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AN EXTRAPOLATED CRANK-NICOLSON CHARACTERISTIC FINITE ELEMENT METHOD FOR NONLINEAR SOBOLEV EQUATIONS†

MI RAY OHM AND JUN YONG SHIN *

ABSTRACT. An extrapolated Crank-Nicolson characteristic finite element method is introduced for approximate solutions of nonlinear Sobolev equations with a convection term. And we obtain the higher order of convergence for approximate solutions in the temporal and the spatial directions with respect to L^2 norm.

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1. Introduction

In this paper, we will consider a nonlinear Sobolev equation with a convection term: find $u(\boldsymbol{x}, t)$ such that

$$\begin{cases} c(u)u_t + \boldsymbol{d}(u)\cdot\nabla u - \nabla\cdot(a(u)\nabla u) - \nabla\cdot(b(u)\nabla u_t) \\ \qquad\qquad\qquad = f(\boldsymbol{x}, t, u), \quad \text{in } \Omega\times(0,T], \\ u(\boldsymbol{x}, t) = \boldsymbol{0}, \qquad \text{on } \partial\Omega\times(0,T], \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^m$, $1 \le m \le 3$, is a bounded convex domain with boundary $\partial\Omega$, $0 < T < \infty$, and $c, \boldsymbol{d}, a, b$ and f are given functions. For the existence, uniqueness, regularity results, and physical applications of Sobolev equations, refer to [2, 3, 4, 21, 24] and the papers cited therein.

For Sobolev equations with no convection term, many numerical techniques

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*Corresponding author.
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such as classical finite element methods [1, 6, 10, 11, 12] or least-squares methods [9, 18, 19, 25, 26] or mixed finite element methods [8] or discontinuous finite element methods [13, 14, 22, 23] are used. However, in many situations, the convection term is used to describe the convection dominated diffusion. And a characteristic method is used to treat both the time derivative term and the convection term effectively. This method performs well for convection dominated diffusion problems as shown in [5, 7]. Gu in [7] and Shi et al [20] introduce a characteristic finite element method for a Sobolev equation and establish the higher order convergence in the space variable and the first order convergence in the time variable for approximate solutions. However the first order convergence in the time variable makes the higher order convergence in the space variable worse. So, Ohm and Shin in [15, 17] use a Crank-Nicolson or an extrapolated Crank-Nicolson characteristic finite element method for a Sobolev equation to obtain the higher order of convergence both in the space variable and in the time variable with respect to L^2 norm when the given functions $c(\cdot)$ and $\boldsymbol{d}(\cdot)$ depend only on $\boldsymbol{x}$. Ohm and Shin [16] introduce a Crank-Nicolson characteristic finite element method to construct approximate solutions of a nonlinear Sobolev equation with a convection term and establish the higher order of convergence in the time variable as well as in the space variable with respect to L^2 norm, which extend previous result [15] to the nonlinear Sobolev equation.

In this paper, we adopt an extrapolated Crank-Nicolson characteristic finite element method to construct approximate solutions of a nonlinear Sobolev equation with a convection term and establish the higher order of convergence in the time variable as well as in the space variable with respect to L^2 norm, which extends our previous result [17] to the nonlinear Sobolev equation. This paper is composed of four main sections. In Section 2, the smoothness assumptions for $u(\boldsymbol{x}, t)$, the conditions for the given functions, and basic notations are described. In Section 3, finite element spaces and basic approximation properties are given. In Section 4, we construct Crank-Nicolson characteristic finite element approximations of $u(\boldsymbol{x}, t)$ and establish the higher order of convergence in L^2 and H^1 normed spaces.

2. Assumptions and notations

Throughout this paper, let $W^{s,p}(\Omega)$ be the Sobolev space on Ω with its usual norm $\|\cdot\|_{s,p}$ for $s \geq 0$ and $1 \leq p \leq \infty$. When $p = 2$, we denote $H^s(\Omega) \equiv W^{s,2}(\Omega)$, $L^2(\Omega) \equiv H^0(\Omega)$, and $\|\cdot\|_s \equiv \|\cdot\|_{s,2}$. And we use $\|\cdot\| \equiv \|\cdot\|_0$ and $\|\cdot\|_\infty \equiv \|\cdot\|_{0,\infty}$. Let $\boldsymbol{H}^s(\Omega) = \{\boldsymbol{w} = (w_1, w_2, \ldots, w_m) \mid w_i \in H^s(\Omega), 1 \leq i \leq m\}$ be the Sobolev space on Ω with its usual norm $\|\boldsymbol{w}\|_s^2 = \sum_{i=1}^{m} \|w_i\|_s^2$ and $H_0^1(\Omega) = \{w \in H^1(\Omega) \mid w(\boldsymbol{x}) = 0 \text{ on } \partial\Omega\}$. For a given Banach space X and $t_1, t_2 \in [0, T]$, we introduce Sobolev spaces with the corresponding norms:

$$W^{s,p}(t_1, t_2; X) = \Big\{ w(\boldsymbol{x}, t) \mid \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot, t)\|_X \in L^p(t_1, t_2), 0 \leq \beta \leq s \Big\},$$

where

$$\|w\|_{W^{s,p}(t_1,t_2;X)} = \begin{cases} \left(\sum_{\beta=0}^{s} \int_{t_1}^{t_2} \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X^p dt \right)^{1/p}, & 1 \le p < \infty, \\ max_{0\le\beta\le s} \operatorname{esssup}_{t\in(t_1,t_2)} \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X, & p = \infty. \end{cases}$$

We simply denote $L^p(X) \equiv W^{0,p}(0,T;X)$ and $W^{s,p}(X) \equiv W^{s,p}(0,T;X)$.

Assume that $c(p), \boldsymbol{d}(p) = (d_1(p), d_2(p), \ldots, d_m(p))^T$, $a(p), b(p)$ and $f(\boldsymbol{x},t,p)$ satisfy

(A1) There exist constants c_*, c^*, d^*, a_*, a^*, b_*, and b^* such that $0 < c_* \le c(p) \le c^*$, $0 < |\boldsymbol{d}(p)| \le d^*$, $0 < a_* \le a(p) \le a^*$, $0 < b_* \le b(p) \le b^*$, for all $p \in \mathbb{R}$, where $|\boldsymbol{d}(p)| = \sum_{i=1}^{m} d_i^2(p)$.

(A2) $a_p(p), a_{pp}(p), a_{ppp}(p), b_p(p), b_{pp}(p)$, and $b_{ppp}(p)$ are bounded.

(A3) $f(\boldsymbol{x},t,p)$ is locally Lipschitz continuous in the third variable p, i.e. if $|\ p^* - p\ | \le \tilde{K}$ then $|f(\boldsymbol{x},t,p^*) - f(\boldsymbol{x},t,p)| \le K(p,\tilde{K})|\ p^* - p\ |$. And $a(p)$ and $b(p)$ are locally Lipschitz continuous.

For each $(\boldsymbol{x},t)$, let $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x},t)$ be the unit vector such that $\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{c(u)}{\psi(u)}\frac{\partial u}{\partial t} + \frac{\boldsymbol{d}(u)}{\psi(u)} \cdot \nabla u$, where $\psi(u) = [c(u)^2 + |\boldsymbol{d}(u)|^2]^{\frac{1}{2}}$. Then we can rewrite the Sobolev equation (1) as follows: find $u(\boldsymbol{x},t)$ such that

$$\begin{cases} \psi(u)\frac{\partial u}{\partial \boldsymbol{\nu}} - \nabla\cdot(a(u)\nabla u) - \nabla\cdot(b(u)\nabla u_t) = f(\boldsymbol{x},t,u), & \text{in } \Omega\times(0,T], \\ u(\boldsymbol{x},t) = 0, & \text{on } \partial\Omega\times(0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \text{in } \Omega. \end{cases} \qquad (2)$$

And the variational formulation of the equation (2) is given as follows: find $u(\boldsymbol{x},t) \in H_0^1(\Omega)$ such that

$$\begin{cases} (\psi(u)\frac{\partial u}{\partial \boldsymbol{\nu}}, \tau) + (a(u)\nabla u, \nabla\tau) + (b(u)\nabla u_t, \nabla\tau) \\ \qquad\qquad = (f(x,t,u),\tau), \qquad \forall \tau \in H_0^1(\Omega), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}). \end{cases} \qquad (3)$$

3. Finite element spaces and an elliptic projection

For $h > 0$, let $\{S_h^r\}$ be a family of finite dimensional subspaces of $H_0^1(\Omega)$ satisfying the following approximation and inverse properties:
(approximation property) For $\phi \in H_0^1(\Omega) \cap W^{s,p}(\Omega)$, there exist a positive constant K_1, independent of h, ϕ, and r, and a sequence $P_h\phi \in S_h^r$ such that for any $0 \le q \le s$ and $1 \le p \le \infty$

$$\|\phi - P_h\phi\|_{q,p} \le K_1 h^{\mu-q}\|\phi\|_{s,p}$$

where $\mu = \min(r+1, s)$.
(inverse property) There exist a positive constant K_2 independent of h and r, such that

$$\|\varphi\|_1 \le K_2 h^{-1}\|\varphi\| \text{ and } \|\varphi\|_\infty \le K_2 h^{-\frac{m}{2}}\|\varphi\|, \ \ \forall \varphi \in S_h^r.$$

And bilinear forms A and B are defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ as follows:

$$A(u : v, w) = (a(u)\nabla v, \nabla w), \quad B(u : v, w) = (b(u)\nabla v, \nabla w). \tag{4}$$

By following the idea in [10, 14] and the assumption (A1), a differentiable function $\tilde{u} : [0, T] \to S_h^r$ can be defined as follows

$$\begin{cases} A(u : u - \tilde{u}, \chi) + B(u : u_t - \tilde{u}_t, \chi) = 0, & \forall \chi \in S_h^r, \\ (\tilde{u}(0), \chi) = (u_0, \chi), & \forall \chi \in S_h^r. \end{cases} \tag{5}$$

Now letting $\eta = u - \tilde{u}$, we obtain some estimates for η, η_t, η_{tt}, and η_{ttt} whose proofs can be found in [15, 17].

Lemma 3.1. *Let $u_0 \in H^s(\Omega)$, $u_t, u_{tt}, u_{ttt} \in H^s(\Omega)$, and $u_t \in L^2(H^s(\Omega))$. Then there exists a constant K, independent of h, such that*

- **(i)** $\|\eta\| + h\|\eta\|_1 \le Kh^\mu(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s)$,
- **(ii)** $\|\eta_t\| + h\|\eta_t\|_1 \le Kh^\mu(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s)$,
- **(iii)** $\|\eta_{tt}\|_1 \le Kh^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s)$,
- **(iv)** $\|\eta_{ttt}\|_1 \le Kh^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s + \|u_{ttt}\|_s)$,

where $\mu = \min(r + 1, s)$ and $s \ge 2$.

Lemma 3.2. *Let $u_0 \in H^s(\Omega)$, $u, u_t, u_{tt}, u_{ttt} \in L^\infty(H^s(\Omega)) \cap L^\infty(W^{1,\infty}(\Omega))$, and $u_t \in L^2(H^s(\Omega))$. If $\mu \ge 1 + \frac{m}{2}$, then there exists a constant K, independent of h, such that*

$$\max\{\|\eta\|_\infty,\ \|\nabla\eta\|_\infty,\ \|\nabla\eta_t\|_\infty,\ \|\nabla\eta_{tt}\|_\infty,\ \|\nabla\eta_{ttt}\|_\infty\} \le K,$$

where $\mu = \min(r + 1, s)$.

Throughout this paper, we use a generic positive constant K depending only on the domain $\Omega, \tilde{K}$, and $u(\boldsymbol{x}, t)$ but independent of the discretization magnitudes of the space variable and the time variable. Therefore any K's in the different places do not need to be equal.

4. The optimal $L^\infty(L^2)$ and $L^\infty(H^1)$ error estimates

Let N be a positive integer, $\Delta t = T/N$ and $t^n = n\Delta t$, for $0 \le n \le N$. Denote $u^j = u(x, t^j)$, $u^{j-\frac{1}{2}} = \frac{1}{2}(u^j + u^{j-1})$, $t^{j-\frac{1}{2}} = \frac{1}{2}(t^j + t^{j-1})$, $\tilde{\boldsymbol{d}}(\cdot) = \boldsymbol{d}(\cdot)/c(\cdot)$. From (3) and the definitions of bilinear forms A and B, we have

$$\begin{aligned} &\Big(\psi(u(t^{n-\frac{1}{2}}))\frac{\partial u(t^{n-\frac{1}{2}})}{\partial \boldsymbol{\nu}}, \chi\Big) + A(u(t^{n-\frac{1}{2}}) :\ u(t^{n-\frac{1}{2}}), \chi) \\ &\quad + B(u(t^{n-\frac{1}{2}}) :\ u_t(t^{n-\frac{1}{2}}), \chi) = (f(u(t^{n-\frac{1}{2}})), \chi), \quad \forall \chi \in S_h^r, \end{aligned} \tag{6}$$

where $f(u(t^{n-\frac{1}{2}})) = f(\boldsymbol{x}, t^{n-\frac{1}{2}}, u(t^{n-\frac{1}{2}}))$ and so, we get

$$\Big(c(u(t^{n-\frac{1}{2}}))\frac{\check{u}^n - \hat{\hat{u}}^{n-1}}{\Delta t}, \chi\Big) + A(u(t^{n-\frac{1}{2}})\ :\ u^{n-\frac{1}{2}}, \chi) \tag{7}$$

$$+ B(u(t^{n-\frac{1}{2}}) \ : \ \frac{u^n - u^{n-1}}{\Delta t}, \chi)$$
$$= (f(u(t^{n-\frac{1}{2}})), \chi) + Q_1 + Q_2 + Q_3, \ \forall \chi \in S_h^r,$$

where $\breve{\breve{u}}^n = u(\breve{\breve{\boldsymbol{x}}}, t^n)$, $\hat{\hat{u}}^{n-1} = u(\hat{\hat{\boldsymbol{x}}}, t^{n-1})$, $\breve{\breve{\boldsymbol{x}}} = \boldsymbol{x} + \frac{1}{2}\tilde{\boldsymbol{d}}(u(t^{n-\frac{1}{2}}))\Delta t$, $\hat{\hat{\boldsymbol{x}}} = \boldsymbol{x} - \frac{1}{2}\tilde{\boldsymbol{d}}(u(t^{n-\frac{1}{2}}))\Delta t$, $Q_1 = (c(u(t^{n-\frac{1}{2}}))\frac{\breve{\breve{u}}^n - \hat{\hat{u}}^{n-1}}{\Delta t} - \psi(u(t^{n-\frac{1}{2}}))\frac{\partial u(t^{n-\frac{1}{2}})}{\partial \boldsymbol{\nu}}, \chi)$, $Q_2 = A(u(t^{n-\frac{1}{2}}) : \ u^{n-\frac{1}{2}} - u(t^{n-\frac{1}{2}}), \chi)$, and $Q_3 = B(u(t^{n-\frac{1}{2}}) : \ \frac{u^n - u^{n-1}}{\Delta t} - u_t(t^{n-\frac{1}{2}}), \chi)$.

Now an extrapolated Crank-Nicolson characteristic finite element scheme for (1) is given as follows: Find $\{u_h^n\} \in S_h^r$ such that

$$\Big(c(Eu_h^n)\frac{\breve{u}_h^n - \bar{u}_h^{n-1}}{\Delta t}, \chi\Big) + A(Eu_h^n : u_h^{n-\frac{1}{2}}, \chi) + B(Eu_h^n : \frac{u_h^n - u_h^{n-1}}{\Delta t}, \chi) \tag{8}$$
$$= (f(Eu_h^n), \chi), \quad \forall \chi \in S_h^r, \ n = 2, \dots, N,$$

$$\Big(c(u_h^{\frac{1}{2}})\frac{\breve{u}_h^1 - \hat{u}_h^0}{\Delta t}, \chi\Big) + A(u_h^{\frac{1}{2}} : u_h^{\frac{1}{2}}, \chi) + B(u_h^{\frac{1}{2}} : \frac{u_h^1 - u_h^0}{\Delta t}, \chi) \tag{9}$$
$$= (f(u_h^{\frac{1}{2}}), \chi), \quad \forall \chi \in S_h^r,$$

$$u_h^0(\boldsymbol{x}) = \tilde{u}(\boldsymbol{x}, 0), \tag{10}$$

where $Eu_h^n = \frac{3}{2}u_h^{n-1} - \frac{1}{2}u_h^{n-2}$, $\breve{u}_h^n = u_h^n(\breve{\boldsymbol{x}})$, $\bar{u}_h^{n-1} = u_h^{n-1}(\bar{\boldsymbol{x}})$, $\breve{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{2}\tilde{\boldsymbol{d}}(Eu_h^n)\Delta t$, $\bar{\boldsymbol{x}} = \boldsymbol{x} - \frac{1}{2}\tilde{\boldsymbol{d}}(Eu_h^n)\Delta t$, $\breve{u}_h^1 = u_h^1(\breve{\boldsymbol{x}})$, $\hat{u}_h^0 = u_h^0(\hat{\boldsymbol{x}})$, $\breve{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{2}\tilde{\boldsymbol{d}}(u_h^{\frac{1}{2}})\Delta t$, $\hat{\boldsymbol{x}} = \boldsymbol{x} - \frac{1}{2}\tilde{\boldsymbol{d}}(u_h^{\frac{1}{2}})\Delta t$, and $u_h^{\frac{1}{2}} = \frac{1}{2}(u_h^1 + u_h^0)$.

For our analysis of the convergence, we denote $\xi^n = u_h^n - \tilde{u}^n$ and $\partial_t \xi^n = \frac{\xi^n - \xi^{n-1}}{\Delta t}$. Since the equation (9) is the same as one in [16] for $n = 1$, we have the following theorem whose proof can be found in [16].

Theorem 4.1. *Let u and $\{u_h^n\}$ be solutions of (3) and (8)-(10), respectively. In addition to the assumptions of Lemma 3.2, if $\mu \geq 1 + \frac{m}{2}$, $u \in L^\infty(H^3(\Omega))$, and $\Delta t = O(h)$, then*

$$\|\nabla \xi^1\|^2 + \Delta t(\|\partial_t \xi^1\|^2 + \|\nabla \partial_t \xi^1\|^2) \leq K\Delta t(h^{2\mu} + (\Delta t)^4),$$

where $\mu = \min(r+1, s)$.

Theorem 4.2. *Under the same assumptions of Theorem 4.1, we havee*

$$\|\nabla \xi^2\|^2 + \Delta t(\|\partial_t \xi^2\|^2 + \|\nabla \partial_t \xi^2\|^2) \leq K\Delta t(h^{2\mu} + (\Delta t)^4),$$

where $\mu = \min(r+1, s)$.

Proof. From (7) and (8) with $n = 2$ and $\chi = \partial_t \xi^2$, we get

$$\Big(c(Eu_h^2)\partial_t \xi^2, \partial_t \xi^2\Big) + A(Eu_h^2 \ : \ \xi^{\frac{3}{2}}, \partial_t \xi^2) + B(Eu_h^2 \ : \ \partial_t \xi^2, \partial_t \xi^2)$$
$$= \Big(c(Eu_h^2)\frac{\xi^2 - \breve{\xi}^2}{\Delta t}, \partial_t \xi^2\Big) + \Big(c(Eu_h^2)\frac{\bar{\xi}^1 - \xi^1}{\Delta t}, \partial_t \xi^2\Big)$$
$$- \Big(c(Eu_h^2)\frac{\eta^2 - \breve{\eta}^2}{\Delta t}, \partial_t \xi^2\Big) + \Big(c(Eu_h^2)\frac{\eta^2 - \eta^1}{\Delta t}, \partial_t \xi^2\Big)$$

$$
\begin{aligned}
&- \Big(c(Eu_h^2)\frac{\bar{\eta}^1 - \eta^1}{\Delta t}, \partial_t \xi^2\Big) + A(Eu_h^2 \,:\, \eta^{\frac{3}{2}}, \partial_t \xi^2) \\
&+ \Big[A(u(t^{\frac{3}{2}}) \,:\, u^{\frac{3}{2}}, \partial_t \xi^2) - A(Eu_h^2 \,:\, u^{\frac{3}{2}}, \partial_t \xi^2)\Big] \\
&+ B(Eu_h^2 \,:\, \partial_t \eta^2, \partial_t \xi^2) \\
&+ \Big[B(u(t^{\frac{3}{2}}) \,:\, \frac{u^2 - u^1}{\Delta t}, \partial_t \xi^2) - B(Eu_h^2 \,:\, \frac{u^2 - u^1}{\Delta t}, \partial_t \xi^2)\Big] \\
&+ \Big(c(Eu_h^2)\frac{\check{\breve{u}}^2 - \breve{u}^2 - \hat{\hat{u}}^1 + \bar{u}^1}{\Delta t}, \partial_t \xi^2\Big) \\
&+ \Big([c(u(t^{\frac{3}{2}})) - c(Eu_h^2)]\frac{\check{\breve{u}}^2 - \hat{\hat{u}}^1}{\Delta t}, \partial_t \xi^2\Big) \\
&+ (f(Eu_h^2) - f(u(t^{\frac{3}{2}})), \partial_t \xi^2) - Q_1 - Q_2 - Q_3 = \sum_{i=1}^{15} I_i. \qquad (11)
\end{aligned}
$$

Now let L_1, L_2 and L_3 denote three terms of the left-hand side of (11), respectively. First we can estimate L_1, L_2 and L_3 as follows:

$$
\begin{aligned}
L_1 &= (c(Eu_h^2)\partial_t \xi^2, \partial_t \xi^2) \geq c_* \|\partial_t \xi^2\|^2, \\
L_2 &= A(Eu_h^2 \,:\, \xi^{\frac{3}{2}}, \partial_t \xi^2) \\
&= \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^2)}\nabla \xi^2\|^2 - \|\sqrt{a(Eu_h^2)}\nabla \xi^1\|^2) \\
&\geq \frac{1}{2\Delta t}(a_* \|\nabla \xi^2\|^2 - a^* \|\nabla \xi^1\|^2) \\
L_3 &= B(Eu_h^2 \,:\, \partial_t \xi^2, \partial_t \xi^2) \geq b_* \|\nabla \partial_t \xi^2\|^2.
\end{aligned}
$$

By applying these bounds of $L_1 \sim L_3$ to (11), we get

$$
\begin{aligned}
&c_* \|\partial_t \xi^2\|^2 + b_* \|\nabla \partial_t \xi^2\|^2 + \frac{1}{2\Delta t} a_* \|\nabla \xi^2\|^2 \\
&\leq \frac{1}{2\Delta t} a^* \|\nabla \xi^1\|^2 + \sum_{i=1}^{15} I_i. \qquad (12)
\end{aligned}
$$

By using the assumption (A1) and Cauchy-Schwartz inequality, we can estimate $I_1 \sim I_5$ as follows:

$$
\begin{aligned}
I_1 &\leq \epsilon \|\partial_t \xi^2\|^2 + K \|\nabla \xi^2\|^2, \\
I_2 &\leq \epsilon \|\partial_t \xi^2\|^2 + K \|\nabla \xi^1\|^2, \\
I_3 &\leq \epsilon (\|\partial_t \xi^2\|^2 + \|\nabla \partial_t \xi^2\|^2) + K \|\eta^2\|^2, \\
I_4 &\leq \epsilon \|\partial_t \xi^2\|^2 + K \|\eta_t^2\|^2, \\
I_5 &\leq \epsilon (\|\partial_t \xi^2\|^2 + \|\nabla \partial_t \xi^2\|^2) + K \|\eta^1\|^2,
\end{aligned}
$$

for an $\epsilon > 0$. To estimate the sum of I_6 and I_8, we split it into six terms, by using (5), as follows:

$$\begin{aligned}
I_6 + I_8 =& A(Eu_h^2 \,:\, \eta^{\frac{3}{2}}, \partial_t \xi^2) + B(Eu_h^2 \,:\, \partial_t \eta^2, \partial_t \xi^2) \\
=& (a(Eu_h^2)(\nabla \eta^{\frac{3}{2}} - \nabla \eta(t^{\frac{3}{2}})), \nabla \partial_t \xi^2) \\
&+ ([a(Eu_h^2) - a(u(t^{\frac{3}{2}}))]\nabla \eta(t^{\frac{3}{2}}), \nabla \partial_t \xi^2) \\
&+ (a(u(t^{\frac{3}{2}}))\nabla \eta(t^{\frac{3}{2}}), \nabla \partial_t \xi^2) \\
&+ \Big(b(Eu_h^2)(\frac{\nabla \eta^2 - \nabla \eta^1}{\Delta t} - \nabla \eta_t(t^{\frac{3}{2}})), \nabla \partial_t \xi^2\Big) \\
&+ ((b(Eu_h^2) - b(u(t^{\frac{3}{2}})))\nabla \eta_t(t^{\frac{3}{2}}), \nabla \partial_t \xi^2) \\
&+ \Big(b(u(t^{\frac{3}{2}}))\nabla \eta_t(t^{\frac{3}{2}}), \nabla \partial_t \xi^2\Big) \\
\equiv& \sum_{i=1}^{6} J_i.
\end{aligned}$$

By Taylor expansion and Lemma 3.1, we get

$$\|\nabla \eta^{\frac{3}{2}} - \nabla \eta(t^{\frac{3}{2}})\| + \|\frac{\nabla \eta^2 - \nabla \eta^1}{\Delta t} - \nabla \eta_t(t^{\frac{3}{2}})\| \le K(\Delta t)^2$$

and hence

$$J_1 + J_4 \le \epsilon \|\nabla \partial_t \xi^2\|^2 + K(\Delta t)^4.$$

Note that

$$\begin{aligned}
\|Eu_h^2 - u(t^{\frac{3}{2}})\| &= \|Eu_h^2 - E\tilde{u}^2 + E\tilde{u}^2 - \tilde{u}(t^{\frac{3}{2}}) + \tilde{u}(t^{\frac{3}{2}}) - u(t^{\frac{3}{2}})\| \\
&= \|E\xi^2 + \frac{3}{2}\tilde{u}^1 - \frac{1}{2}\tilde{u}^0 - \tilde{u}(t^{\frac{3}{2}}) - \eta(t^{\frac{3}{2}})\| \\
&\le K(\|\xi^1\| + \|\xi^0\| + (\Delta t)^2 + \|\eta(t^{\frac{3}{2}})\|) \\
&\le K(\|\xi^1\| + h^\mu + (\Delta t)^2). \qquad (13)
\end{aligned}$$

By Lemma 3.2 and (13), we can estimate J_2 and J_5 as follows:

$$J_2 \le \epsilon \|\nabla \partial_t \xi^2\|^2 + K(\|\xi^1\|^2 + h^{2\mu} + (\Delta t)^4)$$

and

$$J_5 \le \epsilon \|\nabla \partial_t \xi^2\|^2 + K(\|\xi^1\|^2 + h^{2\mu} + (\Delta t)^4).$$

From (5), it is obvious that $J_3 + J_6 = 0$. Therefore, we get

$$I_6 + I_8 \le 3\epsilon \|\nabla \partial_t \xi^2\|^2 + K(\|\xi^1\|^2 + h^{2\mu} + (\Delta t)^4).$$

Note that by Taylor expansion, we have

$$\begin{aligned}
&\breve{u}^2 - \breve{u}^2 - \hat{u}^1 + \bar{u}^1 \\
=&\Delta t\Big[\frac{\boldsymbol{d}(u(\boldsymbol{x}, t^{\frac{3}{2}}))(c(Eu_h^2) - c(u(\boldsymbol{x}, t^{\frac{3}{2}})))}{c(Eu_h^2)c(u(\boldsymbol{x}, t^{\frac{3}{2}}))}
\end{aligned}$$

$$+\frac{c(u(\boldsymbol{x},t^{\frac{3}{2}})(\boldsymbol{d}(u(\boldsymbol{x},t^{\frac{3}{2}}))-\boldsymbol{d}(Eu_h^2))}{c(Eu_h^2)c(u(\boldsymbol{x},t^{\frac{3}{2}}))}\Big]\cdot\nabla u(\boldsymbol{x},t^{\frac{3}{2}})+O((\Delta t)^3) \quad (14)$$

and

$$\|\check{u}^2-\hat{u}^1\|_\infty\le K\Delta t. \quad (15)$$

By using (13), (14), and (15), we can estimate I_7, I_9, I_{10}, I_{11}, and I_{12} as follows:

$$\begin{aligned}
I_7 &\le \epsilon\|\nabla\partial_t\xi^2\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4),\\
I_9 &\le \epsilon\|\nabla\partial_t\xi^2\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4),\\
I_{10} &\le \epsilon\|\partial_t\xi^2\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4),\\
I_{11} &\le \epsilon\|\partial_t\xi^2\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4),\\
I_{12} &\le \epsilon\|\partial_t\xi^2\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4).
\end{aligned}$$

By using Taylor expansion, there exist $t_\theta^1\in(t^{\frac{3}{2}},t^2), t_\theta^0\in(t^1,t^{\frac{3}{2}}), \check{\hat{\boldsymbol{x}}}_{\theta i}\in(\check{\hat{\boldsymbol{x}}},\boldsymbol{x})$, and $\hat{\hat{\boldsymbol{x}}}_{\theta i}\in(\hat{\hat{\boldsymbol{x}}},\boldsymbol{x})$, $1\le i\le 3$, such that

$$\begin{aligned}
&\psi(u(t^{\frac{3}{2}}))\frac{\partial u(t^{\frac{3}{2}})}{\partial\boldsymbol{\nu}}-c(u(t^{\frac{3}{2}}))\frac{\check{u}^2-\hat{u}^1}{\Delta t}\\
=&-c(u(t^{\frac{3}{2}}))(\Delta t)^2\Big[\frac{1}{48}\tilde{\boldsymbol{d}}^3\cdot\nabla^3u(\check{\hat{\boldsymbol{x}}}_{\theta1},t^{\frac{3}{2}})+\frac{1}{16}\tilde{\boldsymbol{d}}^2\cdot\nabla^2u_t(\check{\hat{\boldsymbol{x}}}_{\theta2},t^{\frac{3}{2}})\\
&+\frac{1}{16}\tilde{\boldsymbol{d}}\cdot\nabla u_{tt}(\check{\hat{\boldsymbol{x}}}_{\theta3},t^{\frac{3}{2}})+\frac{1}{48}\check{u}_{ttt}(t_\theta^1)\\
&+\frac{1}{48}\tilde{\boldsymbol{d}}^3\cdot\nabla^3u(\hat{\hat{\boldsymbol{x}}}_{\theta1},t^{\frac{3}{2}})+\frac{1}{16}\tilde{\boldsymbol{d}}^2\cdot\nabla^2u_t(\hat{\hat{\boldsymbol{x}}}_{\theta2},t^{\frac{3}{2}})\\
&+\frac{1}{16}\tilde{\boldsymbol{d}}\cdot\nabla u_{tt}(\hat{\hat{\boldsymbol{x}}}_{\theta3},t^{\frac{3}{2}})+\frac{1}{48}\hat{u}_{ttt}(t_\theta^0)\Big]
\end{aligned} \quad (16)$$

where $\boldsymbol{d}^j\cdot(\nabla^ju)=\sum_{l=0}^{j}\binom{j}{l}d_1^{j-l}d_2^l\frac{\partial^ju}{\partial x_1^{j-l}\partial x_2^l}$ for $j=1,2,3$ when $m=2$ and we use similar notations when $m=3$. By the regularity of u, (13), (15), and (16), we have

$$\begin{aligned}
I_{13} &\le \epsilon\|\partial_t\xi^2\|^2+K(\Delta t)^4,\\
I_{14} &\le \epsilon\|\nabla\partial_t\xi^2\|^2+K(\Delta t)^4,\\
I_{15} &\le \epsilon\|\nabla\partial_t\xi^2\|^2+K(\Delta t)^4.
\end{aligned}$$

Now by applying the upper bounds for $I_1\sim I_{15}$ to (12), we get

$$\begin{aligned}
&c_*\|\partial_t\xi^2\|^2+b_*\|\nabla\partial_t\xi^2\|^2+\frac{a_*}{2\Delta t}\|\nabla\xi^2\|^2\\
\le&\frac{a^*}{2\Delta t}\|\nabla\xi^1\|^2+K(\|\xi^1\|^2+\|\nabla\xi^2\|^2+\|\nabla\xi^1\|^2+h^{2\mu}+(\Delta t)^4)\\
&+9\epsilon\|\partial_t\xi^2\|^2+9\epsilon\|\nabla\partial_t\xi^2\|^2.
\end{aligned} \quad (17)$$

Thus, by using Theorem 4.1, we have

$$\|\nabla\xi^2\|^2 + \Delta t(\|\partial_t\xi^2\|^2 + \|\nabla\partial_t\xi^2\|^2) \le K\Delta t[h^{2\mu} + (\Delta t)^4]$$

for sufficiently small ϵ and Δt. □

Theorem 4.3. *Under the same assumptions of Theorem 4.1, we have*

$$\max_{0\le n\le N}\Big[\|u^n - u_h^n\| + h\|\nabla(u^n - u_h^n)\|\Big] \le K(h^\mu + (\Delta t)^2),$$

where $\mu = \min(r+1, s)$.

Proof. To prove this theorem, we will establish the following statement: There exist $0 < \tilde{h} < 1$ and $0 < \tilde{\Delta} t < 1$ such that

$$\|\nabla\xi^n\|^2 + \Delta t(\|\partial_t\xi^n\|^2 + \|\nabla\partial_t\xi^n\|^2) \le K(h^{2\mu} + (\Delta t)^4) \tag{18}$$

for any $0 < h < \tilde{h}$, $0 < \Delta t < \tilde{\Delta} t$ and $n = 0, 1, \ldots, N$. Since $\xi^0 = 0$, it is trivial that (18) holds for $n = 0$. And by Theorem 4.1 and Theorem 4.2, (18) holds for $n = 1$ and $n = 2$. Now we assume that (18) holds with $n \le l - 1$. Notice that $\|\xi^n\|_\infty \le K,\ \ 0 \le n \le l-1$. We subtract (7) from (8) with $3 \le n \le l$ and $\chi = \partial_t\xi^n$ to get

$$\begin{aligned}
&\Big(c(Eu_h^n)\partial_t\xi^n, \partial_t\xi^n\Big) + A(Eu_h^n : \xi^{n-\frac{1}{2}}, \partial_t\xi^n) + B(Eu_h^n : \partial_t\xi^n, \partial_t\xi^n)\\
=&\Big(c(Eu_h^n)\frac{\xi^n - \breve{\xi}^n}{\Delta t}, \partial_t\xi^n\Big) + \Big(c(Eu_h^n)\frac{\bar{\xi}^{n-1} - \xi^{n-1}}{\Delta t}, \partial_t\xi^n\Big)\\
&-\Big(c(Eu_h^n)\frac{\eta^n - \breve{\eta}^n}{\Delta t}, \partial_t\xi^n\Big) + \Big(c(Eu_h^n)\frac{\eta^n - \eta^{n-1}}{\Delta t}, \partial_t\xi^n\Big)\\
&-\Big(c(Eu_h^n)\frac{\bar{\eta}^{n-1} - \eta^{n-1}}{\Delta t}, \partial_t\xi^n\Big) + A(Eu_h^n : \eta^{n-\frac{1}{2}}, \partial_t\xi^n)\\
&+\Big[A(u(t^{n-\frac{1}{2}}) : u^{n-\frac{1}{2}}, \partial_t\xi^n) - A(Eu_h^n : u^{n-\frac{1}{2}}, \partial_t\xi^n)\Big]\\
&+B(Eu_h^n : \partial_t\eta^n, \partial_t\xi^n)\\
&+\Big[B(u(t^{n-\frac{1}{2}}) : \frac{u^n - u^{n-1}}{\Delta t}, \partial_t\xi^n) - B(Eu_h^n : \frac{u^n - u^{n-1}}{\Delta t}, \partial_t\xi^n)\Big]\\
&+\Big(c(Eu_h^n)\frac{\check{\breve{u}}^n - \breve{u}^n - \hat{\hat{u}}^{n-1} + \bar{u}^{n-1}}{\Delta t}, \partial_t\xi^n\Big)\\
&+\Big([c(u(t^{n-\frac{1}{2}})) - c(Eu_h^n)]\frac{\check{\breve{u}}^n - \hat{\hat{u}}^{n-1}}{\Delta t}, \partial_t\xi^n\Big)\\
&+(f(Eu_h^n) - f(u(t^{n-\frac{1}{2}})), \partial_t\xi^n) - Q_1 - Q_2 - Q_3 = \sum_{i=1}^{15} I_i.
\end{aligned} \tag{19}$$

Now let L_1, L_2 and L_3 denote three terms of the left-hand side of (19), respectively. First, we can get the lower bounds of L_1, L_2 and L_3 as follows:

$$L_1 = (c(Eu_h^n)\partial_t\xi^n, \partial_t\xi^n) \ge c_*\|\partial_t\xi^n\|^2,$$

$$
\begin{aligned}
L_2 &= A(Eu_h^n \,:\, \xi^{n-\frac{1}{2}}, \partial_t \xi^n) \\
&\ge \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla \xi^n\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla \xi^{n-1}\|^2) \\
&\quad + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^{n-1})}\nabla \xi^{n-1}\|^2 - \|\sqrt{a(Eu_h^n)}\nabla \xi^{n-1}\|^2), \\
L_3 &= B(Eu_h^n \,:\, \partial_t \xi^n, \partial_t \xi^n) \ge b_* \|\nabla \partial_t \xi^n\|^2.
\end{aligned}
$$

By applying these bounds of $L_1 \sim L_3$ to (19), we get

$$
\begin{aligned}
& c_* \|\partial_t \xi^n\|^2 + b_* \|\nabla \partial_t \xi^n\|^2 \\
&\quad + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla \xi^n\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla \xi^{n-1}\|^2) \\
&\le \frac{1}{2\Delta t}([a(Eu_h^n) - a(Eu_h^{n-1})]\nabla \xi^{n-1}, \nabla \xi^{n-1}) + \sum_{i=1}^{15} I_i.
\end{aligned} \tag{20}
$$

By (18) and the fact that $\Delta t = O(h)$, we obtain

$$
\begin{aligned}
& \|Eu_h^n - Eu_h^{n-1}\|_\infty \\
&= \|E(u_h^n - \tilde{u}^n) - E(u_h^{n-1} - \tilde{u}^{n-1}) + E\tilde{u}^n - E\tilde{u}^{n-1}\|_\infty \\
&\le \Delta t\Big(\frac{3}{2}\|\partial_t \xi^{n-1}\|_\infty + \frac{1}{2}\|\partial_t \xi^{n-2}\|_\infty\Big) + K\Delta t \\
&\le K\Delta t.
\end{aligned} \tag{21}
$$

Therefore, applying the assumption (A2) and (21) to (20), we have

$$
\begin{aligned}
& c_* \|\partial_t \xi^n\|^2 + b_* \|\nabla \partial_t \xi^n\|^2 \\
&\quad + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla \xi^n\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla \xi^{n-1}\|^2) \\
&\le K\|\nabla \xi^{n-1}\|^2 + \sum_{i=1}^{15} I_i.
\end{aligned} \tag{22}
$$

By using the assumption (A1) and Cauchy-Schwartz inequality, we can get the following bounds for $I_1 \sim I_5$:

$$
\begin{aligned}
I_1 &\le \epsilon \|\partial_t \xi^n\|^2 + K\|\nabla \xi^n\|^2, \\
I_2 &\le \epsilon \|\partial_t \xi^n\|^2 + K\|\nabla \xi^{n-1}\|^2, \\
I_3 &\le \epsilon (\|\partial_t \xi^n\|^2 + \|\nabla \partial_t \xi^n\|^2) + K\|\eta^n\|^2, \\
I_4 &\le \epsilon \|\partial_t \xi^n\|^2 + K\|\eta_t^n\|^2, \\
I_5 &\le \epsilon (\|\partial_t \xi^n\|^2 + \|\nabla \partial_t \xi^n\|^2) + K\|\eta^{n-1}\|^2,
\end{aligned}
$$

for an $\epsilon > 0$. The sum of I_6 and I_8 can be split into six terms, by using (5), as follows:

$$
\begin{aligned}
I_6 + I_8 =& (a(Eu_h^n)(\nabla \eta^{n-\frac{1}{2}} - \nabla \eta(t^{n-\frac{1}{2}})), \nabla \partial_t \xi^n) \\
& + ([a(Eu_h^n) - a(u(t^{n-\frac{1}{2}}))]\nabla \eta(t^{n-\frac{1}{2}}), \nabla \partial_t \xi^n)
\end{aligned}
$$

$$
\begin{aligned}
&+ (a(u(t^{n-\frac{1}{2}}))\nabla\eta(t^{n-\frac{1}{2}}), \nabla\partial_t\xi^n) \\
&+ \Big(b(Eu_h^n)(\frac{\nabla\eta^n - \nabla\eta^{n-1}}{\Delta t} - \nabla\eta_t(t^{n-\frac{1}{2}})), \nabla\partial_t\xi^n\Big) \\
&+ ((b(Eu_h^n) - b(u(t^{n-\frac{1}{2}})))\nabla\eta_t(t^{n-\frac{1}{2}}), \nabla\partial_t\xi^n) \\
&+ \Big(b(u(t^{n-\frac{1}{2}}))\nabla\eta_t(t^{n-\frac{1}{2}}), \nabla\partial_t\xi^n\Big) \\
\equiv &\sum_{i=1}^{6} J_i.
\end{aligned}
$$

By Taylor expansion and Lemma 3.1, we get

$$\|\nabla\eta^{n-\frac{1}{2}} - \nabla\eta(t^{n-\frac{1}{2}})\| + \|\frac{\nabla\eta^n - \nabla\eta^{n-1}}{\Delta t} - \nabla\eta_t(t^{n-\frac{1}{2}})\| \le K(\Delta t)^2$$

and so,

$$J_1 + J_4 \le \epsilon\|\nabla\partial_t\xi^n\|^2 + K(\Delta t)^4.$$

Notice that as in (13)

$$\|Eu_h^n - u(t^{n-\frac{1}{2}})\| \le K(\|\xi^{n-1}\| + \|\xi^{n-2}\| + h^\mu + (\Delta t)^2). \tag{23}$$

By (23), we get the bounds for J_2 and J_5 as follows:

$$J_2 \le \epsilon\|\nabla\partial_t\xi^n\|^2 + K(\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + h^{2\mu} + (\Delta t)^4)$$

and

$$J_5 \le \epsilon\|\nabla\partial_t\xi^n\|^2 + K(\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + h^{2\mu} + (\Delta t)^4).$$

From (5), it is obvious that $J_3 + J_6 = 0$. Therefore, we get

$$I_6 + I_8 \le 3\epsilon\|\nabla\partial_t\xi^1\|^2 + K(\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + h^{2\mu} + (\Delta t)^4).$$

Note that by using Taylor expansion, we have

$$
\begin{aligned}
&\check{\bar{u}}^n - \breve{u}^n - \hat{\bar{u}}^{n-1} + \bar{u}^{n-1} \\
=&\Delta t\Big(\frac{\boldsymbol{d}(u(\boldsymbol{x}, t^{n-\frac{1}{2}}))(c(Eu_h^n) - c(u(\boldsymbol{x}, t^{n-\frac{1}{2}})))}{c(u(\boldsymbol{x}, t^{n-\frac{1}{2}}))c(Eu_h^n)} \\
&\quad + \frac{(\boldsymbol{d}(u(\boldsymbol{x}, t^{n-\frac{1}{2}})) - \boldsymbol{d}(Eu_h^n))c(u(\boldsymbol{x}, t^{n-\frac{1}{2}}))}{c(u(\boldsymbol{x}, t^{n-\frac{1}{2}}))c(Eu_h^n)}\Big) \cdot \nabla u(\boldsymbol{x}, t^{n-\frac{1}{2}}) \\
&+ O((\Delta t)^3)
\end{aligned} \tag{24}
$$

and

$$\|\check{\bar{u}}^n - \hat{\bar{u}}^{n-1}\|_\infty \le K\Delta t. \tag{25}$$

By using (23), (24), and (25), we can get the bounds for I_7, I_9, I_{10}, I_{11}, and I_{12} as follows:

$$
\begin{aligned}
I_7 &\le \epsilon\|\nabla\partial_t\xi^n\|^2 + K(\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + h^{2\mu} + (\Delta t)^4), \\
I_9 &\le \epsilon\|\nabla\partial_t\xi^n\|^2 + K(\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + h^{2\mu} + (\Delta t)^4),
\end{aligned}
$$

$$
\begin{aligned}
I_{10} &\le \epsilon\|\partial_t\xi^n\|^2 + K(\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + h^{2\mu} + (\Delta t)^4),\\
I_{11} &\le \epsilon\|\partial_t\xi^n\|^2 + K(\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + h^{2\mu} + (\Delta t)^4),\\
I_{12} &\le \epsilon\|\partial_t\xi^n\|^2 + K(\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + h^{2\mu} + (\Delta t)^4).
\end{aligned}
$$

By similar argument as in (16), there exist $t_\theta^1 \in (t^{n-\frac{1}{2}}, t^n), t_\theta^0 \in (t^{n-1}, t^{n-\frac{1}{2}})$, $\check{\check{\boldsymbol{x}}}_{\theta i} \in (\check{\check{\boldsymbol{x}}}, \boldsymbol{x})$, and $\hat{\hat{\boldsymbol{x}}}_{\theta i} \in (\hat{\hat{\boldsymbol{x}}}, \boldsymbol{x}),\ 1 \le i \le 3$, such that

$$
\begin{aligned}
&\psi(u(t^{n-\frac{1}{2}}))\frac{\partial u(t^{n-\frac{1}{2}})}{\partial \boldsymbol{\nu}} - c(u(t^{n-\frac{1}{2}}))\frac{\check{\check{u}}^n - \hat{\hat{u}}^{n-1}}{\Delta t}\\
&= -c(u(t^{n-\frac{1}{2}}))(\Delta t)^2\Big[\frac{1}{48}\tilde{\boldsymbol{d}}^3 \cdot \nabla^3 u(\check{\check{\boldsymbol{x}}}_{\theta 1}, t^{n-\frac{1}{2}}) + \frac{1}{16}\tilde{\boldsymbol{d}}^2 \cdot \nabla^2 u_t(\check{\check{\boldsymbol{x}}}_{\theta 2}, t^{n-\frac{1}{2}})\\
&\qquad + \frac{1}{16}\tilde{\boldsymbol{d}} \cdot \nabla u_{tt}(\check{\check{\boldsymbol{x}}}_{\theta 3}, t^{n-\frac{1}{2}}) + \frac{1}{48}\check{\check{u}}_{ttt}(t_\theta^1)\\
&\qquad + \frac{1}{48}\tilde{\boldsymbol{d}}^3 \cdot \nabla^3 u(\hat{\hat{\boldsymbol{x}}}_{\theta 1}, t^{n-\frac{1}{2}}) + \frac{1}{16}\tilde{\boldsymbol{d}}^2 \cdot \nabla^2 u_t(\hat{\hat{\boldsymbol{x}}}_{\theta 2}, t^{n-\frac{1}{2}})\\
&\qquad + \frac{1}{16}\tilde{\boldsymbol{d}} \cdot \nabla u_{tt}(\hat{\hat{\boldsymbol{x}}}_{\theta 3}, t^{n-\frac{1}{2}}) + \frac{1}{48}\hat{\hat{u}}_{ttt}(t_\theta^0)\Big],
\end{aligned}
\tag{26}
$$

where $\boldsymbol{d}^j \cdot (\nabla^j u) = \sum_{l=0}^{j} \binom{j}{l} d_1^{j-l} d_2^l \frac{\partial^j u}{\partial x_1^{j-l}\partial x_2^l}$ for $j = 1, 2, 3$ when $m = 2$ and we use similar notations when $m = 3$. By the regularity of u, (23), (25), and (26), we get

$$
\begin{aligned}
I_{13} &\le \epsilon\|\partial_t\xi^n\|^2 + K(\Delta t)^4,\\
I_{14} &\le \epsilon\|\nabla\partial_t\xi^n\|^2 + K(\Delta t)^4,\\
I_{15} &\le \epsilon\|\nabla\partial_t\xi^n\|^2 + K(\Delta t)^4.
\end{aligned}
$$

Now applying the estimates for $I_1 \sim I_{15}$ to (20), we obtain

$$
\begin{aligned}
&c_*\|\partial_t\xi^n\|^2 + b_*\|\nabla\partial_t\xi^n\|^2\\
&\quad + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla\xi^n\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\xi^{n-1}\|^2)\\
&\le K(\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\nabla\xi^n\|^2 + \|\nabla\xi^{n-1}\|^2 + h^{2\mu} + (\Delta t)^4)\\
&\quad + 9\epsilon\|\partial_t\xi^n\|^2 + 9\epsilon\|\nabla\partial_t\xi^n\|^2.
\end{aligned}
\tag{27}
$$

Hence, by Poincare's inequality, (27) can be estimated as follows:

$$
\begin{aligned}
&\Delta t[c_*\|\partial_t\xi^n\|^2 + b_*\|\nabla\partial_t\xi^n\|^2]\\
&\qquad + (\|\sqrt{a(Eu_h^n)}\nabla\xi^n\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\xi^{n-1}\|^2)\\
&\le K\Delta t(\|\nabla\xi^n\|^2 + \|\nabla\xi^{n-1}\|^2 + \|\nabla\xi^{n-2}\|^2 + h^{2\mu} + (\Delta t)^4).
\end{aligned}
\tag{28}
$$

for sufficiently small ϵ. Now we add both sides of (28) from $n = 3$ to l to get

$$
\Delta t \sum_{n=3}^{l}[c_*\|\partial_t\xi^n\|^2 + b_*\|\nabla\partial_t\xi^n\|^2] + \|\sqrt{a(Eu_h^l)}\nabla\xi^l\|^2
$$

$$\leq K\Delta t\sum_{n=1}^{l}(\|\nabla\xi^n\|^2+h^{2\mu}+(\Delta t)^4)+K\|\nabla\xi^2\|^2.$$

So, by Theorem 4.2, we have

$$\begin{aligned}&\|\nabla\xi^l\|^2+\Delta t\{\|\partial_t\xi^l\|^2+\|\nabla\partial_t\xi^l\|^2\}\\ \leq& K\Big[\Delta t\sum_{n=1}^{l-1}\{\|\nabla\xi^n\|^2+\Delta t(\|\partial_t\xi^n\|^2+\|\nabla\partial_t\xi^n\|^2)\}+K\Delta t\sum_{n=1}^{l}\{h^{2\mu}+(\Delta t)^4\}\Big],\end{aligned}$$

for sufficiently small Δt. Therefore, by Gronwall's inequality, we have

$$\|\nabla\xi^l\|^2+\Delta t\{\|\partial_t\xi^l\|^2+\|\nabla\partial_t\xi^l\|^2\}\leq K[h^{2\mu}+(\Delta t)^4],$$

which completes the proof of the statement (18) by the mathematical induction. By using the triangle inequality and the Poincare's inequality, we finally have $\|u^l-u_h^l\|\leq K(h^\mu+(\Delta t)^2)$ and $\|\nabla(u^l-u_h^l)\|\leq K(h^{\mu-1}+(\Delta t)^2)$. Thus the proof of this theorem is completed. □

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Mi Ray Ohm received her BS degree from Busan National University and Ph.D. from Busan National University under the direction of Professor Ki Sik Ha. She is currently a professor at Dongseo University. Her research interest is numerical analysis on partial differential equations.

Division of Mechatronics Engineering, Dongseo University, Busan 47011, Korea.
e-mail: mrohm@dongseo.ac.kr

Jun Yong Shin received his BS degree from Busan National University and Ph.D. degree from University of Texas at Arlington under the direction of Professor R. Kannan. He is currently a professor at Pukyong National University. His research interest is numerical analysis on partial differential equations.

Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 48513, Korea.
e-mail: jyshin@pknu.ac.kr