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# SCHWARZ METHOD FOR SINGULARLY PERTURBED SECOND ORDER CONVECTION-DIFFUSION EQUATIONS

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ABSTRACT. In this paper, we have constructed an overlapping Schwarz method for singularly perturbed second order convection-diffusion equations. The method splits the original domain into two overlapping subdomains. A hybrid difference scheme is proposed in which on the boundary layer region we use the central finite difference scheme on a uniform mesh while on the non-layer region we use the mid-point difference scheme on a uniform mesh. It is shown that the numerical approximations which converge in the maximum norm to the exact solution. When appropriate subdomains are used, the numerical approximations generated from the method are shown to be first order convergent. Furthermore it is shown that, two iterations are sufficient to achieve the expected accuracy. Numerical examples are presented to support the theoretical results. The main advantages of this method used with the proposed scheme is it reduces iteration counts very much and easily identifies in which iteration the Schwarz iterate terminates.

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## 1. Introduction

Consider the following singularly perturbed convection-diffusion equations as in [5, 6].

$$Ly := \quad -\varepsilon y^{''}(x) + a(x)y'(x) = f(x), \quad x \in \Omega = (0, 1), \tag{1}$$

$$y(0) = q_0, \quad y(1) = q_1, \tag{2}$$

where $q_0$, $q_1$ are given constants and the functions $y(x)$, $a(x)$ and $f(x) \in C^{(2)}(\bar{\Omega})$ with $a(x) \geq \alpha$, $\alpha > 0$, $0 < \varepsilon \ll 1$.

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The classical numerical methods fail to produce good approximations for singularly perturbed problems (SPPs). Several non-classical approaches are used to design the numerical methods for singularly perturbed problems. Such approaches can be either iterative or non-iterative. With an iterative approach numerical methods for SPPs comprising domain decomposition and Schwarz iterative techniques have been examined by various authors, for example, in [1]-[8]. In [8], Miller et al. examined a continuous overlapping Schwarz method for a singularly perturbed convection-diffusion equation with arbitrary fixed interface positions and found it to be uniformly convergent with respect to the perturbation parameter. The authors of [15] found a flaw in the analysis of domain decomposition methods explored in [3], [4], [11], [13] and [14]. The authors observation is that the constant $C$ is not independent of the iteration number $k$ and it is growing at each induction step in their proof of [3], [4], [11], [13] and [14]. But in [15] the authors present an alternate analysis of overlapping domain decomposition methods for singularly perturbed reaction-diffusion problems with two parameters and problems in [13].

In this paper, a discrete Schwarz method is discussed based on a central finite difference scheme and a midpoint difference scheme on the uniform mesh. The method splits the original domain into two overlapping subdomains. In this proposed scheme on the boundary layer region we use the central finite difference scheme on a uniform mesh while on the non-layer region we use the mid-point difference scheme on a uniform mesh.

In [5, 6] it is shown that the numerical approximations generated by this method fail to converge to the solution of the continuous problem. But in this paper, of primary interest we have proved that discrete Schwarz method converge to the solution of the continuous problem. When appropriate subdomains are used, the method is shown to be first order convergent. Furthermore we improve upon the analysis of the reference [8] to show that, just two iterations are required to achieve the expected accuracy. Iteration counts for the method are presented.

An outline of the rest of the paper is as follows. In Section 2, the continuous Schwarz method is described and derivative estimates of smooth and singular components are given. In Section 3, the discrete Schwarz method is described. The maximum pointwise error bounds are obtained in Section 4. Numerical experiments are presented in Section 5 and finally, conclusions are included in Section 6.

Assumption: In this paper we assume that $\varepsilon \leq CN^{-1}$ is generally the case for discretization of convection-diffusion equations.

**Notations**: Through out the paper we use $C$, with or without subscript to denote a generic positive constant independent of the iteration parameter $k$ and the discretization parameter $N$.

Let $y : D \to \mathbb{R}, \ D \subseteq \mathbb{R}$. The appropriate norm for studying the convergence of the numerical solution to the exact solution of a SPP is $||y||_D = \sup_{x \in D} |y(x)|$.

## 2. Continuous Schwarz method

In this section, a continuous Schwarz method is described. This process generates a sequence of iterates $\{y^{[k]}\}$, which converges as $k \to \infty$ to the exact solution $y$. Further we state, the maximum principle for (1)-(2). Using this principle, a stability result is stated. Finally, the bounds on the derivatives of the regular and singular components of $y$ is presented.

First, we split the domain into two overlapping subdomains as

$$\Omega_c = (0, 1-\tau) \text{ and } \Omega_r = (1-2\tau, 1),$$

where the subdomain parameter is an appropriate constant, defined in Section 3.

The iterative process is defined as follows:

$$y^{[0]}(x) \equiv 0,\ 0 < x < 1, \quad y^{[0]}(0) = y(0), \quad y^{[0]}(1) = y(1).$$

For $k \geq 1,$ the iterates $y^{[k]}(x)$ are defined by

$$y^{[k]}(x) = \begin{cases} y_c^{[k]}(x), & \text{for } x \in \bar{\Omega}_c, \\ y_r^{[k]}(x), & \text{for } x \in \bar{\Omega}_r \backslash \bar{\Omega}_c, \end{cases}$$

where $y_p^{[k]},\ p = \{c, r\}$ are the solutions of the problems

$$\begin{aligned} Ly_r^{[k]}(x) &= f \text{ in } \Omega_r, \quad y_r^{[k]}(1-2\tau) = y^{[k-1]}(1-2\tau), \quad y_r^{[k]}(1) = y(1) \text{ and} \\ Ly_c^{[k]}(x) &= f \text{ in } \Omega_c, \quad y_c^{[k]}(0) = y(0), \quad y_c^{[k]}(1-\tau) = y_r^{[k]}(1-\tau). \end{aligned}$$

Let $\Omega_p = (d, e),\ \bar{\Omega}_p = [d, e],\ p = \{c, r\}$. The BVP (1)-(2) satisfies the following maximum principle on each $\bar{\Omega}_p$.

**Theorem 2.1. (Maximum Principle)**. *Consider the BVP (1)-(2). Let* $y(d) \geq 0,\ y(e) \geq 0,\ Ly(x) \geq 0,\ for\ x \in \Omega_p$. *Then,* $y(x) \geq 0,\ \forall x \in \bar{\Omega}_p$ .

*Proof.* Please refer [8]. □

An immediate consequence of this is the following stability result.

**Lemma 2.2. (Stability Result)**. *If* $y(x)$ *is the solution of the BVP (1)-(2) then*

$$||y(x)|| \leq \left[\max\{|y(d)|,\ |y(e)|\} + \frac{1}{\alpha}||f||\right], \quad \forall x \in \bar{\Omega}_p.$$

*Proof.* Please refer [8]. □

In Section 4 we establish the convergence of the discrete Schwarz method described in Section 3. We need sharper bounds on the derivatives of components of the exact solution $y$ of (1)-(2). For this we decompose the solution $y$ into regular and singular components as,

$$y(x) = v(x) + w(x) \tag{3}$$

where the regular component $v = v_0$ is the solution of the reduced problem

$$a(x)v_0'(x) = f(x), \quad \text{for all} \quad x \in \Omega,$$
$$v_0(0) = q_0.$$

Further, the singular component $w = w_0$ is the solution of the homogeneous problem

$$Lw_0 = 0, \quad w_0(0) = w_0(1)e^{-\alpha/\varepsilon}, \quad w_0(1) = q_1 - v_0(1).$$

The following lemma provides the bounds on the derivatives of the regular and the singular components of the solution.

**Lemma 2.3.** *The solution $y(x)$ of the BVP (1)-(2) has the decomposition $y(x) = v(x) + w(x)$ into smooth and singular components. They satisfy*

$$|v^{(l)}(x)| \le C \quad \text{and} \quad |w^{(l)}(x)| \le C\varepsilon^{-(l)}e^{-\alpha(1-x)/\varepsilon},$$

*for* $0 \le l \le 3, \quad \forall x \in \bar{\Omega} = (\bar{\Omega}_r \backslash \bar{\Omega}_c) \cup \bar{\Omega}_c.$

*Proof.* Please refer [8]. □

## 3. Discrete Schwarz Method

The continuous overlapping Schwarz method described in Section 2 is discretized by introducing uniform meshes on each subdomain. The domain $\Omega = (0,1)$ is divided into two overlapping subdomains as $\Omega_c = (0, 1-\tau)$ and $\Omega_r = (1-2\tau, 1)$. The subdomain parameter $\tau$ is chosen to be the Shishkin transition point $\tau = \min\left\{\frac{1}{3}, \frac{2\varepsilon}{\alpha}\ln N\right\}$ as in [8]. In each subdomain, $\Omega_p = (d, e), \quad p = \{c, r\}$, construct a uniform mesh $\bar{\Omega}_p^N = \{d = x_0 < x_1 < x_2 < \cdots < x_N = e\}$ with $h_p = x_i - x_{i-1} = (e-d)/N$.

In this proposed scheme we use the central finite difference scheme on a uniform mesh in the subdomain $\Omega_r$ and the mid-point difference scheme on a uniform mesh in the subdomain $\Omega_c$. Then in each subdomain $\Omega_p^N$, $p = \{c, r\}$, the corresponding discretization is,

$$L^N Y_c(x_i) = -\varepsilon\delta^2 Y_c(x_i) + a_{i-1/2}D^- Y_c(x_i) = f_{i-1/2}, \; i = 1, \ldots, N-1,$$
$$L^N Y_r(x_i) = -\varepsilon\delta^2 Y_r(x_i) + a_i D^0 Y_r(x_i) = f_i, \; i = 1, \ldots, N-1,$$

where $\delta^2 Y_p(x_i) = \dfrac{1}{h_p^2}\left(Y_p(x_{i+1}) - 2Y_p(x_i) + Y_p(x_{i-1})\right)$,

$D^0 Y_r(x_i) = \dfrac{Y_r(x_{i+1}) - Y_r(x_{i-1})}{2h_r}, \quad D^- Y_c(x_i) = \dfrac{Y_c(x_i) - Y_c(x_{i-1})}{h_c}$,

$a_{i-1/2} \equiv a((x_{i-1} + x_i)/2)$, and $a_i \equiv a(x_i)$; Similarly for $f_{i-\frac{1}{2}}$ and $f_i$. The discrete problem is

$$L^N Y_p(x_i) = f(x_i),$$

where

$$f(x_i) = \begin{cases} f_{i-\frac{1}{2}}, & \text{for } x_i \in \bar{\Omega}_c^N, \\ f_i, & \text{for } x_i \in \bar{\Omega}_r^N. \end{cases}$$

Then the algorithm for discrete Schwarz method is defined as follows.

*Step1:* We choose the initial mesh function

$$Y^{[0]}(x_i) \equiv 0,\ 0 < x_i < 1,\ Y^{[0]}(0) = y(0),\ Y^{[0]}(1) = y(1).$$

*Step2:* We compute the mesh functions $Y_p^{[k]}$, $p = \{r, c\}$ which are the solutions of the following discrete problems

$$\begin{aligned} L^N Y_r^{[k]}(x_i) &= f_i,\ x_i \in \Omega_r^N, \quad Y_r^{[k]}(1-2\tau) = \bar{Y}^{[k-1]}(1-2\tau), \quad Y_r^{[k]}(1) = y(1), \\ L^N Y_c^{[k]}(x_i) &= f_{i-\frac{1}{2}},\ x_i \in \Omega_c^N, \quad Y_c^{[k]}(0) = y(0), \quad Y_c^{[k]}(1-\tau) = \bar{Y}_r^{[k]}(1-\tau), \end{aligned}$$

where $\bar{Y}^{[k]}$ denotes the piecewise linear interpolant of $Y^{[k]}$ on the mesh $\bar{\Omega}^N := (\bar{\Omega}_r^N \backslash \bar{\Omega}_c) \cup \bar{\Omega}_c^N$.

*Step3:* We compute the mesh function $Y^{[k]}$ by combining together the solutions on the subdomains

$$Y^{[k]}(x_i) = \begin{cases} Y_c^{[k]}(x_i), & \text{for } x_i \in \bar{\Omega}_c^N, \\ Y_r^{[k]}(x_i), & \text{for } x_i \in \bar{\Omega}_r^N \backslash \bar{\Omega}_c. \end{cases} \tag{4}$$

*Step4:* If the stopping criterion

$$||Y^{[k+1]} - Y^{[k]}||_{\bar{\Omega}^N} \le tol$$

is reached, then stop; otherwise go to Step 2. Here *tol* is the user prescribed accuracy. The following are analogous results for the discrete problem.

**Lemma 3.1.** *(Discrete maximum principle). Assume that* $Y(x_0) \ge 0$ *and* $Y(x_N) \ge 0$ *then* $L^N Y(x_i) \ge 0$, $\forall x_i \in \Omega_p^N$ *implies that* $Y(x_i) \ge 0, \forall x_i \in \bar{\Omega}_p^N$.

*Proof.* Please refer to [10, 12]. □

An immediate consequence of this lemma is the following stability result.

**Lemma 3.2.** *If* $Y(x_i)$ *is any mesh function then for all* $x_i \in \bar{\Omega}_p^N$, *then*

$$|Y(x_i)| \le C \max\{|Y(x_0)|,\ |Y(x_N)|,\ ||L^N Y||_{\Omega_p^N}\}.$$

## 4. Error Estimates

In this section, we estimate the error in discrete Schwarz iterates and prove that two iterations are required to attain first order convergence. Following the method of analysis adapted in [13] and [15], we derive error estimates. The analysis proceeds as follows.

**Lemma 4.1.** *Let* $y$ *be the solution of (1)-(2) and let* $Y^{[k]}$ *be the* $k^{th}$ *iterate of the discrete Schwarz method as described in Section 3. Then, there are constants* $C$ *such that*

$$||Y^{[k]} - y||_{\bar{\Omega}^N} \le C 2^{-k} + C N^{-1} \ln^3 N,$$

*where $C$ is independent of $k$ and $N$.*

*Proof.* At the first iteration $(Y^{[0]}-y)(0)=0$ and $(Y^{[0]}-y)(1)=0$. Since $Y^{[0]}(x_i)=0$ for $x_i\in\Omega^N:=\{x_1<x_2<x_3\cdots<x_{N-1}\}$, we can use Lemma 2.3 to show that

$$||Y^{[0]}-y||_{\Omega^N}=||y||_{\Omega^N}\le C.$$

Clearly, there are constants $C$ such that

$$||Y^{[0]}-y||_{\bar{\Omega}^N}\le C2^0+CN^{-1}\ln^3 N.$$

Thus, the result holds for $k=0$ and the proof is now completed by induction. Assume that, for an arbitrary integer $k\ge 0$, there exists $C$ such that

$$||Y^{[k]}-y||_{\bar{\Omega}^N}\le C2^{-k}+CN^{-1}\ln^3 N.$$

**Case (i):** Error bound estimation on $\bar{\Omega}_r^N$.
In the proposed scheme we use the central finite difference scheme on $\bar{\Omega}_r^N$. One can deduce the following truncation error estimate as in [9] on $x_i\in\bar{\Omega}_r^N$ as

$$||(L^N-L)y||_{\Omega_r^N}\le C(\varepsilon h_r||y^{(3)}||_{\Omega_r}+||a||h_r^2||y^{(3)}||_{\Omega_r}). \tag{5}$$

In order to find a bound on $||L^N(Y_r^{[k+1]}-y)||$ we must decompose $y$ as in (3). Consider

$$\begin{aligned}||L^N(Y_r^{[k+1]}-y)||_{\Omega_r^N} &= ||(L^N-L)y||_{\Omega_r^N},\\ &\le ||(L^N-L)v||_{\Omega_r^N}+||(L^N-L)w||_{\Omega_r^N}.\end{aligned} \tag{6}$$

For the first term on the right-hand side of (6), we use the local truncation error estimate (5), $h_r\le CN^{-1}$, $\varepsilon\le CN^{-1}$ and Lemma 2.3 to get

$$\begin{aligned}||(L^N-L)v||_{\Omega_r^N} &\le C(\varepsilon h_r||v^{(3)}||_{\Omega_r}+||a||h_r^2||v^{(3)}||_{\Omega_r}),\\ &\le CN^{-2}+CN^{-2},\\ &\le CN^{-1}.\end{aligned}$$

For the second term on the right-hand side of (6), when $\tau=\frac{2\varepsilon}{\alpha}\ln N$, using local truncation error estimate (5) and $h_r\le C\varepsilon N^{-1}\ln N$, we have

$$\begin{aligned}||(L^N-L)w||_{\Omega_r^N} &\le C(\varepsilon h_r||w^{(3)}||_{\Omega_r}+||a||h_r^2||w^{(3)}||_{\Omega_r}),\\ &\le Ch_r\varepsilon^{-2}+Ch_r^2\varepsilon^{-3},\\ &\le C\varepsilon^{-1}N^{-1}\ln N+CN^{-2}\varepsilon^{-1}\ln^2 N,\\ &\le C\varepsilon^{-1}N^{-1}\ln^2 N.\end{aligned}$$

Using the above estimates in (6), we have

$$||L^N(Y_r^{[k+1]}-y)||_{\Omega_r^N} \le CN^{-1}\ln^3 N+C\varepsilon^{-1}N^{-1}\ln^2 N, \tag{7}$$

for some $C$. The end point of the subdomain $\Omega_r^N$ is $1-2\tau$, which in general is not in $\Omega^N=\{x_1<x_2<x_3<\cdots<x_{N-1}\}$, so we use a piecewise linear

interpolant of the previous iterate to determine $Y_r^{[k+1]}(1-2\tau)$.
Now, using our inductive argument we have

$$\begin{aligned}|(Y_r^{[k+1]}-y)(1-2\tau)| &= |(\bar{Y}^{[k]}-y)(1-2\tau)|,\\ &= |(Y^{[k]}-y)(1-2\tau)|,\\ &\leq |(Y^{[k]}-\bar{y})(1-2\tau)|+|(\bar{y}-y)(1-2\tau)|, \end{aligned} \tag{8}$$

where $\bar{y}$ is the piecewise linear interpolant of $y$ using grid points of $\bar{\Omega}_c^N$.
For the second term on the right-hand side of (8), using solution decomposition as in (3) we get

$$|(\bar{y}-y)(1-2\tau)| \leq |(\bar{v}-v)(1-2\tau)|+|(\bar{w}-w)(1-2\tau)|. \tag{9}$$

Note that $(1-2\tau)$ lies in $\bar{\Omega}_c$. For any $z \in C^2(\bar{\Omega}_c)$, standard argument of piecewise linear interpolant $\bar{z}$ gives

$$|(z-\bar{z})(1-2\tau)| \leq Ch_c^2||z^{(2)}||_{\bar{\Omega}_c} \quad \text{and} \quad |(z-\bar{z})(1-2\tau)| \leq C||z||_{\bar{\Omega}_c}. \tag{10}$$

For the first term on the right-hand side of (9), we use the first bound of (10), $h_c \leq CN^{-1}$ and Lemma 2.3 to get

$$\begin{aligned}|(\bar{v}-v)(1-2\tau)| &\leq Ch_c^2||v^{(2)}||_{\bar{\Omega}_c},\\ &\leq CN^{-2},\\ &\leq CN^{-1}.\end{aligned}$$

For the second term on the right-hand side of (9), when $\tau = \frac{2\varepsilon}{\alpha}\ln N$, note that the layer function $w$ is monotonicaly increasing in the region $(1/3, 1-\tau) \subset \bar{\Omega}_c$. Hence using the second bound of (10), we have

$$\begin{aligned}|(\bar{w}-w)(1-2\tau)| &\leq C||w||_{\bar{\Omega}_c},\\ &\leq Ce^{-\alpha\tau/\varepsilon} \leq CN^{-1}.\end{aligned}$$

$$\therefore\ |(\bar{y}-y)(1-2\tau)| \leq CN^{-1}. \tag{11}$$

Now, using (11) in (8) we have

$$\begin{aligned}|(Y_r^{[k+1]}-y)(1-2\tau)| &\leq |(Y^{[k]}-\bar{y})(1-2\tau)|+|(\bar{y}-y)(1-2\tau)|,\\ &\leq C2^{-k}+CN^{-1}\ln^3 N+CN^{-1},\\ &\leq C2^{-k}+CN^{-1}\ln^3 N.\end{aligned}$$

Consider the mesh function

$$\Psi^{\pm}(x_i) = C\left(\frac{1+x_i}{4}\right)2^{-k}+C(1+x_i)N^{-1}\ln^3 N+C(x_i-(1-2\tau))\varepsilon^{-1}N^{-1}\ln^2 N \\ \pm(Y_r^{[k+1]}-y)(x_i),$$

where $C$ is positive constants to be chosen suitably, so that the following expressions are satisfied.

Note that $\quad \Psi^{\pm}(1-2\tau) \quad \geq \quad C\left(\frac{1+(1-2\tau)}{4}\right)2^{-k}+C(1+(1-2\tau))N^{-1}\ln^3 N$

$$
\begin{aligned}
&- \quad C2^{-k} - CN^{-1}\ln^3 N,\\
&\geq \quad C\left(\frac{1}{4}\right)2^{-k} + CN^{-1}\ln^3 N - C2^{-k} - CN^{-1}\ln^3 N > 0,
\end{aligned}
$$

$$
\Psi^{\pm}(1) \;=\; C\left(\frac{1}{2}\right)2^{-k} + 2CN^{-1}\ln^3 N + 2C\tau\varepsilon^{-1}N^{-1}\ln^2 N \pm 0 > 0 \quad \text{and}
$$

$$
\begin{aligned}
L^N\Psi^{\pm}(x_i) \;&\geq\; \alpha\left(\left(\frac{C}{4}\right)2^{-k} + CN^{-1}\ln^3 N + C\varepsilon^{-1}N^{-1}\ln^2 N\right) - CN^{-1}\ln^3 N\\
&- \quad C\varepsilon^{-1}N^{-1}\ln^2 N > 0.
\end{aligned}
$$

Using the discrete maximum principle for the operator $L^N$ on $\bar{\Omega}_r^N$ we get,

$$
\begin{aligned}
||(Y_r^{[k+1]} - y)||_{\bar{\Omega}_r^N} \;&\leq\; C\left(\frac{1+x_i}{4}\right)2^{-k} + C(1+x_i)N^{-1}\ln^3 N\\
&+ \quad C(x_i - (1-2\tau))\varepsilon^{-1}N^{-1}\ln^2 N.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
||(Y_r^{[k+1]} - y)||_{\bar{\Omega}_r^N\setminus\bar{\Omega}_c} \;&\leq\; \left(\frac{C}{2}\right)2^{-k} + 2CN^{-1}\ln^3 N + 2CN^{-1}\tau\varepsilon^{-1}\ln^2 N,\\
&\leq\; C2^{-(k+1)} + CN^{-1}\ln^3 N + CN^{-1}\tau\varepsilon^{-1}\ln^2 N.
\end{aligned}
$$

But since $\tau = \frac{2\varepsilon}{\alpha}\ln N,$ this gives

$$
||(Y_r^{[k+1]} - y)||_{\bar{\Omega}_r^N\setminus\bar{\Omega}_c} \;\leq\; C2^{-(k+1)} + CN^{-1}\ln^3 N. \tag{12}
$$

**Case (ii):** Error bound estimation on $\bar{\Omega}_c^N$.
We use solution decomposition at each point $x_i \in \bar{\Omega}_c^N$, $(Y_c^{[k+1]} - y)$ can be written in the form

$$
(Y_c^{[k+1]} - y)(x_i) = (V_c^{[k+1]} - v)(x_i) + (W_c^{[k+1]} - w)(x_i). \tag{13}
$$

Suppose that $(1-\tau)$ lies in $\bar{\Omega}_r$. For any $z \in C^2(\bar{\Omega}_r)$, standard argument of piecewise linear interpolant $\bar{z}$ gives

$$
|(z - \bar{z})(1-\tau)| \leq Ch_r^2||z^{(2)}||_{\bar{\Omega}_r}. \tag{14}
$$

In the proposed scheme we use the midpoint difference scheme on $\bar{\Omega}_c^N$. One can deduce the following truncation error estimates as in [8] on $x_i \in \bar{\Omega}_c^N$ as

$$
||(L^N - L)y||_{\Omega_c^N} \;\leq\; C\varepsilon h_c||y^{(3)}||_{\Omega_c} + Ch_c||y^{(2)}||_{\Omega_c}. \tag{15}
$$

**Subcase (i):** For the first term on the right-hand side of (13), we use the above local truncation error estimate (15), $h_c \leq CN^{-1}$, $\varepsilon \leq CN^{-1}$ and Lemma 2.3 to get

$$
\begin{aligned}
||L^N(V_c^{[k+1]} - v)||_{\Omega_c^N} \;&=\; ||(L^N - L)v||_{\Omega_c^N},\\
&\leq\; C\varepsilon h_c||v^{(3)}||_{\Omega_c} + Ch_c||v^{(2)}||_{\Omega_c},\\
&\leq\; CN^{-1}.
\end{aligned}
$$

Now, using our inductive argument, the bound of (14), $h_r \leq CN^{-1}$ and Lemma 2.3, we get

$$\begin{aligned}
|(V_c^{[k+1]} - v)(1-\tau)| &= |(\bar{V}_r^{[k+1]} - v)(1-\tau)|, \\
&\leq Ch_r^2 ||v^{(2)}||_{\bar{\Omega}_r}, \\
&\leq CN^{-2}, \\
&\leq CN^{-1},
\end{aligned}$$

where we have use the fact that $(1-\tau)$ is the mesh point of $\bar{\Omega}_r^N$.
Consider the mesh function

$$\Phi^{\pm}(x_i) = C\left(\frac{x_i}{2(1-\tau)}\right)2^{-k} + (1+x_i)CN^{-1} \pm (V_c^{[k+1]} - v)(x_i),$$

where $C$ is positive constants to be chosen suitably, so that the following expressions are satisfied. Note that

$$\Phi^{\pm}(0) = CN^{-1} \pm 0 > 0,$$

$$\begin{aligned}
\Phi^{\pm}(1-\tau) &\geq \left(\frac{C}{2}\right)2^{-k} + C(2-\tau)N^{-1} - CN^{-1}, \\
&\geq \left(\frac{C}{2}\right)2^{-k} + CN^{-1} - CN^{-1} > 0 \ \text{ and}
\end{aligned}$$

$$L^N\Phi^{\pm}(x_i) \geq \alpha\left(\left(\frac{C}{2}\right)2^{-k} + CN^{-1}\right) - CN^{-1} > 0.$$

We use the discrete maximum principle for the operator $L^N$ on $\bar{\Omega}_c^N$ to get,

$$\begin{aligned}
||V_c^{[k+1]} - v||_{\bar{\Omega}_c^N} &\leq C\left(\frac{1}{2}\right)2^{-k} + C(2-\tau)N^{-1}, \\
&\leq C2^{-(k+1)} + CN^{-1}.
\end{aligned}$$

**Subcase (ii):** For the second term on the right-hand side of (13), when $\tau = \frac{2\varepsilon}{\alpha}\ln N$, using the arguments discussed as in ([8], page no 63) we get

$$||W_c^{[k+1]} - w||_{\bar{\Omega}_c^N} \leq CN^{-1}.$$

Now, using error bound for the regular and layer parts we get,

$$||(Y_c^{[k+1]} - y)||_{\bar{\Omega}_c^N} \leq C2^{-(k+1)} + CN^{-1}\ln^3 N. \tag{16}$$

On combining the error bounds (12) and (16) , we have

$$||Y^{[k+1]} - y||_{\bar{\Omega}^N} \leq C2^{-(k+1)} + CN^{-1}\ln^3 N.$$

This completes the proof. $\square$

Our next lemma shows that solutions to the discrete problem can be bounded by solutions to a problem with constant coefficients.

**Lemma 4.2.** *Suppose that $Y_{c,i}$ is the solution of*

$$-\varepsilon\delta^2 Y_{c,i} + aD^- Y_{c,i} = 0,\ i = 1,\ldots,N-1, \quad Y_{c,0} = e^{-a/\varepsilon} Y_{c,N}, \quad Y_{c,N} = Y_{c,N} \tag{17}$$

*and $Z_{c,i}$ is the solution of the problem*

$$-\varepsilon\delta^2 Z_{c,i} + b_i D^- Z_{c,i} = 0,\ i = 1,\ldots,N-1, \quad Z_{c,0} = e^{-b_0/\varepsilon} Z_{c,N}, \quad Z_{c,N} = Y_{c,N}, \tag{18}$$

*where it is assumed that for all $i$, $0 \le i \le N$, $b_i \ge a$. Then, $Z_{c,i} \le Y_{c,i}$, for $x_i \in \bar{\Omega}_c^N$. Similarly, $Z_{r,i} \le Y_{r,i}$, for $x_i \in \bar{\Omega}_r^N$.*

*Proof.* Please refer to ([8], page no 53). □

In the following lemma we show that the iterative process converges much faster than is shown in Lemma 4.1.

**Lemma 4.3.** *Let $Y^{[k]}$ be the $k^{th}$ iterate of the discrete Schwarz method described in Section 3. Then there exists some $C$ such that*

$$||Y^{[k+1]} - Y^{[k]}||_{\bar{\Omega}^N} \le C\nu^k \quad \textit{where} \quad \nu = \left(1 + \frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2} < 1,$$

*and $C$ is independent of $k$ and $N$. Furthermore if $\tau = \dfrac{2\varepsilon}{\alpha}\ln N$ then $\nu \le 2N^{-1/2}$.*

*Proof.* At the first iteration $||Y^{[0]}||_{\Omega^N} = 0$.
Then clearly

$$||Y^{[1]} - Y^{[0]}||_{\Omega^N} = ||Y^{[1]}||_{\Omega^N}.$$

$Y_r^{[1]}$ satisfies

$$L^N Y_r^{[1]} = f_i \quad \text{for } x_i \in \Omega_r^N,$$

$$Y_r^{[1]}(1-2\tau) = \bar{Y}^{[0]}(1-2\tau), \quad Y_r^{[1]}(1) = y(1).$$

Therefore, we use Lemma 3.2 to obtain $||Y_r^{[1]}||_{\bar{\Omega}_r^N} \le C.$

$$\text{Consequently, } ||Y_r^{[1]}||_{\bar{\Omega}_r^N \setminus \bar{\Omega}_c} \le C.$$

Also $Y_c^{[1]}$ satisfies

$$L^N Y_c^{[1]} = f_{i-1/2} \quad \text{for } x_i \in \Omega_c^N,$$

$$Y_c^{[1]}(0) = y(0), \quad Y_c^{[1]}(1-\tau) = \bar{Y}_r^{[1]}(1-\tau).$$

Therefore, we can apply Lemma 3.2 to get $||Y_c^{[1]}||_{\bar{\Omega}_c^N} \le C.$
Combining all these estimates we obtain

$$||Y^{[1]} - Y^{[0]}||_{\bar{\Omega}^N} \le C\nu^0.$$

Thus, the result holds for $k = 0$ and the proof is now completed by an induction argument.
Assume that, for an arbitrary integer $k \geq 0$,

$$||Y^{[k+1]} - Y^{[k]}||_{\bar{\Omega}^N} \leq C\nu^k \quad \text{where} \quad \nu = \left(1 + \frac{\alpha\tau}{2\varepsilon N}\right)^{-N/2}.$$

Let $\Phi_c^{[k+1]}(x_i)$ be the solution of

$$\begin{cases} -\varepsilon\delta^2\Phi_c^{[k+1]}(x_i) + \alpha D^-\Phi_c^{[k+1]}(x_i) = 0 \text{ for } x_i \in \Omega_c^N, \\ \Phi_c^{[k+1]}(0) = 0, \quad \Phi_c^{[k+1]}(1-\tau) = C\nu^k. \end{cases} \tag{19}$$

Then by using Lemma 4.2 we have

$$(Y_c^{[k+1]} - Y_c^{[k]})(x_i) \leq \Phi_c^{[k+1]}(x_i) \text{ for } x_i \in \Omega_c^N. \tag{20}$$

The exact solution to the difference problem (19) is

$$\Phi_c^{[k+1]}(x_i) = C\nu^k \frac{m_1^i - m_2^i}{m_1^N - m_2^N},$$

where

$$\begin{aligned} m_1 &= \left(1 + \frac{\alpha h_c}{2\varepsilon}\right) + \sqrt{\left(1 + \frac{\alpha h_c}{2\varepsilon}\right)^2 + \left(-1 - \frac{\alpha h_c}{\varepsilon}\right)}, \\ &= 1 + \frac{\alpha h_c}{\varepsilon} \geq 1 + \frac{\alpha h_c}{2\varepsilon} = \left(1 + \frac{\alpha(1-\tau)}{2\varepsilon N}\right) \geq \left(1 + \frac{\alpha\tau}{2\varepsilon N}\right). \end{aligned}$$

$$\begin{aligned} m_2 &= \left(1 + \frac{\alpha h_c}{2\varepsilon}\right) - \sqrt{\left(1 + \frac{\alpha h_c}{2\varepsilon}\right)^2 + \left(-1 - \frac{\alpha h_c}{\varepsilon}\right)}, \\ &= 1. \end{aligned}$$

Now

$$L^N(Y_r^{[k+2]} - Y_r^{[k+1]})(x_i) = 0, \quad \forall\ x_i \in \bar{\Omega}_r^N,$$
$$(Y_r^{[k+2]} - Y_r^{[k+1]})(1) = 0.$$

Using our inductive hypothesis and (20)

$$\begin{aligned} |(Y_r^{[k+2]} - Y_r^{[k+1]})(1-2\tau)| &= |(\bar{Y}_c^{[k+1]} - \bar{Y}_c^{[k]})(1-2\tau)|, \\ &= |(Y_c^{[k+1]} - Y_c^{[k]})(1-2\tau)|, \\ &\leq \Phi_c^{[k+1]}(1-2\tau), \end{aligned}$$

where we have used the fact that $(1-2\tau)$ is the mesh point of $\bar{\Omega}_c^N$.
Using Lemma 3.2 we obtain

$$||Y_r^{[k+2]} - Y_r^{[k+1]}||_{\bar{\Omega}_r^N} \quad \leq \quad \Phi_c^{[k+1]}(1-2\tau).$$

Here we used

$$\Phi_c^{[k+1]}(1-2\tau) \quad = \quad C\nu^k \frac{m_1^{N/2} - m_2^{N/2}}{m_1^N - m_2^N},$$

$$
\begin{aligned}
&\le \frac{C\nu^k}{m_1^{N/2}},\\
&\le \frac{C\nu^k}{\left(1+\dfrac{\tau\alpha}{2\varepsilon N}\right)^{N/2}},\\
&= C\nu^k\left(1+\frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2},\\
&= C\nu^{k+1}.
\end{aligned}
$$

$$\therefore\ ||Y_r^{[k+2]}-Y_r^{[k+1]}||_{\bar{\Omega}_r^N}\le C\nu^{k+1}. \tag{21}$$

$$\text{Consequently, } ||Y_r^{[k+2]}-Y_r^{[k+1]}||_{\bar{\Omega}_r^N\setminus\Omega_c}\le C\nu^{k+1}. \tag{22}$$

Finally note that

$$L^N(Y_c^{[k+2]}-Y_c^{[k+1]})(x_i)=0 \quad \text{for } x_i\in\Omega_c^N, \quad (Y_c^{[k+2]}-Y_c^{[k+1]})(0)=0.$$

Using our inductive hypothesis and (21), we have

$$
\begin{aligned}
|(Y_c^{[k+2]}-Y_c^{[k+1]})(1-\tau)| &= |(\bar{Y}_r^{[k+2]}-\bar{Y}_r^{[k+1]})(1-\tau)|,\\
&= |(Y_r^{[k+2]}-Y_r^{[k+1]})(1-\tau)|,\\
&\le C\nu^{k+1},
\end{aligned}
$$

where we have used the fact that $(1-\tau)$ is the mesh point of $\bar{\Omega}_r^N$.
Therefore, we can apply Lemma 3.2 to get

$$||Y_c^{[k+2]}-Y_c^{[k+1]}||_{\bar{\Omega}_c^N}\le C\nu^{k+1}. \tag{23}$$

Combining the estimates (22) and (23) we obtain,

$$||Y^{[k+2]}-Y^{[k+1]}||_{\bar{\Omega}^N}\le C\nu^{k+1}.$$

For $\tau=2\varepsilon\ln N/\alpha$, using the arguments given in Lemma 5.1 of [8] we obtain,

$$
\begin{aligned}
\nu &= \left(1+\frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2},\\
&= \left(1+\frac{\ln N}{N}\right)^{-N/2}\le 2N^{-1/2},\ N\ge 1.
\end{aligned}
$$

□

The following theorem which is the main result of this paper, combining Lemmas 4.1 and 4.3 to prove that, two iterations are sufficient to attain first order convergence.

**Theorem 4.4.** *Let $y$ be the solution of (1)-(2) and $Y^{[k]}$ be the $k^{th}$ iterate of the discrete Schwarz method described in Section 3. If $\tau=\dfrac{2\varepsilon}{\alpha}\ln N$ and $N>2$, then*

$$||Y^{[k]}-y||_{\bar{\Omega}^N}\le CN^{-k/2}+CN^{-1}\ln^3 N,$$

*where $C$ is independent of $k$ and $N$.*

*Proof.* From Lemma 4.3 there exists $Y$ such that

$$Y := \lim_{k\to\infty} Y^{[k]}.$$

We know from Lemma 4.1 that there exists $C$ such that

$$||Y^{[k]} - y||_{\bar{\Omega}^N} \leq C2^{-k} + CN^{-1}\ln^3 N.$$

This implies that

$$||Y - y||_{\bar{\Omega}^N} \leq CN^{-1}\ln^3 N. \tag{24}$$

Also from Lemma 4.3 there exists $C$ such that

$$||Y^{[k+1]} - Y^{[k]}||_{\bar{\Omega}^N} \leq CN^{-k/2}.$$

Consequently, for $N \geq 2$, there exists $C$ such that

$$\begin{aligned} ||Y^{[k]} - Y||_{\bar{\Omega}^N} &\leq C\sum_{l=k}^{\infty} N^{-l/2}, \\ &= C\left[\frac{N^{-k/2}}{1 - N^{-1/2}}\right], \\ &\leq CN^{-k/2}. \end{aligned} \tag{25}$$

Using (24) and (25), we can conclude that

$$\begin{aligned} ||Y^{[k]} - y||_{\bar{\Omega}^N} &= ||Y^{[k]} - Y + Y - y||_{\bar{\Omega}^N}, \\ &\leq ||Y^{[k]} - Y||_{\bar{\Omega}^N} + ||Y - y||_{\bar{\Omega}^N}, \\ &\leq CN^{-k/2} + CN^{-1}\ln^3 N. \end{aligned}$$

□

## 5. Numerical Experiments

In this section, we present two examples to illustrate the theoretical results for the BVP (1)-(2). The stopping criterion for the iterative procedure is taken to be

$$||Y^{[k+1]} - Y^{[k]}||_{\bar{\Omega}^N} \leq 10^{-14}.$$

We normally omit the superscript $k$ on the final Schwarz iterate and write simply $Y^N$. Let $Y^N$ be a Schwarz numerical approximation for the exact solution $y$ on the mesh $\Omega^N$ and $N$ is the number of mesh points. For a finite set of values of $\varepsilon = \{2^{-4}, \ldots, 2^{-35}\}$, we compute the maximum pointwise errors

$$D_\varepsilon^N = ||Y^N - y||_{\Omega^N}, \quad D^N = \max_\varepsilon D_\varepsilon^N.$$

From these quantities the order of convergence are computed from

$$p^N = \frac{\ln D^N - \ln D^{2N}}{\ln(2\ln N) - \ln(\ln(2N))}.$$

**Example 5.1.**

$$-\varepsilon y''(x) + y'(x) = 0, \quad x \in \Omega$$
$$y(0) = 0, \quad y(1) = 1.$$

In [5, 6], the authors consider the above problem as a test problem whose exact solution in closed form is easy to determine. The exact maximum pointwise errors at each of the mesh points in $\bar{\Omega}^N$ are given in Table 1, for various values of $\varepsilon$ and $N$ (as in [6]). From this table it is obvious that the maximum pointwise errors are unacceptably large. The iteration counts given in Table 2 (as in [6]), also increase with increasing $N$. These numerical results indicate that the method proposed in [8] does not produce satisfactory approximations to the solution of the convection-diffusion problem for example 5.1. In [5] the authors employ a different finite difference operator, but this is not sufficient to overcome the difficulty of Schwarz method with the chosen decomposition of the domain of Shishkin mesh type overlap.

The authors of [5] and [6] conclude that the discrete Schwarz iterates constructed by them did not converge to the solutions of continuous problem. But it is not true in our case, because of our new scheme proposed in the present paper, the discrete Schwarz iterates converges to the solutions of the continuous problem. We checked this for example 5.1. Numerical approximations and iteration counts $(k)$ are given in Table 3 and 4. Also we have presented the graphs (Fig. 1, Fig. 2) of comparison of the exact and numerical solutions of the example 5.1 for various values of $\varepsilon$ and $N$. Fig. 3 and Fig. 4 represent the graphs of comparison of the exact and the numerical solution at iterations, $k = 1, 2$ for $\varepsilon = 2^{-10}$, $N = 16$ and $N = 64$ respectively.

**Example 5.2.**

$$-\varepsilon y''(x) + (1 + x)y'(x) = e^{-1/\varepsilon}, \quad x \in \Omega$$
$$y(0) = 0, \quad y(1) = 1.$$

Numerical approximations and iteration counts $(k)$ are given in Table 5 and 6. Also we have presented the graph (Fig. 5) of example 5.2.

From the above examples number of iterations taken by using proposed scheme in this method is not more than two which is very much reduced when comparing iteration counts presented in [5, 6]. This illustrates the efficiency of the new scheme proposed in this paper.

TABLE 1. *Computed nodal maximum pointwise errors $D_\varepsilon^N$, when the standard Schwarz method with a uniform mesh in each subdomain is applied to example 5.1 as in* [6].

| Number of mesh points N | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\varepsilon$ | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| $2^0$ | 4.78e-03 | 2.45e-03 | 1.24e-03 | 6.25e-04 | 3.14e-04 | 1.57e-04 | 7.86e-05 |
| $2^{-2}$ | 4.69e-02 | 2.52e-02 | 1.31e-02 | 2.32e-02 | 1.36e-02 | 7.83e-03 | 4.44e-03 |
| $2^{-4}$ | 9.97e-02 | 5.89e-02 | 3.96e-02 | 2.54e-02 | 1.40e-02 | 7.88e-03 | 4.44e-03 |
| $2^{-6}$ | 3.23e-01 | 8.96e-02 | 4.03e-02 | 6.51e-02 | 186e-02 | 8.26e-03 | 4.48e-03 |
| $2^{-8}$ | 6.84e-01 | 2.99e-01 | 7.86e-02 | 2.23e-01 | 5.38e-02 | 1.40e-02 | 5.03e-03 |
| $2^{-10}$ | 8.90e-01 | 6.51e-01 | 2.61e-01 | 6.51e-02 | 1.86e-02 | 8.26e-03 | 4.48e-03 |
| $2^{-12}$ | 9.59e-01 | 8.82e-01 | 6.00e-01 | 2.23e-01 | 5.38e-02 | 1.40e-02 | 5.03e-03 |
| $2^{-14}$ | 9.78e-01 | 9.65e-01 | 8.58e-01 | 5.44e-01 | 1.91e-01 | 4.56e-02 | 1.10e-02 |
| $2^{-16}$ | 9.83e-01 | 9.88e-01 | 8.28e-01 | 9.60e-01 | 4.91e-01 | 1.65e-01 | 3.98e-02 |
| $2^{-18}$ | 9.84e-01 | 9.94e-01 | 9.89e-01 | 9.51e-01 | 7.95e-01 | 4.46e-01 | 1.46e-01 |
| $2^{-20}$ | 9.84e-01 | 9.96e-01 | 9.96e-01 | 9.87e-01 | 9.40e-01 | 7.63e-01 | 4.08e-01 |
| $2^{-22}$ | 9.84e-01 | 9.96e-01 | 9.98e-01 | 9.97e-01 | 9.84e-01 | 9.28e-01 | 7.34e-01 |
| $2^{-24}$ | 9.84e-01 | 9.96e-01 | 9.99e-01 | 9.99e-01 | 9.96e-01 | 9.81e-01 | 9.17e-01 |
| $2^{-26}$ | 9.84e-01 | 9.96e-01 | 9.99e-01 | 1.00e+00 | 1.00e+00 | 9.99e-01 | 9.94e-01 |
| $2^{-28}$ | 9.84e-01 | 9.96e-01 | 9.99e-01 | 1.00e+00 | 1.00e+00 | 9.99e-01 | 9.94e-01 |
| $2^{-30}$ | **9.84e-01** | **9.96e-01** | **9.99e-01** | **1.00e+00** | **1.00e+00** | **1.00e+00** | **9.99e-01** |

TABLE 2. *Computed Iteration counts for various of $\varepsilon$and N, when the standard Schwarz with a uniform mesh in each subdomain is applied to example 5.1 as in* [6].

| Number of mesh points N | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| $2^0$ | 16 | 16 | 16 | 16 | 16 | 16 | 16 | 16 |
| $2^{-2}$ | 13 | 13 | 12 | 12 | 12 | 12 | 12 | 12 |
| $2^{-4}$ | 16 | 10 | 7 | 6 | 5 | 4 | 4 | 4 |
| $2^{-6}$ | 44 | 21 | 11 | 7 | 6 | 5 | 4 | 4 |
| $2^{-8}$ | 82 | 70 | 34 | 12 | 8 | 6 | 4 | 4 |
| $2^{-10}$ | 103 | 140 | 116 | 57 | 22 | 9 | 6 | 4 |
| $2^{-12}$ | 110 | 186 | 254 | 197 | 97 | 40 | 14 | 7 |
| $2^{-14}$ | 112 | 203 | 358 | 466 | 339 | 165 | 70 | 28 |
| $2^{-16}$ | 112 | 207 | 399 | 705 | 856 | 585 | 283 | 123 |
| $2^{-18}$ | 112 | 208 | 411 | 808 | 1381 | 1561 | 1017 | 489 |
| $2^{-20}$ | 112 | 209 | 414 | 838 | 1629 | 2669 | 2828 | 1774 |
| $2^{-22}$ | 112 | 209 | 415 | 846 | 1706 | 3243 | 5085 | 5097 |
| $2^{-24}$ | 112 | 209 | 415 | 848 | 1727 | 3427 | 6351 | 9574 |
| $2^{-26}$ | 112 | 209 | 415 | 849 | 1732 | 3476 | 6773 | 12266 |
| $2^{-28}$ | 112 | 209 | 415 | 849 | 1733 | 3489 | 6887 | 13194 |

TABLE 3. *Computed nodal maximum pointwise errors $D_{\varepsilon}^{N}$, when the new scheme proposed in Schwarz method with a uniform mesh in each subdomain is applied to example 5.1.*

| | Number of mesh points N | | |
|---|---|---|---|
| $\varepsilon$ | 16 | 32 | 64 |
| $2^{-4}$ | **4.528739968419103e-005** | **3.241793964611053e-005** | **2.742391726118322e-005** |
| $2^{-5}$ | 3.051757811233613e-005 | 1.470766842011089e-006 | 7.729764832148481e-008 |
| $2^{-6}$ | 3.051757812500001e-005 | 1.470766854675236e-006 | 7.729766098564910e-008 |
| $2^{-7}$ | 3.051757812500001e-005 | 1.470766854675236e-006 | 7.729766098564856e-008 |
| $2^{-8}$ | 3.051757812500001e-005 | 1.470766854675215e-006 | 7.729766098565076e-008 |
| $2^{-9}$ | 3.051757812500088e-005 | 1.470766854675257e-006 | 7.729766098565076e-008 |
| $2^{-10}$ | 3.051757812499741e-005 | 1.470766854675173e-006 | 7.729766098564197e-008 |
| $2^{-11}$ | 3.051757812499741e-005 | 1.470766854675340e-006 | 7.729766098565076e-008 |
| $2^{-12}$ | 3.051757812500435e-005 | 1.470766854675340e-006 | 7.729766098563318e-008 |
| $2^{-13}$ | 3.051757812500435e-005 | 1.470766854675340e-006 | 7.729766098570347e-008 |
| $2^{-14}$ | 3.051757812503211e-005 | 1.470766854676678e-006 | 7.729766098556288e-008 |
| $2^{-15}$ | 3.051757812497660e-005 | 1.470766854676678e-006 | 7.729766098570347e-008 |
| $2^{-16}$ | 3.051757812486557e-005 | 1.470766854665977e-006 | 7.729766098598469e-008 |
| $2^{-17}$ | 3.051757812486557e-005 | 1.470766854676678e-006 | 7.729766098542227e-008 |
| $2^{-18}$ | 3.051757812530966e-005 | 1.470766854676678e-006 | 7.729766098542227e-008 |
| $2^{-19}$ | 3.051757812530966e-005 | 1.470766854676678e-006 | 7.729766098542227e-008 |
| $2^{-20}$ | 3.051757812708602e-005 | 1.470766854676678e-006 | 7.729766098992158e-008 |
| $2^{-21}$ | 3.051757812708602e-005 | 1.470766854847898e-006 | 7.729766098092295e-008 |
| $2^{-22}$ | 3.051757811998059e-005 | 1.470766854505458e-006 | 7.729766096292568e-008 |
| $2^{-23}$ | 3.051757813419145e-005 | 1.470766855190337e-006 | 7.729766096292568e-008 |
| $2^{-24}$ | 3.051757810576974e-005 | 1.470766852450821e-006 | 7.729766110690380e-008 |
| $2^{-25}$ | 3.051757804892632e-005 | 1.470766855190337e-006 | 7.729766096292568e-008 |
| $2^{-26}$ | 3.051757816261316e-005 | 1.470766855190337e-006 | 7.729766096292568e-008 |
| $2^{-27}$ | 3.051757816261316e-005 | 1.470766866148404e-006 | 7.729766096292568e-008 |
| $2^{-28}$ | 3.051757816261316e-005 | 1.470766844232270e-006 | 7.729766211475060e-008 |
| $2^{-29}$ | 3.051757907210787e-005 | 1.470766888064539e-006 | 7.729766211475060e-008 |
| $2^{-30}$ | 3.051757725311847e-005 | 1.470766712735473e-006 | 7.729765290015176e-008 |
| $2^{-31}$ | 3.051757725311847e-005 | 1.470766888064539e-006 | 7.729765290015176e-008 |
| $2^{-32}$ | 3.051757725311847e-005 | 1.470766888064539e-006 | 7.729765290015176e-008 |
| $2^{-33}$ | 3.051756270120713e-005 | 1.470766888064539e-006 | 7.729768975855370e-008 |
| $2^{-34}$ | 3.051759180503675e-005 | 1.470765485432601e-006 | 7.729761604176741e-008 |
| $2^{-35}$ | 3.051753359740525e-005 | 1.470768290697814e-006 | 7.729761604176741e-008 |
| $p^{N}$ | 7.113070154773608e-001 | 3.275020460528117e-001 | 1.552700457434692e-001 |
| $k$ | 2 | 2 | 2 |

TABLE 4. *Computed nodal maximum pointwise errors $D_\varepsilon^N$, when the new scheme proposed in Schwarz method with a uniform mesh in each subdomain is applied to example 5.1.*

| | Number of mesh points N | | |
|---|---|---|---|
| $\varepsilon$ | 128 | 256 | 512 |
| $2^{-4}$ | **2.522222408662630e-005** | **2.418829880664464e-005** | **2.368726861785616e-005** |
| $2^{-5}$ | 4.335200262317575e-009 | 5.905177357288623e-010 | 5.664178695156151e-010 |
| $2^{-6}$ | 4.335212926483100e-009 | 2.539036173059733e-010 | 1.527869733109676e-011 |
| $2^{-7}$ | 4.335212926483070e-009 | 2.539036173059715e-010 | 1.527869733109676e-011 |
| $2^{-8}$ | 4.335212926483131e-009 | 2.539036173059715e-010 | 1.527869733109655e-011 |
| $2^{-9}$ | 4.335212926483008e-009 | 2.539036173059787e-010 | 1.527869733109655e-011 |
| $2^{-10}$ | 4.335212926482761e-009 | 2.539036173059931e-010 | 1.527869733109741e-011 |
| $2^{-11}$ | 4.335212926483254e-009 | 2.539036173059643e-010 | 1.527869733109568e-011 |
| $2^{-12}$ | 4.335212926484240e-009 | 2.539036173059065e-010 | 1.527869733110262e-011 |
| $2^{-13}$ | 4.335212926482269e-009 | 2.539036173059065e-010 | 1.527869733110262e-011 |
| $2^{-14}$ | 4.335212926486211e-009 | 2.539036173059065e-010 | 1.527869733107483e-011 |
| $2^{-15}$ | 4.335212926470440e-009 | 2.539036173068302e-010 | 1.527869733110262e-011 |
| $2^{-16}$ | 4.335212926501983e-009 | 2.539036173059065e-010 | 1.527869733115821e-011 |
| $2^{-17}$ | 4.335212926470440e-009 | 2.539036173059065e-010 | 1.527869733115821e-011 |
| $2^{-18}$ | 4.335212926470440e-009 | 2.539036173059065e-010 | 1.527869733093587e-011 |
| $2^{-19}$ | 4.335212926470440e-009 | 2.539036173059065e-010 | 1.527869733138054e-011 |
| $2^{-20}$ | 4.335212926218097e-009 | 2.539036173206857e-010 | 1.527869733138054e-011 |
| $2^{-21}$ | 4.335212926218097e-009 | 2.539036173206857e-010 | 1.527869732782319e-011 |
| $2^{-22}$ | 4.335212927227468e-009 | 2.539036172615691e-010 | 1.527869733138054e-011 |
| $2^{-23}$ | 4.335212927227468e-009 | 2.539036172615691e-010 | 1.527869733849524e-011 |
| $2^{-24}$ | 4.335212931264949e-009 | 2.539036174980353e-010 | 1.527869733849524e-011 |
| $2^{-25}$ | 4.335212923189986e-009 | 2.539036170251030e-010 | 1.527869731003645e-011 |
| $2^{-26}$ | 4.335212923189986e-009 | 2.539036170251030e-010 | 1.527869725311887e-011 |
| $2^{-27}$ | 4.335212955489839e-009 | 2.539036170251030e-010 | 1.527869748078920e-011 |
| $2^{-28}$ | 4.335212890890133e-009 | 2.539036170251030e-010 | 1.527869702544854e-011 |
| $2^{-29}$ | 4.335212890890133e-009 | 2.539036170251030e-010 | 1.527869748078920e-011 |
| $2^{-30}$ | 4.335212890890133e-009 | 2.539036170251030e-010 | 1.527869839147056e-011 |
| $2^{-31}$ | 4.335212374092515e-009 | 2.539036170251030e-010 | 1.527869474874543e-011 |
| $2^{-32}$ | 4.335213407687812e-009 | 2.539035564897706e-010 | 1.527869839147056e-011 |
| $2^{-33}$ | 4.335211340497464e-009 | 2.539035564897706e-010 | 1.527869110602118e-011 |
| $2^{-34}$ | 4.335215474879147e-009 | 2.539035564897706e-010 | 1.527869110602118e-011 |
| $2^{-35}$ | 4.335223743654342e-009 | 2.539035564897706e-010 | 1.527872024783956e-011 |
| $p^N$ | 7.479509893086782e-002 | 3.637918932637768e-002 | |
| $k$ | 2 | 2 | 2 |

TABLE 5. *Computed nodal maximum pointwise errors $D_\varepsilon^N$, when the new scheme proposed in Schwarz method with a uniform mesh in each subdomain is applied to example 5.2.*

| Number of mesh points N | | | |
|---|---|---|---|
| $\varepsilon$ | 16 | 32 | 64 |
| $2^{-4}$ | **9.973696333253496e-007** | **7.646444355134572e-007** | **6.715263892627045e-007** |
| $2^{-5}$ | 2.730406312569372e-008 | 6.065170063375438e-010 | 2.603856665331250e-011 |
| $2^{-6}$ | 5.042707030935151e-009 | 3.622156646483836e-011 | 3.944835777638760e-013 |
| $2^{-7}$ | 2.167114878146073e-009 | 8.851715523369058e-012 | 4.854905899901432e-014 |
| $2^{-8}$ | 1.420662876196369e-009 | 4.375800948326315e-012 | 1.703165144981671e-014 |
| $2^{-9}$ | 1.150258843704314e-009 | 3.076611174090747e-012 | 1.008775980293497e-014 |
| $2^{-10}$ | 1.035017887667948e-009 | 2.579764965556072e-012 | 7.763610144571725e-015 |
| $2^{-11}$ | 9.818021816617765e-010 | 2.362293768211346e-012 | 6.810801325219683e-015 |
| $2^{-12}$ | 9.562293321107735e-010 | 2.260532660417098e-012 | 6.379188837374893e-015 |
| $2^{-13}$ | 9.436937869373168e-010 | 2.211307949009760e-012 | 6.173750603508164e-015 |
| $2^{-14}$ | 9.374877744824940e-010 | 2.187099027898693e-012 | 6.073525975676230e-015 |
| $2^{-15}$ | 9.344000897878102e-010 | 2.175094137197675e-012 | 6.024025465001347e-015 |
| $2^{-16}$ | 9.328600631023310e-010 | 2.169116424841472e-012 | 5.999426705680422e-015 |
| $2^{-17}$ | 9.320910018464335e-010 | 2.166133732134631e-012 | 5.987165019407192e-015 |
| $2^{-18}$ | 9.317067090337078e-010 | 2.164643924166083e-012 | 5.981043577290349e-015 |
| $2^{-19}$ | 9.315146220330612e-010 | 2.163899404390695e-012 | 5.977985203525555e-015 |
| $2^{-20}$ | 9.314185934924161e-010 | 2.163527240599339e-012 | 5.976456604279724e-015 |
| $2^{-21}$ | 9.313705829344505e-010 | 2.163341183084948e-012 | 5.975692448456694e-015 |
| $2^{-22}$ | 9.313465779871744e-010 | 2.163248159194747e-012 | 5.975310405794378e-015 |
| $2^{-23}$ | 9.313345769381871e-010 | 2.163201651771628e-012 | 5.975119397796284e-015 |
| $2^{-24}$ | 9.313285738695823e-010 | 2.163178386850970e-012 | 5.975023912780948e-015 |
| $2^{-25}$ | 9.313255710487433e-010 | 2.163166770601318e-012 | 5.974976134675396e-015 |
| $2^{-26}$ | 9.313240757134963e-010 | 2.163160960485284e-012 | 5.974952251330496e-015 |
| $2^{-27}$ | 9.313233263120483e-010 | 2.163158075579098e-012 | 5.974940320823040e-015 |
| $2^{-28}$ | 9.31322965489340e-010 | 2.163156592835516e-012 | 5.974934622678630e-015 |
| $2^{-29}$ | 9.313227989558353e-010 | 2.163156012631783e-012 | 5.974931417474788e-015 |
| $2^{-30}$ | 9.313226324223564e-010 | 2.163154981158863e-012 | 5.974928568406150e-015 |
| $2^{-31}$ | 9.313225214000537e-010 | 2.163155239027047e-012 | 5.974928568406150e-015 |
| $2^{-32}$ | 9.313225214000537e-010 | 2.163155239027047e-012 | 5.974927143872341e-015 |
| $2^{-33}$ | 9.313216332221082e-010 | 2.163155239027047e-012 | 5.974932842009617e-015 |
| $2^{-34}$ | 9.313234095788461e-010 | 2.163151113139794e-012 | 5.974921445740498e-015 |
| $2^{-35}$ | 9.313198568687584e-010 | 2.163159364922169e-012 | 5.974921445740498e-015 |
| $p^N$ | 5.653371478035641e-001 | 2.542112521190105e-001 | 1.191897052610551e-001 |
| $k$ | 2 | 2 | 2 |

TABLE 6. *Computed nodal maximum pointwise errors $D_\varepsilon^N$, when the new scheme proposed in Schwarz method with a uniform mesh in each subdomain is applied to example 5.2.*

| Number of mesh points N | | | |
|---|---|---|---|
| $\varepsilon$ | 128 | 256 | 512 |
| $2^{-4}$ | **6.297421690292230e-007** | **6.099348336990207e-007** | **6.002898484073085e-007** |
| $2^{-5}$ | 2.018098282743000e-012 | 4.603565822072004e-013 | 4.474237919406331e-013 |
| $2^{-6}$ | 6.170211470271201e-015 | 1.323841346302853e-016 | 3.775808596125031e-018 |
| $2^{-7}$ | 3.405339824232550e-016 | 2.921371974521934e-018 | 2.968870901312856e-020 |
| $2^{-8}$ | 8.000012427327956e-017 | 4.339726074748408e-019 | 2.632582464869418e-021 |
| $2^{-9}$ | 3.877535331929915e-017 | 1.672630635591109e-019 | 7.839300633851615e-022 |
| $2^{-10}$ | 2.699530973903139e-017 | 1.038410119362168e-019 | 4.277844437671695e-022 |
| $2^{-11}$ | 2.252447049514132e-017 | 8.181884492381085e-020 | 3.160085414831977e-022 |
| $2^{-12}$ | 2.057489976598139e-017 | 7.262657441635648e-020 | 2.716037352958272e-022 |
| $2^{-13}$ | 1.966433648629274e-017 | 6.842529340607499e-020 | 2.517991135533304e-022 |
| $2^{-14}$ | 1.922427991920145e-017 | 6.641668915154731e-020 | 2.424451083615993e-022 |
| $2^{-15}$ | 1.900795843820936e-017 | 6.543460715307705e-020 | 2.378992323786804e-022 |
| $2^{-16}$ | 1.890071222739985e-017 | 6.494902529660335e-020 | 2.356583583860070e-022 |
| $2^{-17}$ | 1.884731624845231e-017 | 6.470758733489777e-020 | 2.345458492648709e-022 |
| $2^{-18}$ | 1.882067485410859e-017 | 6.458720512729605e-020 | 2.339915657488776e-022 |
| $2^{-19}$ | 1.880736828142638e-017 | 6.452709803528843e-020 | 2.337149154067038e-022 |
| $2^{-20}$ | 1.880071852240611e-017 | 6.449706548045581e-020 | 2.335767129137230e-022 |
| $2^{-21}$ | 1.879739452408912e-017 | 6.448205444141129e-020 | 2.335076422213073e-022 |
| $2^{-22}$ | 1.879573275516678e-017 | 6.447455020579473e-020 | 2.334731146299111e-022 |
| $2^{-23}$ | 1.879490192142066e-017 | 6.447079842299368e-020 | 2.334558529118597e-022 |
| $2^{-24}$ | 1.879448655332727e-017 | 6.446892276356613e-020 | 2.334472226944896e-022 |
| $2^{-25}$ | 1.879427882021234e-017 | 6.446798456405261e-020 | 2.334429066183903e-022 |
| $2^{-26}$ | 1.879417498952277e-017 | 6.446751552945592e-020 | 2.334407464361784e-022 |
| $2^{-27}$ | 1.879412317941345e-017 | 6.446728113351692e-020 | 2.334396750488739e-022 |
| $2^{-28}$ | 1.879409657427768e-017 | 6.446716393586701e-020 | 2.334391254430091e-022 |
| $2^{-29}$ | 1.879408425191176e-017 | 6.446710437648722e-020 | 2.334388610760919e-022 |
| $2^{-30}$ | 1.879407753062467e-017 | 6.446707747872084e-020 | 2.334387636778295e-022 |
| $2^{-31}$ | 1.879407080933998e-017 | 6.446706210857366e-020 | 2.334385688814266e-022 |
| $2^{-32}$ | 1.879407529019617e-017 | 6.446701599815411e-020 | 2.334386245375251e-022 |
| $2^{-33}$ | 1.879405736677782e-017 | 6.446701599815411e-020 | 2.334384019132106e-022 |
| $2^{-34}$ | 1.879409321363163e-017 | 6.446701599815411e-020 | 2.334384019132106e-022 |
| $2^{-35}$ | 1.879416490754436e-017 | 6.446701599815411e-020 | 2.334392924117426e-022 |
| $p^N$ | 5.710767876545902e-002 | 2.770332901541028e-002 | |
| $k$ | 2 | 2 | 2 |

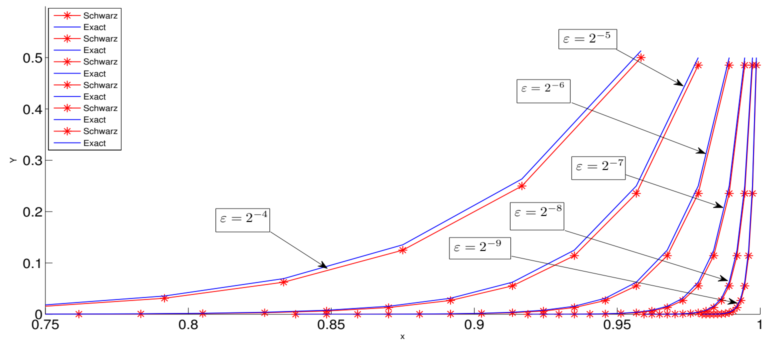


FIGURE 1. Comparison of the exact and the numerical solution of example 5.1 with $N = 16$ and $\alpha = 1$ (within the layer region ).

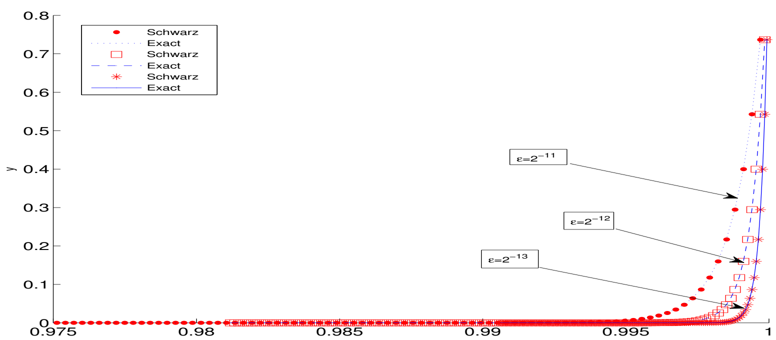


FIGURE 2. Comparison of the exact and the numerical solution of example 5.1 with $N = 64$ and $\alpha = 1$ (within the layer region ).

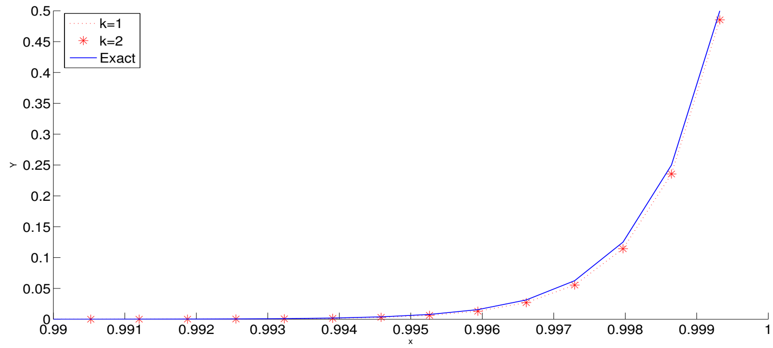


FIGURE 3. Comparison of the exact and the numerical solution at iterations, $k = 1, 2$ of example 5.1 with $N = 16$, $\varepsilon = 2^{-10}$, $\alpha = 1$ (within the layer region ).

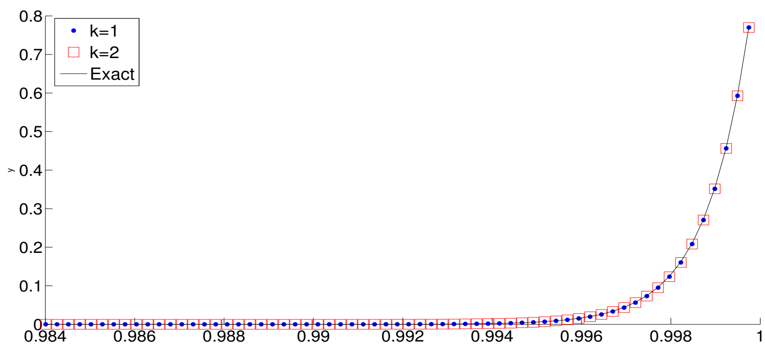


FIGURE 4. Comparison of the exact and the numerical solution at iterations, $k = 1, 2$ of example 5.1 with $N = 64$, $\varepsilon = 2^{-10}$, $\alpha = 1$ (within the layer region ).

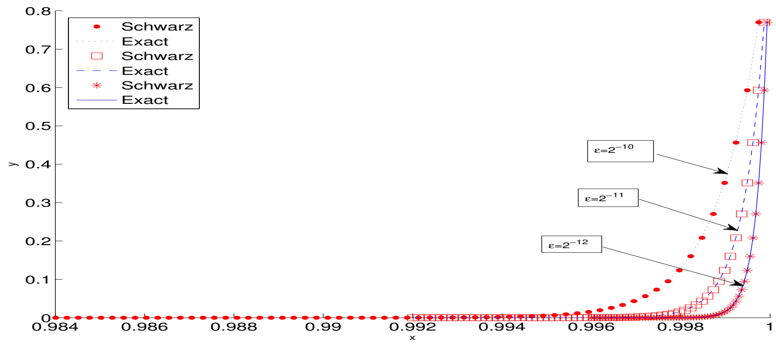


FIGURE 5. Comparison of the exact and the numerical solution of example 5.2 with $N = 64$ and $\alpha = 1$ (within the layer region ).

## 6. Conclusion

A singularly perturbed second order convection-diffusion equations is considered. It is shown that a designed discrete Schwarz method produced numerical approximations which converged in the maximum norm to the exact solution. This convergence is shown to be of first order. Note that from Theorem 4.4, for $k \geq 2$ the $N^{-1} + N^{-1}\ln^3 N$ term dominated the error bound. Thus, two iterations are sufficient to attain the desired accuracy. Numerical experiments validated the theoretical results.

In [5, 6], the authors used same scheme in both the domains $\Omega_r$ and $\Omega_c$ whereas in our case, we used different schemes in each subdomain $\Omega_r$ and $\Omega_c$, which helped us to overcome the fundamental difficulty mentioned in [5, 6].

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