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NEW INEQUALITIES FOR GENERALIZED LOG h-CONVEX FUNCTIONS

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Abstract. In the paper, we introduce some new classes of generalized log$h$-convex functions in the first sense and in the second sense. We establish Hermite-Hadamard type inequality for different classes of generalized convex functions. It is shown that the classes of generalized log h-convex functions in both senses include several new and known classes of log h convex functions. Several special cases are also discussed. Results proved in this paper can be viewed as a new contributions in this area of research.

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1. Introduction

Convex analysis plays a border role onto classical convex from one side and geometry on the other side. This unique quality allows us to study different problems related to pure and applied sciences in a unified framework. In recent years, several new classes of convex functions has been introduced and investigated using novel and innovative ideas, see [1, 3, 6, 7, 14, 15, 23]. The necessary and sufficient condition for a function to be convex is to satisfy Hermite-Hadamard inequality. This classical Hermite-Hadamard inequality is one of the most important inequality related to convex function, see [12, 13]. For the applications, generalizations Hermite-Hadamard type inequalities and other aspects of these inequlaities, see [4, 8, 19, 20] and the references therein.

A significant generalization of convex functions was the introduction of h-convex functions by Varosanec [25], which include s-convex, p convex and Godunova -Levine functions as its special cases . For different properties and other aspects of h-convex functions, see, [18, 24, 26]. Gordji et al. [9] introduced a

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class of convex functions, which is called generalized convex (φ-convex) function. These generalized convex functions are nonconvex functions. For recent developments, see [5, 10, 16, 17, 19, 21, 22] and the references therein.

Inspired and motivated by the ongoing research, we introduce some new classes of generalized convex functions, which are called generalized log h-convex functions in the first sense and in the second sense, respectively. We derive some new Hermite-Hadamard integral inequalities for these nonconvex functions. Our results include a wide class of known new error estimates for various classes of convex functions. Results obtained in this paper continue to hold for the various classes of convex functions, which can be obtained as special cases. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminaries

Let $I = [a, b]$ and J be the intervals in real line $\mathbb{R}$, $[0, 1] \subseteq J$. Let $f : I = [a, b] \to \mathbb{R}$ and $h : J \to \mathbb{R}$ be two nonnegative and continuous functions and $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous bifunction. First of all, we recall the following well known results and concepts.

Definition 2.1. ([9]). A function $f : I = [a, b] \to \mathbb{R}$ is said to be a generalized((φ-convex)) convex function with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a + tb) \leq (1-t)f(a) + t[f(a) + \eta(f(b), f(a))], \quad \forall a, b \in I, t \in [0, 1].$$

Definition 2.2. Let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized h-convex function in the first sense with respect to a bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a nonnegative function h, if

$$f((1-t)a + tb) \leq h(1-t)f(a) + h(t)[f(a) + \eta(f(b), f(a))], \quad \forall a, b \in I, t \in [0, 1].$$

If $\eta(f(b), f(a)) = f(b) - f(a)$, then the Definition 2.2 reduces to

Definition 2.3. ([25]). Let $h : J \to \mathbb{R}$ be a non-negative function. A non-negative function $f : I \to (0, \infty)$is said to be h-convex, or $f \in SX(h, I)$, if

$$f((1-t)a + tb) \leq h(1-t)f(a) + h(t)f(b), \quad \forall a, b \in I, t \in [0, 1].$$

We now introduce a new class of generalized convex functions.

Definition 2.4. Let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : I = [a, b] \to \mathbb{R}_+$ is said to be generalized log h-convex in the first sense with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a nonnegative function h, if

$$f((1-t)a + tb) \leq [f(a)]^{h(1-t)}[f(a) + \eta(f(b), f(a))]^{h(t)}, \quad \forall a, b \in I, t \in [0, 1]. \quad (1)$$

If $t = \frac{1}{2}$, then

$$f(\frac{a+b}{2}) \quad \leq \quad [f(a)]^{h(\frac{1}{2})}[f(a) + \eta(f(b), f(a))]^{h(\frac{1}{2})}, \quad \forall a, b \in I. \quad (2)$$

The function f is known as generalized Jensen log h-convex function.

From Definition 2.4, we have

$$\log f((1-t)a+tb) \leq h(1-t)\log[f(a)] + h(t)\log[f(a)+\eta(f(b),f(a))],$$

and

$$f((1-t)a+tb) \leq h(1-t)[f(a)] + h(t)[f(a)+\eta(f(b),f(a))].$$

This means that every generalized log h-convex function is a generalized h-convex function. However the converse is not true.

Now we will discuss some special cases of generalized log h-convex functions in the first sense.

I. If $\eta(f(b),f(a)) = f(b)-f(a)$, then Definition 2.4 reduces to

Definition 2.5. ([18]). Let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : I \to (0,\infty)$ is said to be log h-convex or multiplicatively h-convex in the first sense if log (f) is convex, or equivalently if one has the following inequality

$$f((1-t)a+tb) \leq [f(a)]^{h(1-t)}[f(b)]^{h(t)}, \quad \forall a,b \in I, t \in [0,1].$$

II. If $h(t) = t$, then Definition 2.4 reduces to

Definition 2.6. A function $f : I = [a,b] \to \mathbb{R}_+$ is said to be generalized log convex with respect to a bifunction $\eta(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a+tb) \leq [f(a)]^{1-t}[f(a)+\eta(f(b),f(a))]^{t}, \quad \forall a,b \in I, t \in [0,1]$$

III. If $h(t) = t^s$, then Definition 2.4 reduces to

Definition 2.7. A function $f : I = [a,b] \to \mathbb{R}_+$ is said to be generalized log s-convex in the second sense for $s \in (0,1)$ with respect to a bifunction $\eta(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a+tb) \leq [f(a)]^{(1-t)^s}[f(a)+\eta(f(b),f(a))]^{t^s}, \quad \forall a,b \in I, t \in [0,1].$$

IV. If $h(t) = 1$, then Definition 2.4 reduces to

Definition 2.8. A function $f : I = [a,b] \to \mathbb{R}_+$ is said to be a generalized log P-convex with respect to a bifunction $\eta(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a+tb) \leq [f(a)][f(a)+\eta(f(b),f(a))], \quad \forall a,b \in I, t \in [0,1].$$

V. If $h(t) = \frac{1}{t}$, then Definition 2.4 reduces to

Definition 2.9. A function $f : I = [a,b] \to \mathbb{R}_+$ is said to be a generalized log Godunova-Levine convex with respect to a bifunction $\eta(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a+tb) \leq [f(a)]^{\frac{t}{1-t}}[f(a)+\eta(f(b),f(a))]^{\frac{1}{t}}, \quad \forall a,b \in I, t \in (0,1).$$

We now introduce a new class of generalized convex functions, which is called the generalized h-convex functions in the second sense.

Definition 2.10. Let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized h-convex function in the second sense with respect to a bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and non negative function h, if

$$f((1-t)a+tb) \leq h(1-t)h(t)[2f(a)+\eta(f(b), f(a))], \quad \forall a, b \in I, t \in [0,1].$$

If $\eta(f(b), f(a)) = f(b) - f(a)$, then the definition 2.10 reduces to the following new concept.

Definition 2.11. Let $h : J \to \mathbb{R}$ be a non-negative function. A non-negative function $f : I \to (0, \infty)$is said to be h-convex in the second sense, if

$$f((1-t)a+tb) \leq h(1-t)h(t)[f(a)+f(b)], \quad \forall a, b \in I, t \in [0,1].$$

Definition 2.12. Let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : I = [a, b] \to \mathbb{R}_+$ is said to be generalized log h-convex in the second sense with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a+tb) \leq \left\{[f(a)][f(a)+\eta(f(b), f(a))]\right\}^{h(t)h(1-t)}, \quad \forall a, b \in I, t \in [0,1].$$

If $t = \frac{1}{2}$, then

$$f(\frac{a+b}{2}) \quad \leq \quad \left\{[f(a)][f(a)+\eta(f(b), f(a))]\right\}^{h^2(\frac{1}{2})}, \forall a, b \in I.$$

The function f is known as the generalized Jensen log h-convex functions in the second sense.

From which, we have

$$\log f((1-t)a+tb) \leq h(t)h(1-t)\left\{\log[f(a)] + \log[f(a)+\eta(f(b), f(a))]\right\}.$$

WE now discuss some special cases of generalized log h-convex functions in the second sense.

I. If $h(t) = t$, then Definition 2.12 reduces to

Definition 2.13. A function $f : I = [a, b] \to \mathbb{R}_+$ is said to be generalized log tgs-convex with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a+tb) \leq \left\{[f(a)][f(a)+\eta(f(b), f(a))]\right\}^{t(1-t)}, \quad \forall a, b \in I, t \in [0,1].$$

II. If $h(t) = t^s$, then Definition 2.12 reduces to

Definition 2.14. A function $f : I = [a, b] \to \mathbb{R}_+$ is said to be generalized log (tgs, s)-convex in the second sense for $s \in (0, 1]$ with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a + tb) \leq \left\{[f(a)][f(a) + \eta(f(b), f(a))]\right\}^{t^s(1-t)^s}, \quad \forall a, b \in I, t \in [0, 1].$$

III. If $h(t) = t^p$ and $h(1-t) = (1-t)^q$ then, Definition 2.12 reduces to

Definition 2.15. A function $f : I = [a, b] \to \mathbb{R}_+$ is said to be generalized log beta-convex in the second sense with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a + tb) \leq \left\{[f(a)][f(a) + \eta(f(b), f(a))]\right\}^{t^p(1-t)^q}, \quad \forall a, b \in I, t \in [0, 1].$$

IV. If $h(t) + h(1-t) = 1$ and $h(t) = t^p$, then Definition 2.12 reduces to

Definition 2.16. A function $f : I = [a, b] \to \mathbb{R}_+$ is said to be generalized log Toader-convex with respect to a bifunction $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a + tb) \leq \left\{[f(a)][f(a) + \eta(f(b), f(a))]\right\}^{t^p(1-t^p)}, \quad \forall a, b \in I, t \in [0, 1].$$

We would like to point out that for appropriate and suitable choice of the bifunction $\eta(.,.)$ and the non-negative function $h(.)$, one can obtain several new and known classes of convex functions as special of the concepts introduced in this paper. This clearly shows that these concepts are quite flexible and unifying one.

3. Main results

In this section, we establish several new integral inequalities of Hermite-Hadamard type for generalized log h-convex function in the first sense and in the second sense.

Theorem 3.1. *Let f be a generalized log h-convex function in the first sense on I. Then*

$$\begin{aligned} \log f\Big(\frac{a+b}{2}\Big)^{\frac{1}{h(\frac{1}{2})}} \quad - \quad & \frac{1}{(b-a)} \int_a^b \log[f(x) + \eta(f(a+b-x), f(x)]\mathrm{d}x \\ & \leq \frac{1}{(b-a)} \int_a^b \log f(x)\mathrm{d}x \\ & \leq \log\left\{[f(a)][f(a) + \eta(f(b), f(a)]\right\} \int_0^1 h(t)\mathrm{d}t. \end{aligned}$$

Proof. Let f be generalized log h-convex function in the first sense. Then

$$\log f((1-t)a+tb) \leq \big\{h(1-t)\log[f(a)] + h(t)\log[f(a)+\eta(f(b),f(a))]\big\}. \tag{3}$$

Integrating (3) with respect to t on [0,1], we have

$$\begin{aligned}\int_0^1 \log f((1-t)a+tb)\mathrm{d}t &\leq \int_0^1 \big\{h(1-t)\log[f(a)] + h(t)\log[f(a)+\eta(f(b),f(a))]\big\}\mathrm{d}t \\ &= \log\big\{[f(a)][f(a)+\eta(f(b),f(a))]\big\}\int_0^1 h(t)\mathrm{d}t.\end{aligned}$$

Thus

$$\frac{1}{b-a}\int_a^b \log f(x)\mathrm{d}x \leq \log\big\{[f(a)][f(a)+\eta(f(b),f(a))]\big\}\int_0^1 h(t)\mathrm{d}t. \tag{4}$$

Consider

$$\begin{aligned}& f(\frac{a+b}{2}) = \frac{f[((1-t)a+tb)+(ta+(1-t)b)]}{2} \\ \leq\ & [f(1-t)a+tb)]^{h(\frac{1}{2})}[f((1-t)a+tb)+\eta(f(1-t)b+ta),(1-t)a+tb))]^{h(\frac{1}{2})} \\ =\ & \Big\{[f(1-t)a+tb)][f((1-t)a+tb)+\eta(f(1-t)b+ta),(1-t)a+tb))]\Big\}^{h(\frac{1}{2})}.\end{aligned}$$

This implies that

$$\begin{aligned}& \log f(\frac{a+b}{2}) \\ \leq\ & h(\frac{1}{2})\log\Big\{[f(1-t)a+tb)][f((1-t)a+tb)+\eta(f(1-t)b+ta),(1-t)a+tb))]\Big\}.\end{aligned} \tag{5}$$

Integrating (5) with respect to t on [0,1], we have

$$\frac{1}{h(\frac{1}{2})}\log f(\frac{a+b}{2}) \leq \frac{1}{(b-a)}\int_a^b \big\{\log f(x) + \log[f(x)+\eta(f(a+b-x),f(x))]\big\}\mathrm{d}x.$$

Thus

$$\begin{aligned}\frac{1}{h(\frac{1}{2})}\log f(\frac{a+b}{2}) &- \frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x),f(x))]\mathrm{d}x \\ &\leq \frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x.\end{aligned} \tag{6}$$

Combining(4) and (6), we have

$$\begin{aligned}\log f\big(\frac{a+b}{2}\big)^{\frac{1}{h(\frac{1}{2})}} &- \frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x),f(x)]\mathrm{d}x \\ &\leq \frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x\end{aligned}$$

$$\leq \log\left\{[f(a)][f(a)+\eta(f(b),f(a)]\right\}\int_0^1 h(t)\mathrm{d}t.$$

This completes the proof. □

Corollary 3.2. ([18]). *If $\eta(f(b),f(a)) = f(b)-f(a)$, then under the assumptions of Theorem 3.1, we have*

$$f\Big(\frac{a+b}{2}\Big)^{\frac{1}{2h(\frac{1}{2})}} \leq \exp\frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x \leq \big[f(a)f(b)\big]\int_0^1 h(t)\mathrm{d}t.$$

We now discuss some special cases Theorem 3.1.

(I). If $\eta(f(b),f(a)) = f(b)-f(a)$ and $h(t)=t$, then Theorem 3.1 becomes

Theorem 3.3. *Let $f : I \to (0,\infty)$ be a generalized log convex function on I. Then*

$$f(\frac{a+b}{2}) \leq \exp\frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x \leq \sqrt{\big[f(a)f(b)\big]}.$$

(II). If $h(t)=t^s$, then Theorem 3.1 becomes

Theorem 3.4. *Let $f : I \to (0,\infty)$ be a generalized log s-convex function on I with $s\in(0,1)$. Then*

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big)^{2^s} \quad - \quad & \exp\frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x),f(x)]\mathrm{d}x \\ & \leq \exp\frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x \\ & \leq \big\{[f(a)][f(a)+\eta(f(b),f(a)]\big\}\int_0^1 t^s\mathrm{d}t \\ & = \big\{[f(a)][f(a)+\eta(f(b),f(a)]\big\}\frac{1}{(s+1)}. \end{aligned}$$

Corollary 3.5. ([18]). *If $\eta(f(b),f(a)) = f(b)-f(a)$, then under the assumptions theorem 3.4, we have*

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big)^{2^{s-1}} \quad & \leq \quad \exp\frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x \\ & \leq \quad \big\{[f(a)][f(b)]\big\}\int_0^1 t^s\mathrm{d}t \\ & = \quad \big\{[f(a)][f(b)]\big\}\frac{1}{(s+1)}. \end{aligned}$$

Corollary 3.6. ([6]). *If $\eta(f(b), f(a)) = f(b) - f(a)$ and $s = 1$, then under the assumptions of Theorem 3.4, we have*

$$f(\frac{a+b}{2}) \leq \exp \frac{1}{(b-a)} \int_a^b \log f(x)\mathrm{d}x \leq \sqrt{\big[f(a)f(b)\big]}.$$

(III). If $h(t) = 1$, then Theorem 3.1 becomes

Theorem 3.7. *Let $f : I \to (0, \infty)$ be a generalized log P-convex function on I. Then*

$$\begin{aligned} & f\big(\frac{a+b}{2}\big) - \exp \frac{1}{(b-a)} \int_a^b \log[f(x) + \eta(f(a+b-x), f(x)]\mathrm{d}x \\ \leq\ & \exp \frac{1}{(b-a)} \int_a^b \log f(x)\mathrm{d}x \\ \leq\ & \big\{[f(a)][f(a) + \eta(f(b), f(a)]\big\}. \end{aligned}$$

Corollary 3.8. ([18]). *If $\eta(f(b), f(a)) = f(b) - f(a)$, then under the assumptions theorem 3.7, we have*

$$f\big(\frac{a+b}{2}\big) \leq \exp \frac{2}{(b-a)} \int_a^b \log f(x)\mathrm{d}x \leq \big[f(a)f(b)\big]^2.$$

(IV). If $h(t) = \frac{1}{t}$, then Theorem 3.1 becomes

Theorem 3.9. *Let $f : I \to (0, \infty)$ be a generalized log Godunova-Levin-convex function on I. Then*

$$\begin{aligned} \frac{1}{2} \log f(\frac{a+b}{2}) \quad - \quad & \frac{1}{(b-a)} \int_a^b \log[f(x) + \eta(f(a+b-x), f(x))]\mathrm{d}x \\ & \leq \frac{1}{(b-a)} \int_a^b \log f(x)\mathrm{d}x, \end{aligned}$$

and

$$\frac{1}{b-a} \int_a^b \tau x \log f(x)\mathrm{d}x \leq \frac{1}{2} \log \bigg\{[f(a)][f(a) + \eta(f(b), f(a)]\bigg\},$$

where $\tau x = (\frac{x-a}{b-a})(\frac{b-x}{b-a})$.

Corollary 3.10. ([7]). *If $\eta(f(b), f(a)) = f(b) - f(a)$, then, under the assumptions of Theorem 3.9, we have*

$$f(\frac{a+b}{2})^{\frac{1}{4}} \leq \exp \frac{1}{(b-a)} \int_a^b \log f(x)\mathrm{d}x,$$

and

$$\frac{1}{b-a}\int_a^b \tau x \log f(x)\mathrm{d}x \leq \frac{1}{2}\log\Big\{[f(a)][f(b)]\Big\},$$

where $\tau x = (\frac{x-a}{b-a})(\frac{b-x}{b-a})$.

Theorem 3.11. *Let f be a generalized log h-convex function in the second sense. Then*

$$\begin{aligned}\log f\big(\frac{a+b}{2}\big)^{\frac{1}{h^2(\frac{1}{2})}} \quad - \quad & \frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x),f(x)]\mathrm{d}x \\ & \leq \frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x \\ & \leq \log\Big\{[f(a)][f(a)+\eta(f(b),f(a)]\Big\}\int_0^1 h(t)h(1-t)\mathrm{d}t.\end{aligned}$$

Proof. Let f be generalized log h-convex function in the second sense. Then

$$\log f((1-t)a+tb) \leq h(t)h(1-t)\big\{\log[f(a)]+\log[f(a)+\eta(f(b),f(a)]\big\}. \tag{7}$$

Integrating (7) with respect to t on [0,1], we have

$$\begin{aligned} & \int_0^1 \log f((1-t)a+tb)\mathrm{d}t \\ \leq & \int_0^1 h(t)h(1-t)\big[\log[f(a)]+\log[f(a)+\eta(f(b),f(a)]\big]\mathrm{d}t \\ = & \log\big\{[f(a)][f(a)+\eta(f(b),f(a)]\big\}\int_0^1 h(t)h(1-t)\mathrm{d}t.\end{aligned}$$

Thus

$$\frac{1}{b-a}\int_a^b \log f(x)\mathrm{d}x \leq \log\big\{[f(a)][f(a)+\eta(f(b),f(a)]\big\}\int_0^1 h(t)h(1-t)\mathrm{d}t. \tag{8}$$

Consider

$$\begin{aligned} & f(\frac{a+b}{2}) = \frac{((1-t)a+tb)+f(ta+(1-t)b)}{2} \\ \leq & \Big\{[f(1-t)a+tb)][f((1-t)a+tb)+\eta(f(1-t)b+ta),(1-t)a+tb))]\Big\}^{h^2(\frac{1}{2})}.\end{aligned}$$

This implies

$$\begin{aligned} & \log f(\frac{a+b}{2}) \\ \leq & h^2(\frac{1}{2})\log\Big\{[f(1-t)a+tb)][f((1-t)a+tb)+\eta(f(1-t)b+ta),(1-t)a+tb))]\Big\}.\end{aligned} \tag{9}$$

Integrating (9) with respect to t on [0,1], we have

$$\frac{1}{h^2(\frac{1}{2})}\log f(\frac{a+b}{2}) \quad \leq \quad \frac{1}{(b-a)}\int_a^b \big\{\log f(x)+\log[f(x)+\eta(f(a+b-x),f(x))]\big\}\mathrm{d}x.$$

Thus

$$\begin{aligned}\frac{1}{h^2(\frac{1}{2})}\log f(\frac{a+b}{2}) \quad &- \quad \frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x),f(x))]\mathrm{d}x\\ &\leq \frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x.\end{aligned} \tag{10}$$

Combining(8) and (10), we have

$$\begin{aligned}\log f\big(\frac{a+b}{2}\big)^{\frac{1}{h^2(\frac{1}{2})}} \quad &- \quad \frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x),f(x)]\mathrm{d}x\\ &\leq \frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x\\ &\leq \log\Big\{[f(a)][f(a)+\eta(f(b),f(a)]\Big\}\int_0^1 h(t)h(1-t)\mathrm{d}t.\end{aligned} \tag{11}$$

This completes the proof. □

Corollary 3.12. *If $\eta(f(b),f(a))=f(b)-f(a)$, then, under the assumption of Theorem 3.11, we have*

$$f\big(\frac{a+b}{2}\big)^{\frac{1}{2h^2(\frac{1}{2})}} \leq \exp\frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x \leq \big[f(a)f(b)\big]\int_0^1 h(t)h(1-t)\mathrm{d}t.$$

Now we will discuss some special cases Theorem 3.11.

I. If $\eta(f(b),f(a))=f(b)-f(a)$ and $h(t)=t$, then Theorem 3.11 becomes

Theorem 3.13. *Let f be generalized log tgs-convex function in the second sense on I. Then*

$$f\big(\frac{a+b}{2}\big)^2 \leq \exp\frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x \leq \big[[f(a)][f(b)]\big]^{\frac{1}{6}}.$$

II. If $\eta(f(b),f(a))=f(b)-f(a)$ and $h(t)=t^P$, then Theorem 3.11 becomes

Theorem 3.14. *Let f be generalized log Toader-convex function in the second sense on I. Then*

$$f\big(\frac{a+b}{2}\big)^{2^{p-1}} \leq \exp\frac{1}{(b-a)}\int_a^b \log f(x)\mathrm{d}x \leq \big\{[f(a)][f(b)]\beta(p+1,p+1)\big\}.$$

4. Conclusion

In this paper, we have derived several Hermite-Hadamard type inequalities for new classes of generalized convex functions involving an arbitrary non-negative function. It is shown that these classes of generalized convex functions are quite flexible and unifying ones. For different and appropriate choice of the arbitrary functions, one can obtain new and known classes of convex functions. The interested readers are expected to explore the applications of the generalized convex functions.

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References

1. G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl **335** (2007), 1294-1308.
2. W.W. Breckner, *Stetigkeitsaussagen fiir eine Klasse verallgemeinerter convexer funktionen in topologischen linearen Raumen*, Pupl. Inst. Math **23**(1978), 13-20.
3. G. Cristescu and L. Lupsa, *Non-connected Convexities and Applications, Kluwer Academic Publishers*, Dordrechet, Holland, 2002.
4. M.R. Delavar and F. Sajadian, *Hermite-Hadamard type integral inequalities for log-η-convex function*, Math. Computer. Sci **1**(4)(2016), 86-92.
5. M.R. Delavar and S.S. Dragomir, *On η-convexity*, Math. Inequal. Appl **20**(1)(2017), 203-216.
6. S.S. Dragomir and B. Mond, *Integral inequalities of Hadamard type for log-convex functions*, Demonstratio **31**(1998), 354-364.
7. S.S. Dragomir and C.E.M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, Victoria University, Australia, 2000.
8. S.S. Dragomir, J. Pecaric and L.E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math **21** (1995), 335-341.
9. M.E. Gordji, M.R. Delavar and M.D. Sen, *On φ convex functions*, J. Math. Inequal **10**(1)(2016), 173-183.
10. M.E. Gordji, M.R. Delavar and S.S. Dragomir, *An inequality related to η-convex functions* (II), Int. J. Nonlinear. Anal. Appl **6(2)** (2015), 27-33.
11. E.K. Godunova and V.I. Levin, *Neravenstva dlja funkcii sirokogo klassa soderzascego vypuklye monotonnye i nekotorye drugie vidy funkii*, Vycislitel. Mat. i.Fiz. Mezvuzov. Sb. Nauc. MGPI Moskva, (1985), 138-142.
12. J. Hadamard, *Etude sur les proprietes des fonctions entieres e.t en particulier dune fonction consideree par Riemann*, J. Math. Pure. Appl **58** (1893), 171-215.
13. C. Hermite, *Sur deux limites d'une integrale definie*, Mathesis **3** (1883).
14. D.H. Hyers and S.M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc **3** (1952), 821-828.
15. M.A. Noor, K.I. Noor and F. Safdar, *Generalized geometrically convex functions and inequalities*, J. Inequal. Appl **(**2017)(2017):22.
16. M.A. Noor, K.I. Noor and F. Safdar, *Integral inequaities via generalized convex functions*, J. Math. Computer, Sci **17** (2017), 465-476.
17. M.A. Noor, F. Qi and M.U. Awan. *Hermite-Hadamard type inequalities for* log-h-*convex functions*, Analysis **33** (2013), 367-375.

18. M.A. Noor, K.I. Noor, F. Safdar, M.U. Awan and S. Ullah, *Inequaities via generalized* $\log m$*-convex functions*, J. Nonlinear. Sci. Appl (2017), 5789-5802.
19. M.A. Noor, K.I. Noor, M.U. Awan and F. Safdar, *On strongly generalized convex functions*, Filomat **31**(18) (2017), 5783-5790.
20. M.A. Noor, K.I. Noor, M.U. Awan, *Some characterizations of harmonically log-convex functions*, Proc. Jangjeon. Math. Soc **17**(2014), 51-61.
21. M.A. Noor, K.I. Noor, S. Iftikhar, F. Safdar, *Integral inequaities for relative harmonic* (s,η)*-convex functions*, Appl. Math. Computer. Sci **1**(1) (2015), 27-34.
22. M.A. Noor, K.I. Noor and F. Safdar, *Integral inequaities via generalized* (α,m)*-convex functions,* J. Nonlin . Func. Anal **2017**(2017), Article ID: 32.
23. J.E. Pecaric, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
24. M.Z. Sarikaya, A. Saglam and H. Yildirim, *On some Hadamard-type inequalities for h-convex functions*, Jour. Math. Ineq **2**(3)(2008), 335-341.
25. S. Varosanec, *On h-convexity*, J. Math. Anal. Appl (2007), 303-311.
26. B.Y. Xi and F. Qi, *Some inequalities of Hermite-Hadamard type for h-convex functions*, Adv. Inequal. Appl **2**(1) (2013), 1-15.

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