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OSCILLATION OF SECOND ORDER SUBLINEAR NEUTRAL DELAY DYNAMIC EQUATIONS VIA RICCATI TRANSFORMATION†

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Abstract. In this work, we establish oscillation of the second order sublinear neutral delay dynamic equations of the form:

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\gamma}(\alpha(t))+v(t)x^{\gamma}(\eta(t))=0$$

on a time scale $\mathcal{T}$ by means of Riccati transformation technique, under the assumptions

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t = \infty,$$

and

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t < \infty,$$

for various ranges of $p(t)$, where $0<\gamma\leq 1$ is a quotient of odd positive integers.

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1. Introduction

The theory of time scales which has recently introduced by Stefan Hilger [8] in, 1988 in his Ph.D thesis and in order to unify continuous and discrete analysis and for the last decades it is fast going and simultaneously extending to the other areas of research. Many researchers have contributed on different aspects of this new theory; see the survey paper by Agarwal et al. [1] and the references cited therein.

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A time scale $\mathcal{T}$ is an arbitrary closed subset of the real and the cases when the time scale is equal to the reals or to the integers represented the classical theories of differential and difference equations. Apart from that, there is a time scale followed by 'quantum calculus' which has applied in quantum mechanics. In the last few year, there has been increasing interest in obtaining sufficient conditions for the oscillation / nonoscillation of solutions of different classes of dynamic equations for time scales and we refer the reader to papers ([1], [4], [7],) and the references cited therein.

The objective of this work is to study the the behavior of the solution of the second order sublinear neutral delay dynamic equations of the form:

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\gamma}(\alpha(t))+v(t)x^{\gamma}(\eta(t))=0 \tag{1}$$

on an arbitary time scale $\mathcal{T}$, under the assumption

(A_0) $\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t=\infty$,

(A_1) $\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t<\infty$,

where $0<\gamma\leq 1$ is a quotient of odd positive integers, $r(t)$ is a positive Δ differentiable function defined on $[0,\infty)_{\mathcal{T}}$, $q,v\rightarrow[0,\infty)$ and $p,q,v\rightarrow\mathcal{T}$ are rd-continuous functions and $\tau,\sigma,\eta:\rightarrow\mathcal{T}$ are positive rd-continuous functions such that $\lim_{t\rightarrow\infty}\tau(t)=\infty=\lim\limits_{t\rightarrow\infty}\alpha(t)=\infty=\lim_{t\rightarrow\infty}\eta(t)$. Since the interest is in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup\mathcal{T}=\infty$ and we define the time scale integral as $[t_0,\infty)_{\mathcal{T}}=[t_0,\infty)\bigcap\mathcal{T}$.

In [11], The authors have established the oscillation criteria for

$$(r(t)((x(t)+p(t)x(\alpha(t)))^{\Delta})^{\gamma})^{\Delta}+f(t,x(\beta(t)))=0, \tag{2}$$

by using general Riccati substitution when $0\leq p(t)<1$, $\gamma>1$ is a quotient of odd positive integers, $|f(t,x)|\geq|x|^{\gamma}$ and (A_0) hold. But, the problems are still left for other ranges of $p(t)$ as well as (A_1) when $0<\gamma<1$

Agarwal et al. [1] have considered the second order delay dynamic equations on time scale

$$x^{\Delta\Delta}(t)+p(t)x(\tau(t))=0, \tag{3}$$

and established some sufficient conditions for oscillation of (2).
Erbe et al. [7] considered the half linear delay dynamic equations on time scale

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+p(t)x^{\gamma}(\tau(t))=0, \tag{4}$$

where $\gamma>1$ is the quotient of odd positive integers.

Agarwal et al. [2] have discussed the oscillatory behavior of solution of the delay dynamic equation of the form:

$$(r(t)((x(t)+p(t)x(t-\alpha)^{\Delta})^{\gamma})^{\Delta}+f(t,x(t-\delta))=0, \tag{5}$$

where $0\leq p(t)<1$ for all $t\in[t_0,\infty)_{\mathcal{T}}$. Satisfying the condition $|f(t,x)|\geq q(t)|x|^{\gamma}$. However, the study of (5) by means of Riccati transformation is still left for ranges of $p(t)$ and $0 < \gamma < 1$. It is interesting to see the work [13] and [14] in comparison with [2] for the second order neutral delay dynamic equations

$$(r(t)((x(t)+p(t)x(t-\tau))^{\Delta})^{\gamma})^{\Delta}+q(t)|x(t-\delta)|^{\gamma} sgnx(t-\delta)=0,$$

where $\gamma \geq 1$ is a ratio of odd positive integers.

The motivation of the present work has come under two ways. First is due to the work in [13] and [14] and second is due to the work in [7], where Erbe et al. have discussed the sublinear oscillation of the second order delay dynamic equations

$$(r(t)(x'(t))^{\gamma})^{\Delta}+q(t)x^{\gamma}(\tau(t))=0,$$

Definition 1.1. By a solution of (1), we mean a nontrivial real valued function $x \in C'_{rd}[T_x,\infty)$, $T_x \geq t_0$ which has the property $(x(t)+p(t)x(\tau(t))) \in C'_{rd}[T_x,\infty)$, $r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma} \in C'_{rd}[T_x,\infty)$ and satisfies (1) for $T_x \geq t_0$. The solutions vanishing in some neighbourhood of infinite will be excluded from our discussion. A solution $x(t)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory.

2. Oscillation Results with (A_0)

This section deals with the oscillation results for (1)by means of Riccati transformation technique, under the assumption (A_0). Throughout our discussion, we use the following notation

$$z(t)=x(t)+p(t)x(\tau(t)). \tag{1}$$

Lemma 2.1. [2] *Assume that (A_0) holds and $r(t) \in C'_{rd}[a,\infty),\mathcal{R}$ such that $r^{\Delta}(t)>0$. Let $x(t)$ be an eventually positive real valued function such that $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} \leq 0$ for $t \geq t_1 \geq a$. Then $x^{\Delta}(t)>0$ and $x^{\Delta\Delta}(t)<0$ for $t \geq t_1 > t_0$, where $0<\gamma \leq 1$ is a quotient of odd positive integers.*

Lemma 2.2. [7] *Assume that the assumptions of Lemma 2.1 hold. Then there exists a $t^* \in [t_0,\infty)_{\mathcal{T}}$ sufficiently large so that*
(i) $x(t) > tx^{\Delta}(t)$ for $t \in [t^*,\infty)_{\mathcal{T}}$
(ii) $\frac{x(t)}{t}$ is strictly decreasiong on $[t^*,\infty)_{\mathcal{T}}$,
where $0<\gamma \leq 1$ is a quotient of odd positive integers.

Theorem 2.3. *Let $0 \leq p(t) \leq a < \infty$, $\tau(\alpha(t))=\alpha(\tau(t))$ and $\tau(\eta(t))=\eta(\tau(t))$ be hold for $t \in [t_0,\infty)$. Assume that (A_0) holds, and $r^{\Delta}(t)>0$, $\tau^{\Delta}(t) \geq 1$ for large t and*
(A_1) there exists $\lambda>0$ such that $v^{\gamma}(x)+v^{\gamma}(y) \geq \lambda v^{\gamma}(x+y)$
and
there exists $\rho>0$ such that $v^{\gamma}(x)+v^{\gamma}(y) \geq \rho v^{\gamma}(x+y), x,y \in \mathcal{R}$ hold, where $v \in C'_{rd}[T_v,\infty), T_v \geq t_0$ is nontrival real valued function .
Furthermore, assume that there exists a positive Δ -differentiable function $\delta(t)$ such that
(A_2) $\int_{t_0}^{\infty}\left[\delta(s)Q(s)\left(\frac{\alpha(s)}{\beta(s)}\right)^{\gamma}+\delta(s)V(s)\left(\frac{\eta(s)}{\beta(s)}\right)^{\gamma}-\frac{(r(s)+a^{\gamma})r(\tau(s)))((\delta^{\Delta}(s))_+)^{\gamma+1}}{\lambda\rho(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s$

$= \infty$,
where $Q(t) = min\{q(t), q(\tau(t))\}, V(t) = min\{v(t), v(\tau(t))\}$ *and* $(\delta^{\Delta}(t))_{+} = \max\{\delta^{\Delta}(t), 0\}$. *Then every solution of (1) oscillates.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_{0\mathcal{T}}$. Hence, there exists $t_1 > t_0$ such that $x(t) > 0,\ x(\tau(t)) > 0,\ x(\alpha(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. Using (1), (1) becomes

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} = -q(t)x^{\gamma}(\alpha(t)) - v(t)x^{\gamma}(\eta(t)) \leq 0,\ \not\equiv 0\ for\ t \geq t_1. \tag{2}$$

So, $r(t)(z^{\Delta}(t))^{\gamma}$ is nonincreasing on $[t_1, \infty)$, that is, either $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$ for $t \geq t_2 > t_1$. By Lemma 2.1, it follows that $z^{\Delta}(t) > 0$ for $t \geq t_2$. From(1), it is easy to see that

$$\begin{aligned}&(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + q(t)x^{\gamma}(\alpha(t)) + v(t)x^{\gamma}(\eta(t)) + \\ &a^{\gamma}(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta} + a^{\gamma}q(\tau(t))x^{\gamma}(\alpha(\tau(t)) + a^{\gamma}v(\tau(t))x^{\gamma}(\eta(\tau(t)) = 0\end{aligned} \tag{3}$$

for $t \geq t_2$. Using (A_1), (3) yields that

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + a^{\gamma}(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta} + Q(t)z^{\gamma}(\alpha(t)) + V(t)z^{\gamma}(\eta(t)) \leq 0 \tag{4}$$

and therefore,

$$\frac{(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(t))} + \frac{a^{\gamma}(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(t))} + \frac{Q(t)z^{\gamma}(\alpha(t))}{z^{\gamma}(\beta(t))} + \frac{V(t)z^{\gamma}(\eta(t))}{z^{\gamma}(\beta(t))} \leq 0. \tag{5}$$

Let $\delta(t)$ be the positive Δ-differentiable function and consider the general Riccati substitution

$$w(t) = \delta(t)r(t)\left(\frac{z^{\Delta}(t)}{z(t)}\right)^{\gamma} \tag{6}$$

and

$$v(t) = \delta(t)r(\tau(t))\left(\frac{z^{\Delta}(\tau(t))}{z(\tau(t))}\right)^{\gamma}. \tag{7}$$

Due to Lemma2.1, $w(t) > 0$ and $v(t) > 0$ on $[t_2, \infty)_{\mathcal{T}}$. Now,

$$w^{\Delta}(t) = \delta^{\Delta}\left(r\left(\frac{z^{\Delta}}{z}\right)^{\gamma}\right)^{\beta} + \delta\left(r\left(\frac{z^{\Delta}}{z}\right)^{\gamma}\right)^{\Delta} \tag{8}$$

and

$$v^{\Delta}(t) = \delta^{\Delta}\left(r^{\tau}\left(\frac{z^{\tau\Delta}}{z^{\tau}}\right)^{\gamma}\right)^{\beta} + \delta\left(r^{\tau}\left(\frac{z^{\tau\Delta}}{z^{\tau}}\right)^{\gamma}\right)^{\Delta}, \tag{9}$$

where we use the notation $f^{\beta} = f(\beta(t))$. Using (6) and (7)in (8) and (9), we get

$$\begin{aligned}w^{\Delta}(t) + a^{\gamma}v^{\Delta}(t) &= \frac{\delta^{\Delta}}{\delta^{\beta}}\left[w^{\beta} + a^{\gamma}v^{\beta}\right] \\ &\quad + \delta\left[\left(r\left(\frac{z^{\Delta}}{z}\right)^{\gamma}\right)^{\Delta} + a^{\gamma}\left(r^{\tau}\left(\frac{z^{\tau\Delta}}{z^{\tau}}\right)^{\gamma}\right)^{\Delta}\right].\end{aligned} \tag{10}$$

Upon using the fact

$$\left(r\left(\frac{z^{\Delta}}{z}\right)^{\gamma}\right)^{\Delta} = \frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{z^{\gamma\beta}} - \frac{r(z^{\Delta})^{\gamma}(z^{\gamma})^{\Delta}}{z^{\gamma}z^{\gamma\beta}},$$

and

$$\left(r^{\tau}\left(\frac{z^{\tau\Delta}}{z^{\tau}}\right)^{\gamma}\right)^{\Delta} = \frac{(r^{\tau}(z^{\tau\Delta})^{\gamma})^{\Delta}}{z^{\gamma\tau\beta}} - \frac{r^{\tau}(z^{\tau\Delta})^{\gamma}(z^{\gamma\tau})}{z^{\gamma\tau}z^{\gamma\tau\beta}},$$

where $\tau(t) \leq t, \tau(\beta(t)) \leq t$ and $z(t)$ is nondecreasing on $[t_2, \infty)_{\mathcal{T}}$ and $z^{\tau\beta} \leq z^{\beta}$ in (10) and then applying (5), we obtain

$$\begin{aligned} w^{\Delta}(t) + a^{\gamma}v^{\Delta}(t) \leq & \frac{\delta^{\Delta}}{\delta^{\beta}}[w^{\beta} + a^{\gamma}v^{\beta}] - \lambda Q(t)\delta\frac{z^{\gamma\alpha}}{z^{\gamma\beta}} - \rho V(t)\delta\frac{z^{\gamma\eta}}{z^{\gamma\beta}} \\ & - \delta\left[\frac{r(z^{\Delta})^{\gamma}(z^{\gamma})^{\Delta}}{z^{\gamma}z^{\gamma\beta}} + a^{\gamma}\frac{r^{\tau}(z^{\tau\Delta})^{\gamma}(z^{\gamma\tau})^{\Delta}}{z^{\gamma\tau}z^{\gamma\tau\beta}}\right]. \end{aligned} \tag{11}$$

By Lemma 2.2, let there exist $t_3 > t_2$ such that $z(t) > tz^{\Delta}(t)$ and $\frac{z(t)}{t}$ is decreasing on $[t_3, \infty)$ and hence

$$\frac{z(\alpha(t))}{z(\beta(t))} \geq \frac{\alpha(t)}{\beta(t)},\ t \in [t_3, \infty)_{\mathcal{T}}. \tag{12}$$

By the potzsche chain rule [4], we find that

$$\begin{aligned} (z^{\gamma})^{\Delta}(t) &= \gamma\int_0^1 [(1-h)z(t) + hz(\beta(t))]^{\gamma-1} dh z^{\Delta}(t) \\ &\geq \gamma z^{\Delta}(t)\int_0^1 [z(\beta(t))]^{\gamma-1} dh(\ since\ \gamma - 1 \leq 0, \beta(t) \geq t) \end{aligned}$$

that is,

$$(z^{\gamma})^{\Delta}(t) = \gamma z^{\Delta}(z^{\beta})^{\gamma-1} \tag{13}$$

and similarly

$$(z^{\gamma})^{\Delta}(\tau(t)) \geq \gamma z^{\Delta}\tau(t)(z^{\tau\beta})^{\gamma-1}. \tag{14}$$

Using (12), (13), and (14) in (11) we obtain

$$\begin{aligned} w^{\Delta}(t) + a^{\gamma}v^{\Delta}(t) \leq & \frac{\delta^{\Delta}}{\delta^{\beta}}[w^{\beta} + a^{\gamma}v^{\beta}] - \lambda Q\delta\left(\frac{\alpha}{\beta}\right)^{\gamma} - \rho V(t)\delta\left(\frac{\eta}{\beta}\right)^{\gamma} \\ & - \gamma\delta\left[\frac{r(z^{\Delta})^{\gamma}z^{\Delta}}{z^{\gamma}z^{\beta}} + a^{\gamma}\frac{r^{\tau}(z^{\tau\Delta})^{\gamma}z^{\tau\Delta}}{z^{\gamma\tau}z^{\tau\beta}}\right] \\ \leq & \frac{\delta^{\Delta}}{\delta^{\beta}}[w^{\beta} + a^{\gamma}v^{\beta}] - \lambda Q\delta\left(\frac{\alpha}{\beta}\right)^{\gamma} - \rho V\delta\left(\frac{\eta}{\beta}\right)^{\gamma} \\ & - \gamma\delta\left[\frac{r^{\beta}(z^{\beta\Delta})^{\gamma}z^{\Delta}}{z^{\gamma}z^{\beta}} + a^{\gamma}\frac{r^{\tau\beta}(z^{\tau\beta\Delta})^{\gamma}z^{\tau\Delta}}{z^{\gamma\tau}z^{\tau\beta}}\right]. \end{aligned} \tag{15}$$

[Since $r(z^{\Delta})^{\gamma}) \geq r^{\beta}(z^{\beta\Delta})^{\gamma}]$. Since

$$z(\beta(t)) \geq z(t)$$

and

$$z^{\Delta}(t) \geq (r(\beta(t)))^{\frac{1}{\gamma}} z^{\Delta}(\beta(t))/r^{\frac{1}{\gamma(t)}},$$

then the last inequality becomes

$$\begin{aligned}
w^{\Delta}(t) + a^{\gamma} v^{\Delta}(t) &\leq \frac{\delta^{\Delta}}{\delta^{\beta}}[w^{\beta} + a^{\gamma} v^{\beta}] - \lambda Q \delta \left(\frac{\alpha}{\beta}\right)^{\gamma} - \rho V \delta \left(\frac{\eta}{\beta}\right)^{\gamma} \\
&\quad - \gamma\delta \left[\frac{r^{(1+\frac{1}{\gamma})^{\beta}}}{r^{\frac{1}{\gamma}}} \left(\frac{z^{\beta\Delta}}{z^{\beta}}\right)^{\gamma+1} + a^{\gamma} \frac{r^{(1+\frac{1}{\gamma})^{\tau}\beta}}{r^{\frac{1}{\gamma}}\tau} \left(\frac{z^{\tau\beta\Delta})}{z^{\tau\beta}}\right)^{\gamma+1}\right] \\
&\leq \frac{((\delta^{\Delta})_{+})}{\delta^{\beta}}[w^{\beta} + a^{\gamma} v^{\beta}] - \lambda Q \delta \left(\frac{\alpha}{\beta}\right)^{\gamma} - \rho V \delta \left(\frac{\eta}{\beta}\right)^{\gamma} \\
&\quad \gamma\delta \left[\left(\frac{w^{\beta}}{\delta^{\beta}}\right)^{\theta} \frac{1}{r^{\frac{1}{\gamma}}} + \frac{a^{\gamma}}{r^{\frac{1}{\gamma}}\tau} \left(\frac{v^{\beta}}{\delta^{\beta}}\right)^{\theta}\right], \qquad (16)
\end{aligned}$$

where $\theta = \frac{(\gamma+1)}{\gamma}$. Let's define

$$A^{\theta} = \frac{\gamma\delta}{r^{\frac{1}{\gamma}}} \left(\frac{w^{\beta}}{\delta^{\beta}}\right)^{\theta} > 0, B^{\theta-1} = \frac{r^{\frac{1}{(\gamma+1)}}}{\theta(\gamma\delta)^{\frac{1}{\theta}}}((\delta^{\Delta})_{+}) \geq 0.$$

Then using the inequality

$$\theta A B^{\theta-1} - A^{\theta} \leq (\theta-1) B^{\theta}$$

we obtain that

$$\begin{aligned}
&\left(\frac{w}{\delta}\right)^{\beta}((\delta^{\Delta})_{+}) - \frac{\gamma\delta}{r^{\frac{1}{\gamma}}}\left(\frac{w^{\beta}}{\delta^{\beta}}\right)^{\theta} \\
&\qquad = \theta A B^{\alpha-1} - A^{\theta} \leq (\theta-1)B^{\theta} \leq \frac{r((\delta^{\Delta})_{+})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}}, \qquad (17)
\end{aligned}$$

and similarly

$$\left(\frac{v}{\delta}\right)^{\beta}((\delta^{\Delta})_{+}) - \frac{\gamma\delta}{r^{\frac{\tau}{\gamma}}}\left(\frac{v^{\beta}}{\delta^{\beta}}\right)^{\theta} \leq \frac{r^{\tau}((\delta^{\Delta})_{+})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}}. \qquad (18)$$

Using (17) and (18 in (16), we find

$$\begin{aligned}
w^{\Delta} + a^{\gamma} v^{\Delta} &\leq \frac{r((\delta^{\Delta})_{+})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} \\
&\quad + a^{\gamma} \frac{r((\delta^{\Delta})_{+})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} - \lambda\delta Q \left(\frac{\alpha}{\beta}\right)^{\gamma} - \rho\delta V \left(\frac{\eta}{\beta}\right)^{\gamma}. \qquad (19)
\end{aligned}$$

Integrating (19) from $t_4(> t_3)$ to t, we get

$$-w(t_4) - a^{\gamma} v(t_4) \leq w(t) + a^{\gamma} v(t) - w(t_4) - a^{\gamma} v(t_4)$$

$$\leq \int_{t_4}^{t} \left[\frac{(r + r^{\tau} a^{\gamma})((\delta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}} - \rho\lambda\delta\{Q\left(\frac{\alpha}{\beta}\right)^{\gamma} + V\left(\frac{\eta}{\beta}\right)^{\gamma}\}\right]\Delta s$$

which is a contradiction to (A_2). This completes the proof of the theorem. □

Theorem 2.4. *Let $0 \leq p(t) \leq a < 1,\ t \in [t_0, \infty)_{\mathcal{T}}$. Assume that (A_0) holds and $r^{\Delta}(t) > 0$. Furthermore, assume that there exists a positive Δ- differentiable function $\delta(t)$ such that*

(A_3) $\int_{t_0}^{\infty} \left[\delta(s)q(s)\left(\frac{\alpha(s)}{\beta(s)}\right)^{\gamma} + \delta(s)v(s)\left(\frac{\eta(s)}{\beta(s)}\right)^{\gamma} - \frac{r(s)((\delta^{\Delta}(s))_+)^{\gamma+1}}{(1-a)(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\Delta s\right] = \infty.$

Then every solution of (1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.3, we get (2) and by Lemma 2.1 $z(t)$ is nondecreasing on $[t_2, \infty)_{\mathcal{T}}$. Hence there exists $t_3 > t_2$ such that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &= x(t) + p(t)x(\tau(t)) - p(t)x(\tau(t)) \\ &- p(t)p(\tau(t))p(\tau(\tau(t))) \\ &= x(t) - p(t)p(\tau(t))p(\tau(\tau(t))) \\ &\leq x(t), \end{aligned}$$

that is, $x(t) \geq (1-a)z(t)$ on $[t_3, \infty)$. Consequently, (1) reduces to

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + (1-a)q(t)z^{\gamma}(\alpha(t)) + (1-a)v(t)z^{\gamma}(\eta(t)) \leq 0$$

$t \in [t_3, \infty)$.

The rest of the proof follows from the proof of Theorem 2.3 without Riccati substitution (6) and hence the details are omitted. The proof of the theorem is complete. □

Theorem 2.5. *Let $-1 < a \leq p(t) \leq 0$ for $t \in [t_0, \infty)_{\mathcal{T}}$. Assume that (A_0) holds and $r^{\Delta}(t) > 0$. Furthermore, assume that there exists a positive Δ -differentiable function $\delta(t)$ such that*

(A_4) $\int_{t_0}^{\infty} \left[\delta(s)q(s)\left(\frac{\alpha(s)}{\beta(s)}\right)^{\gamma} + \delta(s)v(s)\left(\frac{\eta(s)}{\beta(s)}\right)^{\gamma} - \frac{r(s)((\delta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\Delta s\right] = \infty$

and

(A_5) $\lim_{t\to\infty} sup \int_{t_0}^{t} [q(\tau(s)) + v(\tau(s))]\,\Delta s = \infty$

(A_6) $\int_{t_0}^{\infty} \left[\frac{1}{r(\theta)}\int_{t_0}^{\theta}[q(s) + v(s)]\Delta s\right]^{\frac{1}{\gamma}} \Delta\theta = \infty$

hold. Then every solution of (1) either oscillates or converges to zero as $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 2.3 we get (2) for $t \in [t_2, \infty)\mathcal{T}$. Thus $z(t)$ and $z^{\Delta}(t)$ are monotonic functions on $[t_2, \infty)_{\mathcal{T}}$. In what follows, we consider the following four possible cases:

$$\begin{aligned} (i)\ \ z(t) > 0,\ z^{\Delta}(t) > 0,\ (ii)\ \ z(t) < 0,\ z^{\Delta}(t) > 0, \\ (iii)\ \ z(t) > 0,\ z^{\Delta}(t) < 0,\ (iv)\ \ z(t) < 0,\ z^{\Delta}(t) < 0. \end{aligned}$$

Case(i) In this case, $z(t) \le x(t)$ and $\lim_{t\to\infty} r(t)z^{\Delta}(t)$ exists. Therefore, (1) reduces to

$$\frac{(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(t))} + \frac{q(t)z^{\gamma}(\alpha(t))}{z^{\gamma}(\beta(t))} + \frac{v(t)z^{\gamma}(\eta(t))}{z^{\gamma}(\beta(t))} \le 0 \tag{20}$$

for $t \ge t_3 > t_2$. Upon using the positive Δ- differentiable function $\delta(t)$, we consider the general Riccati substitution (6) and hence

$$\begin{aligned} w^{\Delta}(t) &= \frac{\delta^{\Delta}}{\delta^{\beta}}\left[w^{\beta}\right] + \delta\left(r\left(\frac{z^{\Delta}}{z}\right)^{\gamma}\right)^{\Delta} \\ &\le \frac{\delta^{\Delta}}{\delta^{\beta}}[w^{\beta}] + \delta\left[\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{z^{\gamma\beta}} - \frac{r(z^{\Delta})^{\gamma}(z^{\gamma})^{\Delta}}{z^{\gamma}z^{\gamma\beta}}\right]. \end{aligned} \tag{21}$$

Due to (20), (21) becomes

$$\begin{aligned} w^{\Delta}(t) \le{}& \frac{\delta^{\Delta}}{\delta^{\beta}}[w^{\beta}] \\ &- \delta\left[\frac{q(t)z^{\gamma}(\alpha(t))}{z^{\gamma\beta}} + \frac{v(t)z^{\gamma}(\eta(t))}{z^{\gamma\beta}} + \frac{r(z^{\Delta})^{\gamma}(z^{\gamma})^{\Delta}}{z^{\gamma}z^{\gamma\beta}}\right]. \end{aligned}$$

Using the same type of argument as in the proof of Theorem 2.3, the last inequality yields

$$w^{\Delta}(t) \le \frac{\delta^{\Delta}}{\delta^{\beta}}[w^{\beta}] - \gamma\delta\left[q(t)\left(\frac{\alpha}{\beta}\right)^{\gamma} + v(t)\left(\frac{\eta}{\beta}\right)^{\gamma} + \left(\frac{w^{\beta}}{\delta^{\beta}}\right)^{\theta}\frac{1}{r^{\frac{1}{\gamma}}}\right],$$

where $\theta = \frac{(\gamma+1)}{\gamma}$. The rest of this case is similar to Theorem 2.3.

Case(ii) Let $\lim_{t\to\infty} z(t) = b$, $b \in (-\infty, 0]$. We assert that $b = 0$. If not, then $z(\alpha(t)) \le z(t) \le z(\beta(t)) < b < 0$, for $t \ge t_3 > t_2$. From (1), it follows that $z(t) > ax(\tau(t))$ and hence $x(\alpha(\tau(t))) > \frac{1}{a}z(\alpha(t))$, that is, $x(\alpha(\tau(t))) > \left(\frac{b}{a}\right)$. Also, $x(\eta(\tau(t)) > \left(\frac{b}{a}\right)$. From (2), we have

$$(r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma})^{\Delta} + q(\tau(t))x^{\gamma}(\alpha(\tau(t))) + v(t)x^{\gamma}(\eta(\tau(t))) = 0,$$

that is,

$$(r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma})^{\Delta} + \left(\frac{b}{a}\right)^{\gamma} \le 0,$$

for $t \ge t_3$. Consequently,

$$\left(\frac{b}{a}\right)^{\gamma}\int_{t_3}^{t} q(\tau(s)) + v(\tau(s)) \le -[r(\tau(s))(z^{\Delta}(\tau(s)))^{\gamma}]_{t_3}^{t} < r(\tau(t_3)))(z^{\Delta}(\tau(t_3)))^{\gamma},$$

a contradiction to (A_5). Thus $b = 0$. We claim that $x(t)$ is bounded. If not, there exists a sequence $\{\zeta_n\}$ such that $\zeta_n \to \infty$ as $n \to \infty$ and $x(\zeta_n) = max\{x(t) : t_3 \le t \le \alpha_n\}$. Therefore,

$$\begin{aligned} z(\zeta_n) &= x(\zeta_n) + p(\zeta_n)x(\tau(\zeta_n)) \\ &\ge x(\zeta_n) + ax(\tau(\zeta_n)) \end{aligned}$$

$$
\begin{aligned}
&\geq x(\zeta_n) + ax(\zeta_n) \\
&= (1+a)x(\zeta_n) \ (\because 1+a>0) \\
&\to +\infty \ \ as \ n \to \infty
\end{aligned}
$$

gives a contradiction. Hence,

$$
\begin{aligned}
0 \geq \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty} z(t) \\
&\geq \limsup_{t\to\infty} (x(t) + ax(\tau(t))) \\
&\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty} (ax(\tau(t))) \\
&= \limsup_{t\to\infty} x(t) + a \limsup_{t\to\infty} x(\tau(t)) \\
&= (1+a) \limsup_{t\to\infty} x(t)
\end{aligned}
$$

implies that $\limsup_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$.

Case(iii) Proceeding as in $Case(ii)$, we may show that $x(t)$ is bounded. Let $\lim_{t\to\infty} z(t) = b, b \in [0,\infty)$. We assert that $b = 0$. If not, there exist $t_3 > t_2$ and $b > 0$ such that $z(\alpha(t)) \geq z(t) > b$ and $z(\alpha(t)) \geq z(t) \geq z(\beta(t)) > b > 0$ for $t \geq t_3$. From (1) it follows that $z(t) \leq x(t)$ and hence (2) yields

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq -b^{\gamma} \left[q(t) + v(t)\right], t \geq t_3.$$

Integrating the above inequality from t_3 to t, we get

$$z^{\Delta}(t) < -b \left[\frac{1}{r(t)} \int_{t_3}^{t} [q(s) + v(s)]\Delta s\right]^{\frac{1}{\gamma}},$$

that is,

$$z(t) < z(t_3) - b \int_{t_3}^{t} \left[\frac{1}{r(\theta)} \int_{t_3}^{\theta} [q(s) + v(s)]\Delta s\right]^{\frac{1}{\gamma}} \Delta\theta < 0,$$

for large t due to (A_6). Hence $b = 0$. Using the same type of reasoning as in $Case(ii)$ we can show that $\lim_{t\to\infty} x(t) = 0$

Case(iv) We have $r(t)(z^{\Delta}(t))^{\gamma}$ is nonincreasing and $z(t) < 0$ for $t \geq t_2$. If $x(t)$ is unbounded, then by $Case(ii)$ it follows that $z(t) > 0$ for large t which is absurd. Hence, $x(t)$ is bounded. Consequently, $z(t)$ is bounded and $\lim_{t\to\infty} z(t)$ exists. Since $z(t) < 0$ and nonincreasing, then we can find $b > 0$ and a $t_3 > t_2$ such that $z(t) < b$ for $t \geq t_3$. Proceeding as in $Case(iii)$, we obtain the fact that $\lim_{t\to\infty} z(t) = -\infty$ due to (A_5). This contradiction argues against the $Case(iv)$. This completes the proof of the theorem. □

Remark 2.1. In Theorem 2.5, it is learnt that $x(t)$ is bounded when $z(t) < 0$. Also, $x(t)$ is bounded when $z(t) > 0$ in $Case(iii)$. Hence for unbounded $x(t)$,

$Cases(ii), (iii)$ and (iv) are not existing ultimately. Therefore, we have proved the following result:

Theorem 2.6. *Let $-1 < a \leq p(t) \leq 0$ for $t \in [t_0, \infty)_{\mathcal{T}}$. Assume that $r^{\Delta}(t) > 0$, (A_0) and (A_4) hold. Then every unbounded solution of (1) oscillates.*

Theorem 2.7. *Let $-\infty < a \leq p(t) \leq d < -1$, $\tau(\alpha(t)) = \alpha(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ be hold for all $t \in [t_0, \infty)_{\mathcal{T}}$. Assume that all conditions of Theorem 2.5 hold. If*
(A_7) $\int_{t_0}^{\infty} [q(\tau(s)) + v(\tau(s))]\, ds = \infty$,
then every bounded solution of (1) either oscillates or converges to zero as $t \to \infty$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1). Then proceeding as in the proof of Theorem 2.5, we have four possible cases for $t \in [t_2, \infty)_{\mathcal{T}}$. Among these cases, $Cases(i), (iii)$ and (iv) are similar. For $Case(ii)$, $\lim_{t\to\infty} z(t)$ exists. Let $\lim_{t\to\infty} z(t) = b$, $b \in (-\infty, 0]$ we claim that $b = 0$. If not, then there exist $z(\alpha(t)) \leq z(t) \leq z(\beta(t)) < b < 0$ and $t_3 > t_2$ such that for $t \geq t_3$. From (1), it follows that $z(t) > ax(\tau(t))$ and hence $x(\tau(\alpha(t)) > \frac{b}{a} z(\alpha(t))$, that is, $x(\alpha(\tau(t)) > \left(\frac{b}{a}\right)$ for $t \geq t_3$. Also, $x(\eta(\tau(t)) > \left(\frac{b}{a}\right)$ for $t \geq t_3$. Since (1) can be written as

$$(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta} + q(\tau(t))x^{\gamma}(\alpha(\tau(t)) + v(\tau(t))x^{\gamma}(\eta(\tau(t)) = 0,$$

then for $t \geq t_3$, it follows that

$$(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta} + \left(\frac{b}{a}\right)^{\gamma} q(\tau(t)) + \left(\frac{b}{a}\right)^{\gamma} v(\tau(t)) \leq 0.$$

Consequently,

$$\begin{aligned}\left(\frac{b}{a}\right)^{\gamma} \left[\int_{t_3}^{t} q(\tau(s)) + \int_{t_3}^{t} v(\tau(s))\right] \Delta s &\leq - \left[(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta}\right]_{t_3}^{t} \\ &< -r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma} < \infty \; as \; t \to \infty\end{aligned}$$

contradicts (A_7). So, our claim holds. Therefore,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty} z(t) \\ &\leq \liminf_{t\to\infty} (x(t) + dx(\tau(t))) \\ &\leq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty} (dx(\tau(t))) \\ &= \limsup_{t\to\infty} x(t) + d \limsup_{t\to\infty} x(\tau(t)) \\ &= (1+d) \limsup_{t\to\infty} x(t) \; (\because (1+d) < 0) \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$. Hence the proof of the theorem is complete. □

3. Oscillation Criteria with (A_{00})

This section deals with the sufficient conditions for oscillation of all solutions of (1) under the assumption (A_{00}).

Lemma 3.1. [11] *Assume that (A_{00}) holds. Let $u(t)$ be an eventually positive rd-continuous function on $[t_0,\infty)_{\mathcal{T}}$, $t_0 \geq 0$ such that $r(t)u^{\Delta}(t)$ is continuous and Δ differentiable function with $(r(t)u^{\Delta}(t))^{\gamma})^{\Delta} \leq 0, \not\equiv 0$ for large $t \in [t_0,\infty)_{\mathcal{T}}$, where $r(t)$ is positive and continuous function defined on $[t_0,\infty)_{\mathcal{T}}$. Then the following statements hold:*
(i) If $u^{\Delta}(t) > 0$, then there exists a constant $C > 0$ such that $u(t) > CR(t)$ for large t.
(ii) If $u^{\Delta}(t) < 0$, then $u(t) \geq -(r(t)(u^{\Delta}(t))^{\gamma})^{\frac{1}{\gamma}}R(t)$,
where $R(t) = \int_t^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s$.

Theorem 3.2. *Let $0 \leq p(t) \leq a < \infty, \tau(\eta(t)) = \eta(\tau(t))$ and $\tau(\alpha(t)) = \alpha(\tau(t))$ for $t \in [t_0,\infty)_{\mathcal{T}}$. Assume that (A_{00}) holds, and $r^{\Delta}(t) > 0$. Furthermore, assume that*
(A_8) $\int_{t_0}^{\infty}[R^{\gamma}(\alpha(t))Q(t) + R^{\gamma}(\eta(t))V(t)]\,\Delta t = \infty$
and
(A_9) $\limsup_{t\to\infty}\int_{t_0}^{\beta(t)}\big[Q(s)R^{\gamma}(s) + V(s)R^{\gamma}(s) +$
$\frac{\gamma}{\lambda\rho}\{A^{\frac{\gamma+1}{\lambda}} + a^{\gamma}B^{\frac{\gamma+1}{\gamma}}\}\frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)} - \frac{\gamma}{\lambda\rho}(1+a^{\gamma})\frac{(R(s))^{-1}}{r^{\frac{1}{\gamma}}(\beta(s))}\big]\Delta s = \infty$
hold for any constants $A < 0$, $B < 0$ and where $Q(t)$ and $V(t)$ are defined in Theorem 2.3. Then every solution of (1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.3, we obtained (2) and (4) for $t \geq t_2$. In what follows, we consider two possible cases $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$ for $t \geq t_3 > t_2$. If $z^{\Delta}(t) > 0$ for $t \geq t_3$, then $z(t) \geq CR(t)$ due to Lemma 3.1(i). Therefore, (4) implies that

$$C^{\gamma}R^{\gamma}(\alpha(t))Q(t)+C^{\gamma}R^{\gamma}(\eta(t))V(t) \leq -(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}-(a^{\gamma}r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma})^{\Delta} \tag{1}$$

for $t \geq t_3$. Integrating (1) from t_3 to t, we get

$$\begin{aligned}\int_{t_3}^{t} C^{\gamma}R^{\gamma}(\alpha(s))Q(s) + \int_{t_3}^{t} C^{\gamma}R^{\gamma}(\eta(t))V(s)ds \\ \leq -\left[(r(s)(z^{\Delta}(s))^{\gamma}) + (a^{\gamma}r(\tau(s))(z^{\Delta}(\tau(s)))^{\gamma}ds\right]_{t_3}^{t} \\ \leq r(t_3)z^{\Delta}(t_3)^{\gamma} + a^{\gamma}r(\tau(t_3))(z^{\Delta}(\tau(t_3)))^{\gamma} < \infty,\end{aligned}$$

a contradiction to (A_8). Ultimately, $z^{\Delta}(t) < 0$ for $t \geq t_2$. We consider the Riccati substitutions

$$w(t) = r(t)(z^{\Delta}(t)/z(t))^{\gamma} \tag{2}$$

and

$$v(t) = r(\tau(t))(z^{\Delta}(\tau(t))/z(\tau(t)))^{\gamma} \tag{3}$$

such that $w(t) < 0$ and $v(t) < 0$ for $t \geq t_3 > t_2$. From Lemma 3.1(ii), it is easy to verify that

$$-1 \leq w(t)R^{\gamma}(t) \leq 0 \tag{4}$$

for $t \geq t_3$. On the other hand, $w(t) \leq v(t)$ implies that

$$-1 \leq v(t)R^{\gamma}(t) \leq 0 \tag{5}$$

for $t \geq t_3$, where we have used the fact that both $z(t)$ and $r(t)(z^{\Delta}(t))^{\gamma}$ are nonincreasing functions on $[t_3, \infty)_{\mathcal{T}}$. Since

$$w^{\Delta}(t) = \frac{(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} z^{\gamma}(t) - r(t)(z^{\Delta}(t))^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma}(\sigma(t))},$$

and

$$v^{\Delta}(t) = \frac{(r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma})^{\Delta} z^{\gamma}(\tau(t)) - (r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma}(z^{\gamma}(\tau(t)))^{\Delta}}{z^{\gamma}(\tau(t))z^{\gamma}(\sigma(\tau(t)))},$$

that is,

$$w^{\Delta}(t) = \frac{(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(t))} - \frac{r(t)(z^{\Delta}(t))^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma}(\beta(t))},$$
$$v^{\Delta}(t) = \frac{(r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(\tau(t)))} - \frac{r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma}(z^{\gamma}(\tau(t)))^{\Delta}}{z^{\gamma}(\tau(t))z^{\gamma}(\beta(\tau(t)))},$$

By the potzsche chain rule [4], we find that

$$(z^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [(1-h)z(t) + hz(\beta(t))]^{\gamma-1} dh z^{\Delta}(t)$$
$$\leq \gamma z^{\Delta}(t) z^{\gamma-1}(t)$$

that is,

$$(z^{\gamma}(t))^{\Delta} \leq \gamma z^{\Delta}(t) z^{\gamma-1}(t) \tag{6}$$

and similarly

$$(z^{\gamma}(\tau(t))^{\Delta} \leq \gamma z^{\Delta}(\tau(t)) z^{\gamma-1}(\tau(t)). \tag{7}$$

Upon using (6) and (7) , it follows that

$$w^{\Delta}(t) \leq \frac{(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(t))} - \frac{\gamma r(t)(z^{\Delta}(t))^{\gamma+1}}{z(t)z^{\gamma}(\beta(t))},$$
$$v^{\Delta}(t) \leq \frac{(r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(\tau(t)))} - \frac{\gamma r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma+1}}{z(\tau(t))z^{\gamma}(\beta(\tau(t)))},$$

that is,

$$w^{\Delta}(t) \leq \frac{(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(t))} - \frac{\gamma w^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)}, \tag{8}$$

and

$$v^{\Delta}(t) \leq \frac{(r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(\tau(t)))} - \frac{\gamma v^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \tag{9}$$

Consequently,

$$\begin{aligned} w^{\Delta}(t) + a^{\gamma} v^{\Delta}(t) &\leq \frac{(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(t))} + a^{\gamma} \frac{(r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma})^{\Delta}}{z^{\gamma}((\beta(\tau(t)))} \\ &- \frac{\gamma w^{\frac{\gamma+1}{\gamma}(t)}}{r^{\frac{1}{\gamma}}(t)} - \frac{\gamma a^{\gamma} v^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \\ &\leq \frac{(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(t))} + a^{\gamma} \frac{(r(\tau(t))(z^{\Delta}(\tau(t)))^{\gamma})^{\Delta}}{z^{\gamma}(\beta(t))} \\ &- \frac{\gamma w^{\frac{\gamma+1}{\gamma}(t)}}{r^{\frac{1}{\gamma}}(t)} - \frac{\gamma a^{\gamma} v^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \end{aligned}$$

that is,

$$w^{\Delta}(t) + a^{\gamma} v^{\Delta}(t) \leq -\lambda Q(t) \frac{z^{\gamma}(\alpha(t))}{z^{\gamma}(\beta(t))} - \rho V(t) \frac{z^{\gamma}(\eta(t))}{z^{\gamma}(\beta(t))} - \frac{\gamma w^{\frac{\gamma+1}{\gamma}(t)}}{r^{\frac{1}{\gamma}}(t)} - \frac{\gamma a^{\gamma} v^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \tag{10}$$

for $\beta(t) \geq t \geq \alpha(t)$ then, $(z(\alpha(t)))/(z(\beta(t)))^{\gamma} \geq 1$ and hence the inequality (10)yields

$$w^{\Delta}(t) + a^{\gamma} v^{\Delta}(t) \leq -\lambda Q(t) - \rho V(t) - \frac{\gamma w^{\frac{\gamma+1}{\gamma}(t)}}{r^{\frac{1}{\gamma}}(t)} - \frac{\gamma a^{\gamma} v^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \tag{11}$$

for $t \geq t_3$. Therefore, we find (11) as

$$\begin{aligned} w^{\Delta}(t) R^{\gamma}(t) + a^{\gamma} v^{\Delta}(t) R^{\gamma}(t) &\leq -\lambda Q(t) R^{\gamma}(t) - \rho V(t) R^{\gamma}(t) \\ &- R^{\gamma}(t) \left[\frac{\gamma w^{\frac{\gamma+1}{\gamma}(t)}}{r^{\frac{1}{\gamma}}(t)} + \frac{\gamma a^{\gamma} v^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \right]. \end{aligned}$$

Integrating the preceding inequality from t_4 to $\beta(t)$, we obtain

$$\begin{aligned} & w(\beta(t)) R^{\gamma}(\beta(t)) - w(t_4) R^{\gamma}(t_4) - \int_{t_4}^{\beta(t)} (R^{\gamma}(s))^{\Delta} w(\beta(s)) \Delta s \\ & \quad + a^{\gamma} v(\beta(t)) R^{\gamma}(\beta(t)) - a^{\gamma} v(t_4) R^{\gamma}(t_4) - \int_{t_4}^{\beta(t)} a^{\gamma} (R^{\gamma}(s))^{\Delta} v(\beta(s)) \Delta s \\ & \leq -\lambda \int_{t_4}^{\beta(t)} Q(s) R^{\gamma}(s) - \rho \int_{t_4}^{\beta(t)} V(s) R^{\gamma}(s) - \int_{t_4}^{\beta(t)} R^{\gamma}(s) \left[\frac{\gamma w^{\frac{\gamma+1}{\gamma}(s)}}{r^{\frac{1}{\gamma}}(s)} + \frac{\gamma a^{\gamma} v^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)} \right]. \end{aligned}$$

By the potzsche chain rule we find that

$$(R^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [(1-h)R(t) + hR(\beta(t))]^{\gamma-1} dh R^{\Delta}(t)$$

$$\geq \gamma R^{\Delta}(t)[R(\beta(t))]^{\gamma-1}$$

that is, the above inequality becomes,

$$w(\beta(t))R^{\gamma}(\beta(t)) - w(t_4)R^{\gamma}(t_4) + \gamma \int_{t_4}^{\beta(t)} (R(\beta(s)))^{\gamma-1} \frac{w(\beta(s))}{r^{\frac{1}{\gamma}}(s)} \Delta s$$
$$+ a^{\gamma} v(\beta(t))R^{\gamma}(\beta(t)) - a^{\gamma} v(t_4)R^{\gamma}(t_4) + \gamma \int_{t_4}^{\beta(t)} a^{\gamma} (R(\beta(s)))^{\gamma-1} \frac{v(\beta(s))}{r^{\frac{1}{\gamma}}(s)} \Delta s$$

$$\leq -\lambda \int_{t_4}^{\beta(t)} Q(s)R^{\gamma}(s) - \rho \int_{t_4}^{\beta(t)} V(s)R^{\gamma}(s) - \int_{t_4}^{\beta(t)} R^{\gamma}(s) \left[\frac{\gamma w^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)} + \frac{\gamma a^{\gamma} v^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)} \right] \Delta s.$$

As a result,

$$\gamma \int_{t_4}^{\beta(t)} (R(\beta(s)))^{\gamma-1} \frac{w(\beta(s))}{r^{\frac{1}{\gamma}}(s)} \Delta s + \gamma \int_{t_4}^{\beta(t)} R^{\gamma}(s) \frac{w^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)} \Delta s$$
$$+ \int_{t_4}^{\beta(t)} \frac{a^{\gamma}(R(\beta(s)))^{\gamma-1} v(\beta(s))}{r^{\frac{1}{\gamma}}(s)} \Delta s + \gamma a^{\gamma} \int_{t_4}^{\beta(t)} R^{\gamma}(s) \frac{u^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)} \Delta s$$
$$\leq -w(\beta(t))R^{\gamma}(\beta(t)) + w(t_4)R^{\gamma}(t_4)$$

$$-\lambda \int_{t_4}^{\sigma(t)} Q(s)R^{\gamma}(s)\Delta s - \rho \int_{t_4}^{\beta(t)} V(s)R^{\gamma}(s)\Delta s + a^{\gamma} v(t_4)R^{\gamma}(t_4) - a^{\gamma} v(\beta(t))R^{\gamma}(t). \tag{12}$$

Upon using (4) and (5), (12) reduces to

$$-\gamma \int_{t_4}^{\sigma(t)} (R(\beta(s)))^{-1} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s + \gamma \int_{t_4}^{\beta(t)} R^{\gamma}(s) \frac{w^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)} \Delta s$$
$$- \gamma \int_{t_4}^{\sigma(t)} \frac{a^{\gamma}(R(\beta(s)))^{-1}}{r^{\frac{1}{\gamma}}(s)} \Delta s + \gamma \int_{t_4}^{\beta(t)} R^{\gamma}(s) \frac{v^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)} \Delta s$$

$$\leq -w(\beta(t))R^{\gamma}(\beta(t)) - \lambda \int_{t_4}^{\beta(t)} Q(s)R^{\gamma}(s)\Delta s - \rho \int_{t_4}^{\beta(t)} V(s)R^{\gamma}(s)\Delta s$$
$$- a^{\gamma} v(\beta(t))R^{\gamma}(\beta(t)).$$

We may note that $\gamma > 0$, $\frac{\gamma+1}{\gamma}$ makes $w^{\frac{1+\gamma}{\gamma}}(t) > 0$ and so also $v^{\frac{1+\gamma}{\gamma}}(t) > 0$. Therefore, $w^{\Delta}(t) < 0$ and $v^{\Delta}(t) < 0$ on $[t_3, \infty)_{\mathcal{T}}$. Since $z(t)$ and $(z^{\Delta}))^{\gamma}$ are nonincreasing on $[t_3, \infty)_{\mathcal{T}}$. Hence there exists a constant $A < 0$, $B < 0$ and

$t_4 > t_3$ such that $w(t) \leq A$ and $v(t \leq B$ on $[t_4,\infty)_{\mathcal{T}}$. Therefore the last inequality becomes

$$-\gamma(1+a^{\gamma})\int_{t_4}^{\beta(t)} \frac{(R(\beta(s)))^{-1}}{r^{\frac{1}{\gamma}}(s)}\Delta s + \gamma[A^{\frac{\gamma+1}{\gamma}} + a^{\gamma}B^{\frac{\gamma+1}{\gamma}}]\int_{t_4}^{\beta(t)} \frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)}\Delta s$$
$$+\lambda\int_{t_4}^{\beta(t)} Q(s)R^{\gamma}(s)\Delta s + \rho\int_{t_4}^{\beta(s)} V(s)R^{\gamma}(s)\Delta s$$
$$\leq -w(\beta(t))R^{\gamma}(\beta(t)) - a^{\gamma}v(t)R^{\gamma}(\beta(t)).$$

Therefore,

$$\int_{t_4}^{\beta(t)} \big[Q(s)R^{\gamma}(s) + V(s)R^{\gamma}(s) + \frac{\gamma}{\lambda\rho}\{A^{\frac{\gamma+1}{\gamma}} + a^{\gamma}B^{\frac{\gamma+1}{\gamma}}\}\frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)}$$
$$-\frac{\gamma}{\lambda\rho}(1+a^{\gamma})\frac{R(\beta(s))^{-1}}{r^{\frac{1}{\gamma}}(s)}\big]\Delta s$$
$$\leq \frac{1}{\lambda\rho}(1+a^{\gamma}),$$

due to (3)and (5) a contradiction to (A_9). This completes the proof of the theorem. □

Theorem 3.3. *Let $-1 < a \leq p(t) \leq 0,\ t \in [t_0,\infty)_{\mathcal{T}}$. Assume that (A_{00}) holds and $r^{\Delta}(t) > 0$ for any large t. If (A_5),*
(A_{10}) $\int_T^{\infty}[R^{\gamma}(\alpha(t))q(t) + R^{\gamma}(\eta(t))v(t)]\,\Delta t = \infty$
and
(A_{11}) $\limsup_{t\to\infty}\int_{t_0}^{\beta(t)}\left[q(s)R^{\gamma}(s) + v(s)R^{\gamma}(s) + \gamma\{A^{\frac{\gamma+1}{\gamma}}\frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)} - \frac{(R(\beta(s)))^{-1}}{r^{\frac{1}{\gamma}}(s)}\}\right]\Delta s = \infty$.
hold for any constants $A < 0$ and then every solution of (1) either oscillates or converges to zero as $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 3.2, we have (2). Thus $z(t)$ and $(r(t)z^{\Delta}(t))^{\gamma}$ are monotonic function on $[t_2,\infty)_{\mathcal{T}}$. Here, we consider the four possible cases of Theorem 2.5. It is easy to verify the cases following to Theorem 3.2 and Theorem 2.5. Hence the details are omitted. This completes the proof of the theorem. □

Theorem 3.4. *Let $-\infty < a \leq p(t) \leq d < -1,\ \tau(\alpha(t))) = \alpha(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ be hold for all $t \in [t_0,\infty)_{\mathcal{T}}$. Assume that all conditions of Theorem 3.3 hold. If (A_6) hold, then every bounded solution of (1) either oscillates or converges to zero as $t \to \infty$*

Proof. The proof of the theorem follows from the proof of Theorem 3.3 and Theorem 2.7, Hence the details are omitted. The proof of the theorem is complete. □

Theorem 3.5. *Let $-1 < a \leq p(t) \leq 0$ for $t \in [t_0,\infty)_{\mathcal{T}}$. Assume that $r^{\Delta}(t) > 0$, (A_{00}) and (A_{10}) hold. Then every unbounded solution of (1) oscillates.*

Proof. The proof of the theorem follows from Remark 2.1 and the proof of Theorems 3.3, 2.6, Hence the details are omitted. □

4. Discussion and Examples

In this work, our objective was to establish the sufficient conditions for oscillation of all solutions of (1). But, our method fails to provide the conclusion in the range $-\infty < a \le p(t) \le d < -1$. However, we could manage in Theorems 2.7 and 3.4 with bounded solution. In the literature, we don't find the discussion concerning the oscillation of neutral equations when $-\infty < p(t) \le -1$. So, it is interesting to study the oscillation property of neutral equations in this range, and at the same time it would be interesting to see *an all solution oscillatory problem.*

In our next problem, we study the oscillatory behaviour of solutions of (1) under the key assumptions (A_0) and (A_{00}) in which $\gamma \ge 1$ is a quotient of odd positive integers. We conclude this section with the following examples to illustrate our mail results:

Example 4.1. Consider

$$((t^\gamma((x(t)+(1+t^{-1})x(t-2)))^\Delta)^\gamma)^\Delta+(t+2)^\gamma x^\gamma(t-2)+(t+2)^\gamma x^\gamma(t-2)=0 \quad (1)$$

on $[2,\infty)_{\mathcal{T}}$, where $a=2$ and $t\in[\alpha,\infty)$, $\alpha>1$. If we choose $\delta(t)=1$, then all conditions of Theorem 2.3 are hold true. Hence, (1) is oscillatory.

Example 4.2. Consider

$$((e^{\frac{t}{3}}((x(t)+(1+t^{-1})x(t-1)))^\Delta)^{\frac{1}{3}})^\Delta+e^t x^{\frac{1}{3}}(t-1)+\left(e^t+\frac{1+2^{\frac{1}{3}}}{3}e^{\frac{t+1}{3}}\right)x^{\frac{1}{3}}(t-1)=0 \quad (2)$$

on $[2,\infty)_{\mathcal{T}}$, where $a=2$ and $t\in[\alpha,\infty)$, $\alpha>1$. Clearly, $Q(t)=e^{t-1}$, $V(t)=(e^{t-1}+\frac{1+2^{\frac{1}{3}}}{3}e^{\frac{t}{3}})$ and all conditions of Theorem 3.2 are hold true. Hence, (2) is oscillatory.

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