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SUFFICIENT CONDITIONS FOR UNIVALENCE AND STUDY OF A CLASS OF MEROMORPHIC UNIVALENT FUNCTIONS

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Abstract. In this article we consider the class $\mathcal{A}(p)$ which consists of functions that are meromorphic in the unit disc $\mathbb{D}$ having a simple pole at $z = p \in (0,1)$ with the normalization $f(0) = 0 = f'(0)-1$. First we prove some sufficient conditions for univalence of such functions in $\mathbb{D}$. One of these conditions enable us to consider the class $\mathcal{V}_p(\lambda)$ that consists of functions satisfying certain differential inequality which forces univalence of such functions. Next we establish that $\mathcal{U}_p(\lambda) \subsetneq \mathcal{V}_p(\lambda)$, where $\mathcal{U}_p(\lambda)$ was introduced and studied in [2]. Finally, we discuss some coefficient problems for $\mathcal{V}_p(\lambda)$ and end the article with a coefficient conjecture.

1. Introduction and sufficient condition for univalence

Let $\mathcal{M}$ be the set of meromorphic functions F in $\Delta = \{\zeta \in \mathbb{C} : |\zeta| > 1\} \cup \{\infty\}$ with the following expansion:

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}, \quad \zeta \in \Delta.$$

This means that these functions have simple pole at $z = \infty$ with residue 1. Let $\mathcal{A}$ be the collection of all analytic functions in $\mathbb{D} := \{z : |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$. In [1], Aksentév proved a sufficient condition for a function $F \in \mathcal{M}$ to be univalent which we state now:

Theorem A. *If $F \in \mathcal{M}$ satisfies the inequality*

$$|F'(\zeta) - 1| \le 1, \quad \zeta \in \Delta,$$

then F is univalent in Δ.

This result motivated many authors to consider the classes $\mathcal{U}(\lambda) := \{f \in \mathcal{A} : |U_f(z)| < \lambda, z \in \mathbb{D}\}, \lambda \in (0,1]$ where $U_f(z) := (z/f(z))^2 f'(z) - 1$ and this class has been studied extensively in [6,8] and references therein. In [2], we wanted to see the meromorphic analogue of the class $\mathcal{U}(\lambda)$ by introducing a nonzero

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simple pole for such functions in $\mathbb{D}$. More precisely, we consider the class $\mathcal{A}(p)$ of all functions f that are holomorphic in $\mathbb{D}\setminus\{p\}$, $p\in(0,1)$ possessing a simple pole at the point $z=p$ with nonzero residue m and normalized by the condition $f(0)=0=f'(0)-1$. We define $\Sigma(p):=\{f\in\mathcal{A}(p): f \text{ is one to one in } \mathbb{D}\}$. Therefore, each $f\in\mathcal{A}(p)$ has the Laurent series expansion of the following form

$$f(z)=\frac{m}{z-p}+\sum_{n=0}^{\infty}a_nz^n,\ z\in\mathbb{D}\setminus\{p\}. \tag{1.1}$$

In this context we proved a sufficient condition for a function $f\in\mathcal{A}(p)$ to be univalent (see [2, Theorem 1]), which we recall now.

Theorem B. *Let $f\in\mathcal{A}(p)$. If $|U_f(z)|\leq((1-p)/(1+p))^2$ for $z\in\mathbb{D}$, then f is univalent in $\mathbb{D}$.*

Using Theorem B, we constructed a subclass $\mathcal{U}_p(\lambda)$ of $\Sigma(p)$ which is defined as follows:

$$\mathcal{U}_p(\lambda):=\{f\in\mathcal{A}(p):|U_f(z)|<\lambda\mu,z\in\mathbb{D}\},$$

where $0<\lambda\leq 1$ and $\mu=((1-p)/(1+p))^2$. We urge readers to see the article [2] for many other interesting results on functions in the subclass $\mathcal{U}_p(\lambda)$. In this note, we improve the sufficient condition proved in Theorem B by replacing the number $\mu=((1-p)/(1+p))^2$ with the number 1. We give a proof of this result below.

Theorem 1. *Let $f\in\mathcal{A}(p)$. If $|U_f(z)|<1$ holds for all $z\in\mathbb{D}$, then $f\in\Sigma(p)$.*

Proof. Let $\mathcal{M}_p:=\{f\in\mathcal{M}:F(1/p)=0\}$ where $0<p<1$. Clearly, $\mathcal{M}_p\subseteq\mathcal{M}$. For each $f\in\mathcal{A}(p)$ consider the transformation $F(\zeta):=1/f(1/\zeta)$, $\zeta\in\Delta$. We claim that $F\in\mathcal{M}_p\subseteq\mathcal{M}$. Since f has an expansion of the form (1.1), therefore we have

$$\begin{aligned}
F(\zeta)&=1/f(1/\zeta)\\
&=\left(m\zeta/(1-p\zeta)+\sum_{n=0}^{\infty}a_n\zeta^{-n}\right)^{-1}\\
&=\zeta+(a_1-pa_2-1)/p\\
&\quad+\big(p(a_2-pa_3)+(a_1-pa_2)^2-(a_1-pa_2)\big)/\zeta p^2+\cdots.
\end{aligned}$$

Here we see that $F(1/p)=0, F(\infty)=\infty$ and $F'(\infty)=1$. This proves that each $f\in\mathcal{A}(p)$ can be associated with the mapping $F\in\mathcal{M}_p$. Using the change of variable $\mathbb{D}\ni z=1/\zeta$, the above association quickly yields

$$F'(\zeta)-1=f'(1/\zeta)/(\zeta^2f^2(1/\zeta))-1=z^2f'(z)/f^2(z)-1=U_f(z).$$

Now since $\mathcal{M}_p\subseteq\mathcal{M}$, an application of the Theorem A gives that if any function $F\in\mathcal{M}_p$ satisfies $|F'(\zeta)-1|\leq 1$, $\zeta\in\Delta$, then F is univalent in Δ, i.e., the inequality $|U_f(z)|<1$ forces f to be univalent in $\mathbb{D}$. $\square$

In view of the Theorem 1, it is natural to consider a new subclass $\mathcal{V}_p(\lambda)$ of $\Sigma(p)$ defined as:

$$\mathcal{V}_p(\lambda) := \{f \in \mathcal{A}(p) : |U_f(z)| < \lambda,\ z \in \mathbb{D}\} \quad \text{for } \lambda \in (0,1].$$

We now claim that $\mathcal{U}_p(\lambda) \subsetneq \mathcal{V}_p(\lambda) \subsetneq \Sigma(p)$. To establish the first inclusion, we note that as $\lambda\mu < \lambda$, therefore we have $\mathcal{U}_p(\lambda) \subseteq \mathcal{V}_p(\lambda)$. Now consider the function

$$k_p^\lambda(z) := \frac{-pz}{(z-p)(1-\lambda pz)},\ z \in \mathbb{D}.$$

It is easy to check that $U_{k_p^\lambda}(z) = -\lambda z^2$ so that $|U_{k_p^\lambda}(z)| < \lambda$ but $|U_{k_p^\lambda}(z)| \nless \lambda\mu$ for all $z \in \mathbb{D}$. This proves the first inclusion. Next we wish to establish the second inclusion of our claim. We see that by virtue of the Theorem 1, $\mathcal{V}_p(\lambda) \subseteq \Sigma(p)$. Again considering the following two examples, we see that $\mathcal{V}_p(\lambda) \subsetneq \Sigma(p)$ for $0 < \lambda \leq 1$.

Case 1: $(0 < \lambda < 1)$. Take $a \in \mathbb{C}$ such that $\lambda < |a| < 1$. Consider the functions f_a defined by

$$f_a(z) = \frac{z}{(z-p)(az-1/p)}, \quad z \in \mathbb{D}.$$

It is easy to check that f_a satisfies the normalizations $f_a(p) = \infty$ and $f_a(0) = 0 = f_a'(0) - 1$. Also $f_a(z)$ is univalent in $\mathbb{D}$ and $U_{f_a}(z) = -az^2$. Now as $|z| \to 1^-$, $|U_{f_a}(z)| \to |a| > \lambda$. Therefore $f_a(z) \notin \mathcal{V}_p(\lambda)$. This shows that $\mathcal{V}_p(\lambda)$ is a proper subclass of $\Sigma(p)$ for $0 < \lambda < 1$.

Case 2: $(\lambda = 1)$. It is well-known that the function

$$g(z) = \frac{z - 2pz^2/(1+p^2)}{(1-z/p)(1-zp)}, z \in \mathbb{D},$$

is in $\Sigma(p)$ (Compare [4]). A little calculation shows that

$$U_g(z) = \left(z(1-p^2)/(1+p^2)\right)^2 \left(1 - (2pz/(1+p^2))\right)^{-2}.$$

Now $|U_g(z)| < 1$ holds for all $|z| \leq R$ whenever $R < \frac{1+p^2}{1+2p-p^2} < 1$. From here we can conclude that g does not belongs to the class $\mathcal{V}_p(\lambda)$ for $\lambda = 1$, i.e. $\mathcal{V}_p := \mathcal{V}_p(1) \subsetneq \Sigma(p)$.

Remark. It can be easily seen that similar to the class $\mathcal{U}_p(\lambda)$, the class $\mathcal{V}_p(\lambda)$ is preserved under conjugation and is not preserved under the operations like rotation, dilation, omitted value transformation and the n-th root transformations.

Let $f \in \mathcal{A}(p)$. We see that the function z/f is analytic in $\mathbb{D}$ and non vanishing in $\mathbb{D} \setminus \{p\}$. Therefore it has a Taylor expansion of the following form about the origin.

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots,\ z \in \mathbb{D}. \tag{1.2}$$

Now we prove some sufficient conditions for univalence of functions $f \in \mathcal{A}(p)$ which involves the second and higher order derivatives of z/f. These are the contents of the next two theorems.

Theorem 2. *Let $f \in \mathcal{A}(p)$ and f/z be non-vanishing in $\mathbb{D} \setminus \{0\}$. If $|(z/f(z))''| \leq 2$ for $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$. This condition is only sufficient for univalence but not necessary.*

Proof. First we prove the univalence of f. Using the expansion (1.2), we have

$$U_f(z) = -z(z/f)' + (z/f) - 1 = \sum_{n=2}^{\infty}(1-n)b_n z^n.$$

We also note that $zU_f'(z) = -z^2(z/f)''$. Therefore $|(z/f)''| \leq 2$ yields $|zU_f'(z)| \leq 2|z|$. This implies that $zU_f'(z) \prec 2z$ where $\prec$ denotes usual subordination. Now by a well known result of subordination (compare [5, p. 76, Theorem 3.1d.]), we get $U_f(z) \prec z$, i.e., $|U_f(z)| \leq |z| < 1$. This shows that f is univalent in $\mathbb{D}$ by virtue of the Theorem 1. In order to establish the second claim of the theorem, we consider the function

$$h(z) = \frac{2pz}{(p-z)(2-pz(p+z))},\ z \in \mathbb{D}.$$

Note that $h(0) = 0 = h'(0) - 1$ and $h(p) = \infty$. Also since $|pz(p+z)| < 2$, h has no other poles in $\mathbb{D}$ except at $z = p$. Consequently $h \in \mathcal{A}(p)$. It is easy to check that $U_h(z) = -z^3$ and $(z/h)'' = 3z$. Hence $|U_h(z)| < 1$ but $|(z/h)''| > 2$ for $2/3 < |z| < 1$. This example shows that the boundedness condition in the statement of the theorem is only sufficient but not necessary. $\square$

The following theorem is also a univalence criterion described by a sharp inequality involving the n-th order derivatives of z/f (denoted by $(z/f)^{(n)}$), $n \geq 3$.

Theorem 3. *Let $f \in \mathcal{A}(p)$ and $f(z) \neq 0$ for $\mathbb{D} \setminus \{0\}$. If for $n \geq 3$,*

$$\sum_{k=0}^{n-3} \frac{k+1}{(k+2)!}|\alpha_k| + \frac{n-1}{n!}\left|\left(\frac{z}{f}\right)^{(n)}\right| \leq 1,\ z \in \mathbb{D}, \tag{1.3}$$

where $\alpha_k = -(z/f)^{(k+2)}|_{z=0}$, then f is univalent in $\mathbb{D}$. The result is sharp and equality holds in the above inequality for the function $k_p(z) = -pz/(z-p)(1-pz)$ for all $n \geq 3$ and for the functions

$$f_n(z) = \frac{z}{1 - (1/p + p^{n-1}/(n-1))\,z + z^n/(n-1)},\ z \in \mathbb{D},$$

for each $n \geq 3$.

Proof. Proceeding similarly as the proof of [7, Theorem 1.1], the inequality (1.3) will imply that $|U_f(z)| < 1$ which proves that f is univalent in $\mathbb{D}$. To complete the proof of the remaining assertion of the theorem, we consider the univalent function k_p and compute

$$(z/k_p(z))' = -(1/p+p)+2z, \quad (z/k_p(z))'' = 2 \text{ and } (z/k_p(z))^{(n)} = 0,\, n \geq 3.$$

Therefore we get $\alpha_0 = -2$ and $\alpha_k = 0$ for $k \geq 1$. Taking account of the above computations, it can now be easily checked that the equality holds in the inequality (1.3). Lastly, it can be proved that the functions $f_n \in \mathcal{V}_p(\lambda)$ for $\lambda = 1$, i.e., f_n is univalent in $\mathbb{D}$. Again for the functions f_n, it is easy to check that $\alpha_k = 0$, $0 \leq k \leq n-3$ and $(z/f_n)^{(n)} = n!/(n-1)$ for all $n \geq 3$, which essentially proves the sharpness of the result. □

Now in the following theorem we give sufficient conditions for a function $f \in \mathcal{A}(p)$ to be in the class $\mathcal{V}_p(\lambda)$ by using Theorem 1, Theorem 2 and Theorem 3 in terms of the coefficients b_n defined in (1.2).

Theorem 4. *Let $f \in \mathcal{A}(p)$ and each z/f has the expansion of the form* (1.2). *If f satisfies any one of the following three conditions namely*
(i) $\sum_{n=2}^{\infty}(n-1)|b_n| \leq \lambda$,
(ii) $\sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2\lambda$,
(iii) $\sum_{k=2}^{n}(k-1)|b_k| + (n-1)\sum_{k=n+1}^{\infty}\binom{k}{n}|b_k| \leq \lambda$,
then $f \in \mathcal{V}_p(\lambda)$.

Proof. Since z/f has the form (1.2), it is simple exercise to see that

$$U_f(z) = -\sum_{n=2}^{\infty}(n-1)b_n z^n, \quad (z/f)'' = \sum_{n=2}^{\infty} n(n-1)b_n z^{n-2}$$

and

$$\left(\frac{z}{f}\right)^{(n)} = n!b_n + \sum_{k=n+1}^{\infty}\frac{k!b_k}{(k-n)!}z^{k-n} = \sum_{k=n}^{\infty}\frac{k!b_k}{(k-n)!}z^{k-n}.$$

Therefore condition (i) and (ii) implies that $|U_f(z)| < \lambda$ and $|(z/f)''| < 2\lambda$ respectively. Again following the similar arguments of the proof of Theorem 2, we conclude that $|(z/f)''| < 2\lambda$ implies $|U_f(z)| < \lambda$. Now

$$\alpha_k = -(z/f)^{(k+2)}|_{z=0} = -(k+2)!b_{k+2}.$$

Substituting the value of α_k and $(z/f)^{(n)}$ in terms of the coefficient b_n in the left hand side of the inequality (1.3) we get

$$\begin{aligned} &\sum_{k=0}^{n-3}(k+1)|b_{k+2}| + \frac{n-1}{n!}\left|\sum_{k=n}^{\infty}\frac{k!b_k}{(k-n)!}z^{k-n}\right| \\ &\leq \sum_{k=2}^{n}(k-1)|b_k| + (n-1)\sum_{k=n+1}^{\infty}\binom{k}{n}|b_k| \\ &\leq \lambda \quad \text{(by (iii))}. \end{aligned}$$

Hence an application of Theorem 3 gives $|U_f(z)| < \lambda$. This shows that, in each case $f \in \mathcal{V}_p(\lambda)$. □

In the following section we study some coefficient problems for functions in $\mathcal{V}_p(\lambda)$ which is one of the important problem in geometric function theory.

2. Coefficient problem for the class $\mathcal{V}_p(\lambda)$

Let $f \in \mathcal{V}_p(\lambda)$ with the expansion (1.2). Now proceeding as a similar manner of ([2, Theorem 12]) we have the sharp bounds for $|b_n|$, $n \geq 2$, which is given by

$$|b_n| \leq \frac{\lambda}{n-1}, \quad n \geq 2,$$

and equality holds in the above inequality for the function

$$f(z) = \frac{z}{1 - (1/p + (\lambda p^{n-1})/(n-1))\, z + \lambda z^n/(n-1)}, \; z \in \mathbb{D}. \tag{2.1}$$

Each $f \in \mathcal{V}_p(\lambda)$ has the following Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad |z| < p. \tag{2.2}$$

Now the problem is to find out the region of variability of these Taylor coefficients $a_n(f)$, $n \geq 2$. Here we note that similar to the class $\mathcal{U}_p(\lambda)$, every $f \in \mathcal{V}_p(\lambda)$ has the following representation (see [2, Theorem 3]):

$$\frac{z}{f(z)} = 1 - \left(\frac{f''(0)}{2}\right) z + \lambda z \int_0^z w(t)dt, \tag{2.3}$$

where $w \in \mathcal{B}$. Here $\mathcal{B}$ denotes the class of functions w that are analytic in $\mathbb{D}$ such that $|w(z)| \leq 1$ for $z \in \mathbb{D}$. By using this representation formula in the following theorem we give the exact set of variability for the second Taylor coefficient of $f \in \mathcal{V}_p(\lambda)$.

Theorem 5. *Let each $f \in \mathcal{V}_p(\lambda)$ has the Taylor expansion*

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n,$$

in the disc $\{z : |z| < p\}$. Then the exact region of variability of the second Taylor coefficient $a_2(f)$ is the disc determined by the inequality

$$|a_2(f) - 1/p| \leq \lambda p. \tag{2.4}$$

Proof. Substituting $z = p$ in (2.3) we get

$$a_2(f) = \frac{f''(0)}{2} = \frac{1 + \lambda p \int_0^p w(t)dt}{p}$$

which implies

$$|a_2(f) - 1/p| = \left| \frac{\lambda p \int_0^p w(t)dt}{p} \right| \\ \leq \lambda \int_0^p |w(t)|dt \leq \lambda p.$$

Therefore $|a_2(f) - 1/p| \leq \lambda p$. A point on the boundary of the disc described by (2.4) is attained for the function

$$f_\theta(z) = \frac{z}{1 - \frac{z}{p}\left(1 + \lambda p^2 e^{i\theta}\right) + \lambda e^{i\theta} z^2},\ z \in \mathbb{D},$$

where $\theta \in [0, 2\pi]$. Also the points in the interior of the disc described in (2.4) are attained by the functions

$$f_a(z) = \frac{z}{1 - \frac{z}{p}\left(1 + \lambda a p^2\right) + \lambda a z^2},\ z \in \mathbb{D},$$

where $0 < |a| < 1$. It is easy to see that these functions belong to the class $\mathcal{V}_p(\lambda)$. This shows that the exact region of variability of $a_2(f)$ is given by the disc (2.4). □

Following consequences of the above theorem can be observed easily:

Corollary 1. *Let for some $\lambda \in (0, 1]$, $f \in \mathcal{V}_p(\lambda)$ and has the form $f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n$, in the disc $\{z : |z| < p\}$. Then $|a_2(f)| \leq 1/p + \lambda p$ and equality holds in this inequality for the function k_p^λ.*

Now the function k_p^λ is analytic in the disk $\{z : |z| < p\}$ and has the Taylor expansion as

$$k_p^\lambda(z) = \sum_{n=1}^{\infty} \frac{1 - \lambda^n p^{2n}}{p^{n-1}(1 - \lambda p^2)} z^n, \quad |z| < p.$$

Since the function k_p^λ serves as an extremal function for the class $\mathcal{V}_p(\lambda)$, the above corollary enables us to make the following:

Conjecture 1. *If $f \in \mathcal{V}_p(\lambda)$ for some $0 < \lambda \leq 1$ and has the expansion of the form (2.2), then*

$$|a_n(f)| \leq \frac{1 - \lambda^n p^{2n}}{p^{n-1}(1 - \lambda p^2)},\ n \geq 3.$$

Remark. Here we note that all the results proved in [2] and in [3] for the class $\mathcal{U}_p(\lambda)$ will also be true for the bigger function class $\mathcal{V}_p(\lambda)$ if we substitute λ in place of $\lambda\mu$ and follow the same method of proof. We also remark that the authors of [9] have also considered similar meromorphic functions and arrive at this conjectured bound for $|a_n(f)|$ (compare [9, Remark 2]), but their study of such functions comes from a different perspective.

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