J. Appl. Math. & Informatics Vol. **36**(2018), No. 3 - 4, pp. 205 - 212
https://doi.org/10.14317/jami.2018.205

DIFFERENTIAL EQUATIONS ASSOCIATED WITH TWISTED (h, q)-TANGENT POLYNOMIALS†

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Abstract. In this paper, we study linear differential equations arising from the generating functions of twisted (h, q)-tangent polynomials. We give explicit identities for the twisted (h, q)-tangent polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.
Key words and phrases : Tangent numbers and polynomials, q-tangent numbers and polynomials, linear differential equations, higher-order tangent numbers, twisted (h, q)-tangent numbers and polynomials.

1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, and tangent numbers and polynomials(see [1, 2, 3, 4, 6, 8, 9, 10]). We first give the definitions of the twisted (h, q)-tangent numbers and polynomials. It should be mentioned that the definition of twisted (h, q)-tangent numbers $T^{(h)}_{n,\zeta,q}$ and polynomials $T^{(h)}_{n,\zeta,q}(x)$ can be found in [6]. Let r be a positive integer, and let ζ be rth root of unity. The twisted (h, q)-tangent numbers $T^{(h)}_{n,\zeta,q}$ and polynomials $T^{(h)}_{n,\zeta,q}(x)$ are defined by means of the generating functions:

$$\frac{2}{\zeta q^h e^{2t} + 1} = \sum_{n=0}^{\infty} T^{(h)}_{n,\zeta,q} \frac{t^n}{n!},$$

$$\left(\frac{2}{\zeta q^h e^{2t} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} T^{(h)}_{n,\zeta,q}(x) \frac{t^n}{n!}. \tag{1.1}$$

Received April 20, 2017. Revised January 15, 2018. Accepted March 6, 2018.
†This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

For $k \in \mathbb{N}$, the twisted (h,q)-tangent polynomials of higher order, $T^{(h,k)}_{n,\zeta,q}(x)$ are defined by means of the following generating function

$$\left(\frac{2}{\zeta q^h e^{2t}+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} T^{(h,k)}_{n,\zeta,q}(x)\frac{t^n}{n!}. \tag{1.2}$$

The twisted (h,q)-tangent numbers of higher order, $T^{(h,k)}_{n,\zeta,q}$ are defined by the following generating function

$$\left(\frac{2}{\zeta q^h e^{2t}+1}\right)^k = \sum_{n=0}^{\infty} T^{(h,k)}_{n,\zeta,q}\frac{t^n}{n!}. \tag{1.3}$$

When $k=1$, above (1.2) and (1.3) will become the corresponding definitions of the twisted (h,q)-tangent polynomials $T^{(h)}_{n,\zeta,q}(x)$ and the twisted (h,q)-tangent numbers $T^{(h)}_{n,\zeta,q}$.

Differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials(see [3, 7, 11]). In this paper, we study linear differential equations arising from the generating functions of twisted (h,q)-tangent polynomials. We give explicit identities for the twisted (h,q)-tangent polynomials.

2. Differential equations associated with twisted (h,q)-tangent polynomials

In this section, we study linear differential equations arising from the generating functions of twisted (h,q)-tangent polynomials. Let

$$\begin{aligned} H &= H(t,\zeta,q,h) = \frac{2}{\zeta q^h e^{2t}+1}, \\ F &= F(t,\zeta,q,h,x) = \left(\frac{2}{\zeta q^h e^{2t}+1}\right) e^{xt}. \end{aligned} \tag{2.1}$$

Then, by (2.1), we get

$$H^{(1)} = \frac{d}{dt}H(t,\zeta,q,h) = \frac{d}{dt}\left(\frac{2}{\zeta q^h e^{2t}+1}\right) = -\zeta q^h \left(\frac{2}{\zeta q^h e^{2t}+1}\right)^2 e^{2t}.$$

Hence we have

$$H^{(1)} = -\zeta q^h H^2 e^{2t}.$$

By (2.1), we obtain

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t,\zeta,q,h,x) = \frac{d}{dt}\left(\frac{2}{\zeta q^h e^{2t}+1}\right) e^{xt} \\ &= -\zeta q^h \left(\frac{2}{\zeta q^h e^{2t}+1}\right)^2 e^{(x+2)t} + x\left(\frac{2}{\zeta q^h e^{2t}+1}\right) e^{xt} \\ &= \left(-\zeta q^h H e^{2t} + x\right) F(t,\zeta,q,h,x), \end{aligned} \tag{2.2}$$

$$\begin{aligned}
F^{(2)} &= \left(\frac{d}{dt}\right)^2 F(t,\zeta,q,h,x) \\
&= \left(-1\zeta q^h H^{(1)} e^{2t} - 2x\zeta q^h H e^{2t}\right) F + \left(-\zeta q^h H e^{2t} + x\right) F^{(1)} \\
&= (-1)^2 2\zeta^2 q^{2h} H^2 e^{4t} F + (-1) 2\zeta q^h x H e^{2t} F + (-1) 2\zeta q^h H e^{2t} F + x^2 F, \\
&= \left((-1)^2 2\zeta^2 q^{2h} H^2 e^{4t} + (-1)(2\zeta q^h x + 2\zeta q^h) H e^{2t} + x^2\right) F(t,\zeta,q,h,x)
\end{aligned}$$

and

$$\begin{aligned}
F^{(3)} &= \left(\frac{d}{dt}\right)^3 F(t,\zeta,q,h,x) \\
&= (-1)^2 4\zeta^2 q^{2h} H H^{(1)} e^{4t} F + (-1)^2 8\zeta^2 q^{2h} H^2 e^{4t} F \\
&\quad + (-1)^2 2\zeta^2 q^{2h} H^2 e^{4t} F^{(1)} \\
&\quad + (-2)(\zeta q^h x + \zeta q^h) H^{(1)} e^{2t} F + (-4)(\zeta q^h x + \zeta q^h) H e^{2t} F \\
&\quad + (-2)(\zeta q^h x + \zeta q^h) H e^{2t} F^{(1)} \\
&= (-1)^3 6\zeta^3 q^{3h} H^3 e^{6t} F(t,\zeta,q,h,x) \\
&\quad + (-1)^2 (8\zeta^2 q^{2h} + 2\zeta^2 q^{2h} x + 2\zeta^2 q^{2h} x \\
&\quad + 2\zeta^2 q^{2h}) H^2 e^{4t} F(t,\zeta,q,h,x) \\
&\quad + (-1)(4\zeta q^h x + 4\zeta q^h + \zeta q^h x^2) H e^{2t} F(t,\zeta,q,h,x) \\
&\quad + x^3 F(t,\zeta,q,h,x).
\end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned}
F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,\zeta,q,h,x) \\
&= \left(\sum_{i=0}^{N} (-1)^i a_i(N,\zeta,q,h,x) H^i e^{2it}\right) F(t,\zeta,q,h,x), \\
&\quad (N = 0, 1, 2, \ldots).
\end{aligned} \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we obtain

$$\begin{aligned}
F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\
&= \sum_{i=0}^{N} (-1)^i i a_i(N,\zeta,q,h,x) H^{i-1} H^{(1)} e^{2it} F(t,\zeta,q,h,x) \\
&\quad + \sum_{i=0}^{N} (-1)^i 2i a_i(N,\zeta,q,h,x) H^i e^{2it} F(t,\zeta,q,h,x) \\
&\quad + \left(\sum_{i=0}^{N} (-1)^i a_i(N,\zeta,q,h,x) H^i e^{2it}\right) F^{(1)}(t,\zeta,q,h,x).
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{i=0}^{N}(-1)^{i+1}\zeta q^h(i+1)a_i(N,\zeta,q,h,x)H^{i+1}e^{2(i+1)t}\right)F(t,\zeta,q,h,x)\\
&\qquad + \left(\sum_{i=0}^{N}(-1)^{i}(2i+x)a_i(N,\zeta,q,h,x)H^{i}e^{2it}\right)F(t,\zeta,q,h,x)\\
&= \left(\sum_{i=0}^{N}(-1)^{i}(2i+x)a_i(N,\zeta,q,h,x)H^{i}e^{2it}\right)(t,\zeta,q,h,x)\\
&\qquad + \left(\sum_{i=1}^{N+1}(-1)^{i}\zeta q^h i a_{i-1}(N,\zeta,q,h,x)H^{i}e^{2it}\right)F(t,\zeta,q,h,x).
\end{aligned} \tag{2.5}
$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$
F^{(N+1)} = \left(\sum_{i=0}^{N+1}(-1)^{i}a_i(N+1,\zeta,q,h,x)H^{i}e^{2it}\right)F(t,\zeta,q,h,x). \tag{2.6}
$$

By (2.5) and (2.6), we have

$$
\begin{aligned}
&\left(\sum_{i=0}^{N}(x+2i)a_i(N,\zeta,q,h,x)H^{i}e^{2it} + \sum_{i=1}^{N+1}(-1)^{i}\zeta q^h i a_{i-1}(N,\zeta,q,h,x)H^{i}e^{2it}\right)F\\
&= \left(\sum_{i=0}^{N+1}(-1)^{i}a_i(N+1,\zeta,q,h,x)H^{i}e^{2it}\right)F(t,\zeta,q,h,x).
\end{aligned} \tag{2.7}
$$

Comparing the coefficients on both sides of (2.7), we obtain

$$
\begin{aligned}
&a_0(N+1,\zeta,q,h,x) = xa_0(N,\zeta,q,h,x),\\
&a_{N+1}(N+1,\zeta,q,h,x) = \zeta q^h(N+1)a_N(N,\zeta,q,h,x),
\end{aligned} \tag{2.8}
$$

and

$$
\begin{aligned}
&a_i(N+1,\zeta,q,h,x) = (x+2i)a_i(N,\zeta,q,h,x) + \zeta q^h i a_{i-1}(N,\zeta,q,h,x),\\
&(1 \le i \le N).
\end{aligned} \tag{2.9}
$$

In addition, by (2.2) and (2.4), we get

$$
F = F^{(0)} = a_0(0,\zeta,q,h,x)F(t,\zeta,q,h,x) = F(t,\zeta,q,h,x). \tag{2.10}
$$

Thus, by (2.10), we obtain

$$
a_0(0,\zeta,q,h,x) = 1. \tag{2.11}
$$

It is not difficult to show that

$$
\begin{aligned}
&-\zeta q^h He^{2t}F(t,\zeta,q,h,x) + xF(t,\zeta,q,h,x)\\
&= \sum_{i=0}^{1}(-1)^{i}a_i(1,\zeta,q,h,x)H^{i}e^{2it}F(t,\zeta,q,h,x)\\
&= a_0(1,\zeta,q,h,x)F(t,\zeta,q,h,x) + (-1)a_1(1,\zeta,q,h,x)He^{2t}F.
\end{aligned} \tag{2.12}
$$

Thus, by (2.12), we also get

$$a_0(1,\zeta,q,h,x)=x,\quad a_1(1,\zeta,q,h,x)=\zeta q^h. \tag{2.13}$$

From (2.8), we note that

$$a_0(N+1,\zeta,q,h,x)=xa_0(N,\zeta,q,h,x)=x^2a_0(N-1,\zeta,q,h,x)=\cdots=x^{N+1},$$

and

$$\begin{aligned} a_N(N+1,\zeta,q,h,x)&=\zeta q^h(N+1)a_N(N,\zeta,q,h,x)\\ &=\cdots=\zeta^{(N+1)}q^{(N+1)h}(N+1)!. \end{aligned} \tag{2.14}$$

For $i=1,2,3$ in (2.9), we get

$$\begin{aligned} a_1(N+1,\zeta,q,h,x)&=\zeta q^h\sum_{k=0}^{N}(x+2)^k a_0(N-k,\zeta,q,h,x),\\ a_2(N+1,\zeta,q,h,x)&=2\zeta q^h\sum_{k=0}^{N-1}(x+4)^k a_1(N-k,\zeta,q,h,x),\ \text{and}\\ a_3(N+1,\zeta,q,h,x)&=3\zeta q^h\sum_{k=0}^{N-2}(x+6)^k a_2(N-k,\zeta,q,h,x). \end{aligned}$$

Continuing this process, we can deduce that, for $1\le i\le N$,

$$a_i(N+1,\zeta,q,h,x))=i\zeta q^h\sum_{k=0}^{N-i+1}(x+2i)^k a_{i-1}(N-k,\zeta,q,h,x). \tag{2.15}$$

Now, we give explicit expressions for $a_i(N+1,\zeta,q,h,x)$. By (2.14) and (2.15), we get

$$\begin{aligned} a_1(N+1,\zeta,q,h,x)&=\zeta q^h\sum_{k_1=0}^{N}(x+2)^{k_1}a_0(N-k_1,\zeta,q,h,x)\\ &=\zeta q^h\sum_{k_1=0}^{N}(x+2)^{k_1}x^{N-k_1}, \end{aligned}$$

$$\begin{aligned} a_2(N+1,\zeta,q,h,x)&=2\zeta q^h\sum_{k_2=0}^{N-1}(x+4)^{k_2}a_1(N-k_2,\zeta,q,h,x)\\ &=2!\zeta^2q^{2h}\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_2-1}(x+4)^{k_2}(x+2)^{k_1}x^{N-k_2-k_1-1}, \end{aligned}$$

and

$$\begin{aligned}
&a_3(N+1,\zeta,q,h,x)\\
&=3\zeta q^h\sum_{k_3=0}^{N-2}(x+6)^{k_3}a_2(N-k_3,\zeta,q,h,x)\\
&=3!\zeta^3q^{3h}\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-k_3-2}\sum_{k_1=0}^{N-k_3-k_2-2}(x+6)^{k_3}(x+4)^{k_2}(x+2)^{k_1}x^{N-k_2-k_2-k_1-2}.
\end{aligned}$$

Continuing this process, we have

$$\begin{aligned}
&a_i(N+1)\\
&=i!\zeta^iq^{ih}\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-k_i-i+1}\cdots\\
&\qquad\times\sum_{k_1=0}^{N-k_i-\cdots-k_2-i+1}(x+2i)^{k_i}\cdots(x+2)^{k_1}x^{N-k_i-\cdots-k_1-i+1}.
\end{aligned}\tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.1. *For $N=0,1,2,\ldots,$ the functional equation*

$$F^{(N)}=\left(\sum_{i=0}^{N}(-1)^ia_i(N,\zeta,q,h,x)\left(\frac{2}{\zeta q^he^{2t}+1}\right)^ie^{2it}\right)F$$

has a solution

$$F=F(t,\zeta,q,h,x)=\left(\frac{2}{\zeta q^he^{2t}+1}\right)e^{xt},$$

where

$$\begin{aligned}
&a_0(N,\zeta,q,h,x)=x^N,\\
&a_N(N,\zeta,q,h,x)=N!\zeta^Nq^{Nh},\\
&a_i(N,\zeta,q,h,x)=i!\zeta^iq^{ih}\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\\
&\qquad\times\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(x+2i)^{k_i}\cdots(x+2)^{k_1}x^{N-k_i-\cdots-k_1-i},\\
&(1\le i\le N).
\end{aligned}$$

Here is a plot of the surface for this solution. In Figure 1, we choose $\zeta=e^{\frac{2\pi i}{2}},h=2,q=1/10$.

From (1.1), we note that

$$F^{(N)}=\left(\frac{d}{dt}\right)^NF(t,\zeta,q,h,x)=\sum_{k=0}^{\infty}T_{k+N,\zeta,q}^{(h)}(x)\frac{t^k}{k!}.\tag{2.17}$$

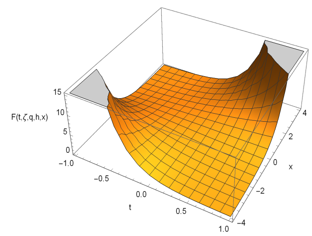

FIGURE 1. The surface for the solution $F(t, \zeta, q, h, x)$

From Theorem 2.1, (1.3), and (2.17), we can derive the following equation:

$$\begin{aligned}
\sum_{k=0}^{\infty} T_{k+N,\zeta,q}^{(h)}(x)\frac{t^k}{k!} &= F^{(N)} \\
&= \left(\sum_{i=0}^{N}(-1)^i a_i(N,\zeta,q,h,x)\left(\frac{2}{\zeta q^h e^{2t}+1}\right)^i e^{2it}\right)F \\
&= \sum_{i=0}^{N}(-1)^i a_i(N,\zeta,q,h,x)e^{(x+2i)t}\left(\frac{2}{\zeta q^h e^{2t}+1}\right)^{i+1} \\
&= \sum_{i=0}^{N}(-1)^i a_i(N,\zeta,q,h,x)\left(\sum_{k=0}^{\infty} T_{k,\zeta,q}^{(h,i+1))}(x+2i)\frac{t^k}{k!}\right) \\
&= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N}(-1)^i a_i(N,\zeta,q,h,x)T_{k,\zeta,q}^{(h,i+1))}(x+2i)\right)\frac{t^k}{k!}.
\end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2.2. *For $k = 0, 1, \ldots,$ and $N = 0, 1, 2, \ldots,$ we have*

$$\begin{aligned}
T_{k+N,\zeta,q}^{(h)}(x) &= \sum_{i=0}^{N}(-1)^i a_i(N,\zeta,q,h,x)T_{k,\zeta,q}^{(h,i+1)}(x+2i) \\
&= \sum_{i=0}^{N}\sum_{l=0}^{k}\binom{k}{l}(-1)^i(2i)^{k-l}a_i(N,\zeta,q,h,x)T_{l,\zeta,q}^{(h,i+1)}(x),
\end{aligned} \tag{2.19}$$

where

$$a_0(N,\zeta,q,h,x) = x^N,$$
$$a_N(N,\zeta,q,h,x) = N!\zeta^N q^{Nh},$$
$$a_i(N,\zeta,q,h,x) = i!\zeta^i q^{ih} \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-k_i-i} \cdots$$
$$\times \sum_{k_1=0}^{N-k_i-\cdots-k_2-i} (x+2i)^{k_i} \cdots (x+2)^{k_1} x^{N-k_i-\cdots-k_1-i},$$
$$(1 \leq i \leq N).$$

Let us take $k=0$ in (2.19). Then, we have the following corollary.

Corollary 2.3. *For $N = 0, 1, 2, \dots,$ we have*

$$T_{N,\zeta,q}^{(h)}(x) = \sum_{i=0}^{N} (-1)^i a_i(N,\zeta,q,h,x) T_{0,\zeta,q}^{(h,i+1)}(x+2i).$$

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