}^{\infty}\left(\frac{1}{a}\right)^{i+1} h\left(\frac{q^i}{p^{i+1}}x\right).$$

□

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