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WEIGHTED INTEGRAL INEQUALITIES FOR GG-CONVEX FUNCTIONS

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Abstract. Some weighted integral inequalities of Hermite-Hadamard type for GG-convex functions defined on positive intervals are given. Applications for special means are also provided.

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1. Introduction

The function $f : I \subset (0, \infty) \to (0, \infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [4]

$$f\left(x^{1-\lambda}y^{\lambda}\right) \leq [f(x)]^{1-\lambda}[f(y)]^{\lambda} \tag{1}$$

for any $x,\ y \in I$ and $\lambda \in [0,1]$. If the inequality is reversed in (1) then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [53], however, the roots of the research in this area can be traced long before him [54].

It is easy to see that [54], the function $f : I \subset (0, \infty) \to (0, \infty)$ is *GG-convex* if and only if the the function $g : \ln I \to \mathbb{R}$, $g = \ln \circ f \circ \exp$ is convex on $\ln I$.

It is known that [54] every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients c_n is a GG-convex function on $(0, r)$, where r is the radius of convergence for f. Therefore functions like exp, sinh, cosh are GG-convex on $\mathbb{R}$, tan, sec, csc, $\frac{1}{x} - \cot x$ are GG-convex on $\left(0, \frac{\pi}{2}\right)$ and $\frac{1}{1-x}$, $\ln \frac{1}{1-x}$ or $\frac{1+x}{1-x}$ are GG-convex on $(0, 1)$. Also, the Γ function is a strictly GG-convex function on $[1, \infty)$.

It is also known that [54], if a function f is GG-convex, then so is $x^{\alpha} f^{\beta}(x)$ for all $\alpha \in \mathbb{R}$ and all $\beta > 0$. If f is continuous, and one of the functions $f(x)^x$ and $f(e^{1/logx})$ is GG-convex, then so is the other.

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Recently [30], we obtained the following weighted inequality:

Theorem 1.1. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$, then*

$$\begin{aligned} & f\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}\right)\right) \\ & \le \exp\left(\frac{\int_a^b w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right) \\ & \le [f(a)]^{\frac{\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt}}{\ln b - \ln a}} [f(b)]^{\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt} - \ln a}{\ln b - \ln a}} . \end{aligned} \tag{2}$$

One can observe that, by the weighted geometric mean - arithmetic mean inequality

$$\alpha^{1-\lambda}\beta^{\lambda} \le (1-\lambda)\alpha + \lambda\beta,\ \ \alpha,\beta > 0 \text{ and } \lambda \in [0,1],$$

we have the further upper bound

$$\begin{aligned} & [f(a)]^{\frac{\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt}}{\ln b - \ln a}} [f(b)]^{\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt} - \ln a}{\ln b - \ln a}} \\ & \le \left(\frac{\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt}}{\ln b - \ln a}\right) f(a) + \left(\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt} - \ln a}{\ln b - \ln a}\right) f(b). \end{aligned}$$

We define the *p-logarithmic mean* of two positive numbers a, b by

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if } b \ne a, \\ \\ b \text{ if } b = a. \end{cases}$$

In particular, by taking $w(t) = t^p$ in (2), we have for any GG-convex function $f : [a,b] \subset (0,\infty) \to (0,\infty)$ that

$$\begin{aligned} & \left[f\left(\left[I\left(a^{p+1}, b^{p+1}\right)\right]^{\frac{1}{p+1}}\right)\right]^{L_p^p(a,b)} \\ & \le \exp\left(\frac{1}{b-a}\int_a^b t^p \ln f(t)\,dt\right) \\ & \le [f(a)]^{\frac{L\left(a^{p+1}, b^{p+1}\right) - a^{p+1}}{(p+1)(b-a)}} [f(b)]^{\frac{b^{p+1} - L\left(a^{p+1}, b^{p+1}\right)}{(p+1)(b-a)}}, \end{aligned} \tag{3}$$

for any $p \in \mathbb{R}$ with $p \ne 0, -1$.

We recall that the *logarithmic mean* is defined by $L(p,q) := \frac{p-q}{\ln p - \ln q}$ if $p \ne q$ and $L(p,p) := p$.

If $p=0$, namely we take $w(t)=1$ in (2), then we get

$$f(I(a,b)) \le \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right) \le [f(b)]^{\frac{b-L(a,b)}{b-a}}[f(a)]^{\frac{L(a,b)-a}{b-a}} \quad (4)$$

that has been obtained by Mitroi and Spiridon in [52].

If $p=-1$, namely we take $w(t)=\frac{1}{t}$ in (2), then we get

$$f\left(\sqrt{ab}\right) \le \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}dt\right) \le \sqrt{f(a)f(b)}. \quad (5)$$

If $p=1$ in (18) then we also have

$$\begin{aligned} f\left(\sqrt{I(a^2,b^2)}\right) &\le \exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right) \quad (6)\\ &\le [f(a)]^{\frac{A(a,b)L(a,b)-a^2}{2(b-a)}}[f(b)]^{\frac{b^2-A(a,b)L(a,b)}{2(b-a)}}. \end{aligned}$$

We recall that the *identric mean* is defined by $I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ for $b \ne a$ and $I(a,a) := a$.

We also recall the classical *Hermite-Hadamard inequality* that states that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2} \quad (7)$$

for any convex function $f:[a,b]\to\mathbb{R}$.

For related results, see [1]-[21], [24]-[31], [32]-[41] and [42]-[57].

Motivated by the above results we establish here some new weighted integral inequalities of Hermite-Hadamard type for GG-convex functions defined on positive intervals. Applications for special means are also provided.

2. New Results for General Weights

We have:

Theorem 2.1. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$, then*

$$f\left(\sqrt{ab}\right) \le \left(\frac{\int_a^b w(t)f^p(t)f^p\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2p}} \le \sqrt{f(a)f(b)} \quad (8)$$

for any $p>0$.

Proof. From the definition of GG-convex functions we have

$$f\left(x^{1-\lambda}y^{\lambda}\right) \le [f(x)]^{1-\lambda}[f(y)]^{\lambda} \quad (9)$$

and

$$f\left(x^{\lambda}y^{1-\lambda}\right) \le [f(x)]^{\lambda}[f(y)]^{1-\lambda} \quad (10)$$

for any $x,y\in[a,b]$ and $\lambda\in[0,1]$.

By multiplication of (9) with (10) we get

$$f\left(x^{1-\lambda}y^{\lambda}\right) f\left(x^{\lambda}y^{1-\lambda}\right) \leq f(x) f(y)$$

for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Therefore

$$f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right) \leq f(a) f(b) \tag{11}$$

for any $\lambda \in [0, 1]$.

From (9) we also have

$$f(\sqrt{xy}) \leq \sqrt{f(x) f(y)} \tag{12}$$

for any $x, y \in [a, b]$.

By taking $x = a^{1-\lambda}b^{\lambda}$, $y = a^{\lambda}b^{1-\lambda}$ in (12) and then squaring we get

$$f^2\left(\sqrt{ab}\right) \leq f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right). \tag{13}$$

Since for any $t \in [a, b]$ there is a unique $\lambda \in [0, 1]$ such that $t = a^{1-\lambda}b^{\lambda}$, we obtain from (11) and (13) that

$$f^2\left(\sqrt{ab}\right) \leq f(t) f\left(\frac{ab}{t}\right) \leq f(a) f(b) \tag{14}$$

for any $t \in [a, b]$.

If we take the power $p > 0$ in (14), multiply by $w(t) \geq 0$ for $t \in [a, b]$ and integrate, we get

$$\begin{aligned} f^{2p}\left(\sqrt{ab}\right) \int_a^b w(t)\, dt &\leq \int_a^b w(t) f^p(t) f^p\left(\frac{ab}{t}\right) dt \\ &\leq f^p(a) f^p(b) \int_a^b w(t)\, dt \end{aligned} \tag{15}$$

that is equivalent to

$$f^{2p}\left(\sqrt{ab}\right) \leq \frac{\int_a^b w(t) f^p(t) f^p\left(\frac{ab}{t}\right) dt}{\int_a^b w(t)\, dt} \leq f^p(a) f^p(b) \tag{16}$$

and by taking the power $\frac{1}{2p}$ we get the desired result (8). □

We observe that for $p = 1$ we get the inequality

$$f\left(\sqrt{ab}\right) \leq \left(\frac{\int_a^b w(t) f(t) f\left(\frac{ab}{t}\right) dt}{\int_a^b w(t)\, dt}\right)^{\frac{1}{2}} \leq \sqrt{f(a) f(b)}, \tag{17}$$

while from $p = \frac{1}{2}$ we get

$$f\left(\sqrt{ab}\right) \leq \frac{\int_a^b w(t) \sqrt{f(t) f\left(\frac{ab}{t}\right)} dt}{\int_a^b w(t)\, dt} \leq \sqrt{f(a) f(b)}. \tag{18}$$

If we take $p=\frac{1}{4}$ in (8), then we get

$$f\left(\sqrt{ab}\right)\leq\left(\frac{\int_a^b w(t)\sqrt[4]{f(t)f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt}\right)^2\leq\sqrt{f(a)f(b)}. \tag{19}$$

Using *Jensen's inequality* for the power $p\geq 1$ $(p\in(0,1))$, namely

$$\left(\frac{\int_a^b w(x)\,g(x)\,dx}{\int_a^b w(x)\,dx}\right)^p\leq(\geq)\,\frac{\int_a^b w(x)\,g^p(x)\,dx}{\int_a^b w(x)\,dx},$$

we can state the following more precise result:

Corollary 2.2. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$.*

(i) If $p\geq 1$, then

$$f\left(\sqrt{ab}\right)\leq\left(\frac{\int_a^b w(t)\,f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2}} \tag{20}$$
$$\leq\left(\frac{\int_a^b w(t)\,f^p(t)\,f^p\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2p}}\leq\sqrt{f(a)f(b)}.$$

(ii) If $p\in(0,1)$, then

$$f\left(\sqrt{ab}\right)\leq\left(\frac{\int_a^b w(t)\,f^p(t)\,f^p\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2p}} \tag{21}$$
$$\leq\left(\frac{\int_a^b w(t)\,f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2}}\leq\sqrt{f(a)f(b)}.$$

If we take in Corollary 2.2 $w(t)=1$, $t\in[a,b]$, then for any $f:[a,b]\subset(0,\infty)\to(0,\infty)$ a *GG*-convex function, we have for $p\geq 1$ that

$$f\left(\sqrt{ab}\right)\leq\left(\frac{1}{b-a}\int_a^b f(t)\,f\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2}} \tag{22}$$
$$\leq\left(\frac{1}{b-a}\int_a^b f^p(t)\,f^p\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2p}}\leq\sqrt{f(a)f(b)}$$

and for $p\in(0,1)$, that

$$f\left(\sqrt{ab}\right)\leq\left(\frac{1}{b-a}\int_a^b f^p(t)\,f^p\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2p}} \tag{23}$$
$$\leq\left(\frac{1}{b-a}\int_a^b f(t)\,f\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2}}\leq\sqrt{f(a)f(b)}.$$

If we take in Corollary 2.2 $w(t) = \frac{1}{t}$, $t \in [a,b]$, then for any $f : [a,b] \subset (0,\infty) \to (0,\infty)$ a GG-convex function, we have for $p \geq 1$ that

$$f\left(\sqrt{ab}\right) \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)\, f\left(\frac{ab}{t}\right)}{t} dt\right)^{\frac{1}{2}} \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f^p(t)\, f^p\left(\frac{ab}{t}\right)}{t} dt\right)^{\frac{1}{2p}} \leq \sqrt{f(a)\, f(b)} \tag{24}$$

and for $p \in (0,1)$, that

$$f\left(\sqrt{ab}\right) \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f^p(t)\, f^p\left(\frac{ab}{t}\right)}{t} dt\right)^{\frac{1}{2p}} \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)\, f\left(\frac{ab}{t}\right)}{t} dt\right)^{\frac{1}{2}} \leq \sqrt{f(a)\, f(b)}. \tag{25}$$

If we take $p = \frac{1}{2}$ in the first inequality in (25), then we get

$$f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t)\, f\left(\frac{ab}{t}\right)} dt \tag{26}$$

that has been obtained by Işcan in [45].

Theorem 2.3. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$, then*

$$\begin{aligned} & f\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\, dt}\right)\right) \\ & \leq \exp\left(\frac{\int_a^b w(t)\ln f(t)\, dt}{\int_a^b w(t)\, dt}\right) \\ & \leq \frac{\int_a^b w(t)\, f(t)\, dt}{\int_a^b w(t)\, dt} \leq \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b - \ln a}} \frac{\int_a^b w(t)\left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b - \ln a}} dt}{\int_a^b w(t)\, dt} \\ & \leq \left(\frac{\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt}}{\ln b - \ln a}\right) f(a) + \left(\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt} - \ln a}{\ln b - \ln a}\right) f(b). \end{aligned} \tag{27}$$

Proof. If we use Jensen's inequality for the exponential function and nonnegative weight w, we have

$$\exp\left(\frac{\int_a^b w(t)\ln f(t)\, dt}{\int_a^b w(t)\, dt}\right) \leq \frac{\int_a^b w(t)\exp(\ln f(t))\, dt}{\int_a^b w(t)\, dt}$$

$$= \frac{\int_a^b w(t) f(t) dt}{\int_a^b w(t) dt},$$

and the second inequality in (27) is proved.

Let $t = a^{1-\lambda} b^{\lambda} \in [a, b]$ with $\lambda \in [0, 1]$, then $\lambda = \frac{\ln t - \ln a}{\ln b - \ln a}$. By the GG-convexity of f we have

$$\begin{aligned} f(t) = f\left(a^{1-\lambda} b^{\lambda}\right) &\leq [f(a)]^{1-\lambda} [f(b)]^{\lambda} \\ &= [f(a)]^{\frac{\ln b - \ln t}{\ln b - \ln a}} [f(b)]^{\frac{\ln t - \ln a}{\ln b - \ln a}} \\ &= \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b - \ln a}} \left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b - \ln a}} \\ &\leq \frac{\ln b - \ln t}{\ln b - \ln a} f(a) + \frac{\ln t - \ln a}{\ln b - \ln a} f(b) \end{aligned} \tag{28}$$

for any $t \in [a, b]$.

Now, if we take the weighted integral mean in (28), then we get the last part of (27). □

By choosing $w(t) = 1$, $t \in [a, b]$ in (27), we deduce

$$\begin{aligned} & f\left(\exp\left(\frac{1}{b-a} \int_a^b \ln t dt\right)\right) \\ & \leq \exp\left(\frac{1}{b-a} \int_a^b \ln f(t) dt\right) \\ & \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b - \ln a}} \frac{\int_a^b \left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b - \ln a}} dt}{b-a} \\ & \leq \left(\frac{\ln b - \frac{1}{b-a} \int_a^b \ln t dt}{\ln b - \ln a}\right) f(a) + \left(\frac{\frac{1}{b-a} \int_a^b \ln t dt - \ln a}{\ln b - \ln a}\right) f(b), \end{aligned}$$

and since $\frac{1}{b-a} \int_a^b \ln t dt = \ln I(a, b)$, hence

$$\begin{aligned} f(I(a, b)) &\leq \exp\left(\frac{1}{b-a} \int_a^b \ln f(t) dt\right) \\ &\leq \frac{1}{b-a} \int_a^b f(t) dt \leq \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b - \ln a}} \frac{\int_a^b \left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b - \ln a}} dt}{b-a} \\ &\leq \frac{\ln b - \ln I(a, b)}{\ln b - \ln a} f(a) + \frac{\ln I(a, b) - \ln a}{\ln b - \ln a} f(b). \end{aligned} \tag{29}$$

If we take $w(t)=\frac{1}{t}$, $t\in[a,b]$ in (27), then we get

$$f\left(\exp\left(\frac{\int_a^b \frac{1}{t}\ln t dt}{\int_a^b \frac{1}{t}dt}\right)\right) \tag{30}$$
$$\leq \exp\left(\frac{\int_a^b \frac{1}{t}\ln f(t)\,dt}{\int_a^b \frac{1}{t}dt}\right)$$
$$\leq \frac{\int_a^b \frac{1}{t}f(t)\,dt}{\int_a^b \frac{1}{t}dt} \leq \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b-\ln a}} \frac{\int_a^b \frac{1}{t}\left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b-\ln a}} dt}{\int_a^b w(t)\,dt}$$
$$\leq \left(\frac{\ln b-\frac{\int_a^b \frac{1}{t}\ln t dt}{\int_a^b \frac{1}{t}dt}}{\ln b-\ln a}\right) f(a) + \left(\frac{\frac{\int_a^b \frac{1}{t}\ln t dt}{\int_a^b \frac{1}{t}dt}-\ln a}{\ln b-\ln a}\right) f(b).$$

This is equivalent, after suitable calculations, to

$$f\left(\sqrt{ab}\right) \leq \exp\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t}\ln f(t)\,dt\right) \tag{31}$$
$$\leq \frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t}f(t)\,dt \leq L(f(a),f(b))\left(\leq \frac{f(a)+f(b)}{2}\right).$$

The third inequality in (31) has been obtained in a different way by Işcan in [45].

3. Other Weighted Inequalities

We have:

Theorem 3.1. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$ and such that $w\left(\frac{ab}{t}\right)=w(t)$ for any $t\in[a,b]$. Then we have the inequalities*

$$f\left(\sqrt{ab}\right) \leq \exp\left(\frac{1}{2}\cdot\frac{\int_a^b \left(1+\frac{ab}{t^2}\right)w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right) \leq \sqrt{f(a)f(b)}. \tag{32}$$

Proof. By taking the log in (14) we get

$$2\ln f\left(\sqrt{ab}\right) \leq \ln f(t)+\ln f\left(\frac{ab}{t}\right) \leq \ln f(a)+\ln f(b) \tag{33}$$

for any $t\in[a,b]$.

If we multiply (33) by $w(t)\geq 0$ with $t\in[a,b]$ and use the fact that $w\left(\frac{ab}{t}\right)=w(t)$ for any $t\in[a,b]$, then we get

$$2w(t)\ln f\left(\sqrt{ab}\right) \leq w(t)\ln f(t)+w\left(\frac{ab}{t}\right)\ln f\left(\frac{ab}{t}\right) \tag{34}$$
$$\leq w(t)\left[\ln f(a)+\ln f(b)\right]$$

for any $t \in [a, b]$.

If we integrate the inequality (34) on $[a, b]$ we get

$$2 \ln f\left(\sqrt{ab}\right) \int_a^b w(t)\,dt \tag{35}$$
$$\le \int_a^b w(t) \ln f(t)\,dt + \int_a^b w\left(\frac{ab}{t}\right) \ln f\left(\frac{ab}{t}\right) dt$$
$$\le [\ln f(a) + \ln f(b)] \int_a^b w(t)\,dt.$$

By changing the variable $u = \frac{ab}{t}$, we have $dt = -\frac{ab}{u^2}du$ and

$$\int_a^b w\left(\frac{ab}{t}\right) \ln f\left(\frac{ab}{t}\right) dt = -\int_b^a w(u) \ln f(u) \frac{ab}{u^2} du = \int_a^b w(t) \ln f(t) \frac{ab}{t^2} dt$$

and by (35) we get

$$2 \ln f\left(\sqrt{ab}\right) \int_a^b w(t)\,dt \le \int_a^b w(t) \ln f(t)\,dt + \int_a^b w(t) \ln f(t) \frac{ab}{t^2} dt$$
$$\le [\ln f(a) + \ln f(b)] \int_a^b w(t)\,dt,$$

which is equivalent to the desired result (32). □

If we take in (32) $w(t) = 1$, $t \in [a, b]$, then we get

$$f\left(\sqrt{ab}\right) \le \exp\left(\frac{1}{2} \cdot \frac{\int_a^b \left(1 + \frac{ab}{t^2}\right) \ln f(t)\,dt}{b - a}\right) \le \sqrt{f(a) f(b)}. \tag{36}$$

Another example of weight w that satisfies the condition $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a, b]$ is $w(t) = \left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right|$, with $t \in [a, b] \subset (0, \infty)$.

The following result also holds:

Theorem 3.2. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be a GG-convex function on $[a, b]$ and $w : [a, b] \to [0, \infty)$ an integrable function on $[a, b]$ and such that $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a, b]$. Then we have the inequalities*

$$f\left(\sqrt{ab}\right) \le \exp\left(\frac{\int_a^b \frac{1}{t} w(t) \ln f(t)\,dt}{\int_a^b \frac{1}{t} w(t)\,dt}\right) \le \sqrt{f(a) f(b)}. \tag{37}$$

Proof. From (33) for $t = a^{1-\lambda} b^{\lambda}$ with $\lambda \in [0, 1]$, we have

$$2 \ln f\left(\sqrt{ab}\right) \le \ln f\left(a^{1-\lambda} b^{\lambda}\right) + \ln f\left(a^{\lambda} b^{1-\lambda}\right) \le \ln f(a) + \ln f(b) \tag{38}$$

for any $\lambda \in [0, 1]$.

Since $w\left(\frac{ab}{t}\right)=w(t)$ for any $t\in[a,b]$, then $w\left(a^{1-\lambda}b^{\lambda}\right)=w\left(a^{\lambda}b^{1-\lambda}\right)$ for any $\lambda\in[0,1]$ and by (38) we have

$$\begin{aligned}
&2\ln f\left(\sqrt{ab}\right)w\left(a^{1-\lambda}b^{\lambda}\right) \\
&\leq w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)+w\left(a^{\lambda}b^{1-\lambda}\right)\ln f\left(a^{\lambda}b^{1-\lambda}\right) \\
&\leq w\left(a^{1-\lambda}b^{\lambda}\right)\left[\ln f(a)+\ln f(b)\right]
\end{aligned} \tag{39}$$

for any $\lambda\in[0,1]$.

Integrating the inequality over $\lambda\in[0,1]$ we have

$$\begin{aligned}
&2\ln f\left(\sqrt{ab}\right)\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)d\lambda \\
&\leq \int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda+\int_0^1 w\left(a^{\lambda}b^{1-\lambda}\right)\ln f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda \\
&\leq \left[\ln f(a)+\ln f(b)\right]\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)d\lambda
\end{aligned} \tag{40}$$

and since

$$\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda=\int_0^1 w\left(a^{\lambda}b^{1-\lambda}\right)\ln f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda,$$

hence by (40) we get

$$\ln f\left(\sqrt{ab}\right)\leq\frac{\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda}{\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)d\lambda}\leq\ln\left(\sqrt{f(a)f(b)}\right). \tag{41}$$

By changing the variable $a^{1-\lambda}b^{\lambda}=t$, then $(1-\lambda)\ln a+\lambda\ln b=\ln t$ which gives that

$$\lambda=\frac{\ln t-\ln a}{\ln b-\ln a}.$$

Therefore $d\lambda=\frac{1}{t}dt$,

$$\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda=\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}w(t)\ln f(t)\,dt$$

and

$$\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)d\lambda=\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}w(t)\,dt$$

and by (41) we get the desired result (37). □

If we take in (37) $w(t)=1$, $t\in[a,b]$, then we recapture (5).

If we take in (37)

$$w(t)=\left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right|=\left|\ln t-\frac{\ln a+\ln b}{2}\right|$$

and since

$$\int_a^b \frac{1}{t} w(t)\, dt = \int_a^b \frac{1}{t}\left|\ln t - \frac{\ln a + \ln b}{2}\right| dt$$
$$= \int_a^b \left|\ln t - \frac{\ln a + \ln b}{2}\right| d\ln t$$
$$= \int_{\ln a}^{\ln b} \left|x - \frac{\ln a + \ln b}{2}\right| dx = \frac{1}{4}(\ln b - \ln a)^2$$

then we get

$$\frac{1}{4}(\ln b - \ln a)^2 \ln f\left(\sqrt{ab}\right) \leq \int_a^b \frac{1}{t}\left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right| \ln f(t)\, dt \tag{42}$$
$$\leq \frac{1}{4}(\ln b - \ln a)^2 \ln\left(\sqrt{f(a) f(b)}\right).$$

We also have:

Theorem 3.3. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$ and such that $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a,b]$. Then we have the inequalities*

$$f\left(\sqrt{ab}\right) \leq \exp\left(\frac{1}{2}\cdot\frac{\int_a^b \left(1+\frac{ab}{t^2}\right) w(t) \ln f(t)\, dt}{\int_a^b w(t)\, dt}\right) \tag{43}$$
$$\leq \frac{\int_a^b w(t)\sqrt{f(t) f\left(\frac{ab}{t}\right)} dt}{\int_a^b w(t)\, dt}$$
$$\leq \frac{\sqrt{ab\int_a^b \frac{w(t)f(t)}{t^2} dt \int_a^b w(t) f(t)\, dt}}{\int_a^b w(t)\, dt}$$
$$\leq \frac{1}{2}\frac{\int_a^b \left(1+\frac{ab}{t^2}\right) w(t) f(t)\, dt}{\int_a^b w(t)\, dt}.$$

Proof. As in the proof of Theorem 3.1 we have

$$\frac{1}{2}\int_a^b \left(1+\frac{ab}{t^2}\right) w(t) \ln f(t)\, dt = \int_a^b w(t) \ln\sqrt{f(t) f\left(\frac{ab}{t}\right)} dt.$$

Then by Jensen's inequality for the exponential and the weight w we have

$$\exp\left(\frac{1}{2}\cdot\frac{\int_a^b \left(1+\frac{ab}{t^2}\right) w(t) \ln f(t)\, dt}{\int_a^b w(t)\, dt}\right)$$
$$= \exp\left(\frac{\int_a^b w(t) \ln\sqrt{f(t) f\left(\frac{ab}{t}\right)} dt}{\int_a^b w(t)\, dt}\right)$$

$$\leq \frac{\int_a^b w(t)\exp\left(\ln\sqrt{f(t)f\left(\frac{ab}{t}\right)}\right)dt}{\int_a^b w(t)\,dt} = \frac{\int_a^b w(t)\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt}$$

that proves the second part of (43).

By Cauchy-Bunyakowsky-Schwarz inequality and the property of w we have

$$\begin{aligned}\int_a^b w(t)\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt &\leq \sqrt{\int_a^b w(t)f(t)\,dt\int_a^b w(t)f\left(\frac{ab}{t}\right)dt} \\ &= \sqrt{\int_a^b w(t)f(t)\,dt\int_a^b w\left(\frac{ab}{t}\right)f\left(\frac{ab}{t}\right)dt} \\ &= \sqrt{ab\int_a^b \frac{w(t)f(t)}{t^2}dt\int_a^b w(t)f(t)\,dt},\end{aligned}$$

which proves the third inequality in (43).

By the geometric mean - arithmetic mean inequality we also have

$$\begin{aligned}&\sqrt{ab\int_a^b \frac{w(t)f(t)}{t^2}dt\int_a^b w(t)f(t)\,dt} \\ &\leq \frac{1}{2}\left(ab\int_a^b \frac{w(t)f(t)}{t^2}+\int_a^b w(t)f(t)\,dt\right) \\ &= \frac{1}{2}\int_a^b\left(1+\frac{ab}{t^2}\right)w(t)f(t)\,dt\end{aligned}$$

that proves the last part of (43). □

If we take $w(t)=1$, $t\in[a,b]$ in (43), then we get

$$\begin{aligned}f\left(\sqrt{ab}\right) &\leq \exp\left(\frac{1}{2(b-a)}\int_a^b\left(1+\frac{ab}{t^2}\right)\ln f(t)\,dt\right) \\ &\leq \frac{1}{b-a}\int_a^b\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt \\ &\leq \frac{1}{b-a}\sqrt{ab\int_a^b\frac{f(t)}{t^2}dt\int_a^b f(t)\,dt} \\ &\leq \frac{1}{2(b-a)}\int_a^b\left(1+\frac{ab}{t^2}\right)f(t)\,dt.\end{aligned} \tag{44}$$

We observe that, if in the first inequality in (23) we take $p=\frac{1}{2}$, then we have

$$f\left(\sqrt{ab}\right) \leq \frac{1}{b-a}\int_a^b\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt. \tag{45}$$

Therefore the first part of (44) is a refinement of (45).

4. Some Particular Cases

We consider a simple example of GG-convex functions, namely $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=\exp x$. By Corollary 2.2, we have for any $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$, that

$$\exp(G(a,b))\leq\left(\frac{\int_a^b w(t)\exp\left(t+\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2}}\tag{46}$$
$$\leq\left(\frac{\int_a^b w(t)\exp\left[p\left(t+\frac{ab}{t}\right)\right]dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2p}}\leq\exp(A(a,b)),$$

for $p\geq 1$.

(ii) If $p\in(0,1)$, then

$$\exp(G(a,b))\leq\left(\frac{\int_a^b w(t)\exp\left[p\left(t+\frac{ab}{t}\right)\right]dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2p}}\tag{47}$$
$$\leq\left(\frac{\int_a^b w(t)\exp\left(t+\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2}}\leq\exp(A(a,b)).$$

From the inequality (27) applied for the GG-convex function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=\exp x$ we have

$$\exp\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}\right)\right)\tag{48}$$
$$\leq\exp\left(\frac{\int_a^b w(t)\,t dt}{\int_a^b w(t)\,dt}\right)\leq\frac{\int_a^b w(t)\exp(t)\,dt}{\int_a^b w(t)\,dt}$$
$$\leq\frac{\int_a^b w(t)\,t^{L(a,b)}dt}{\int_a^b w(t)\,dt}\exp\left(\frac{a\ln b-b\ln a}{\ln b-\ln a}\right),$$

for any $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$.

If $w:[a,b]\to[0,\infty)$ an integrable function on $[a,b]$ and such that $w\left(\frac{ab}{t}\right)=w(t)$ for any $t\in[a,b]$, then from (32) and (37) we get

$$G(a,b)\leq\frac{1}{2}\cdot\frac{\int_a^b\left(1+\frac{ab}{t^2}\right)tw(t)\,dt}{\int_a^b w(t)\,dt}\leq A(a,b)\tag{49}$$

and

$$G(a,b)\leq\frac{\int_a^b w(t)\,dt}{\int_a^b\frac{1}{t}w(t)\,dt}\leq A(a,b).\tag{50}$$

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