

SUMS OF TRIPOTENT AND NILPOTENT MATRICES

MARJAN SHEIBANI ABDOLYOUSEFI AND HUANYIN CHEN

ABSTRACT. Let R be a 2-primal strongly 2-nil-clean ring. We prove that every square matrix over R is the sum of a tripotent and a nilpotent matrices. The similar result for rings of bounded index is proved. We thereby provide a large class of rings over which every matrix is the sum of a tripotent and a nilpotent matrices.

1. Introduction

Throughout, all rings are associative with an identity. A ring R is nil clean provided that every element in R is the sum of an idempotent and a nilpotent (see [8]). A ring R is weakly nil-clean provided that every element in R is the sum or difference of a nilpotent and an idempotent (see [2]). The subjects of nil-clean rings and weakly nil-clean rings are interested for so many mathematicians, e.g., [2–5, 7, 8, 11, 12, 14] and [16].

The purpose of this paper is to consider matrices over a new type of rings which cover (weakly) nil clean rings. An element $e \in R$ is a tripotent if $e = e^3$. We call a ring R is trinil clean provided that every element in R is the sum of a tripotent and a nilpotent. We shall explore when a matrix ring is trinil-clean, i.e., when every matrix over a ring can be written as the sum of a tripotent and a nilpotent. A ring R is 2-primal if its prime radical coincides with the set of nilpotent elements of the ring, e.g., commutative rings, reduced rings, etc. Following the authors, a ring R is strongly 2-nil-clean if every element in R is the sum of two idempotents and a nilpotent that commute. Evidently, a ring R is strongly 2-nil-clean if and only if every element in R is the sum of a tripotent and a nilpotent that commute (see [6, Theorem 2.8]). If R is a 2-primal strongly 2-nil-clean ring, we prove that $M_n(R)$ is trinil clean for all $n \in \mathbb{N}$. Furthermore, we show that the matrix ring over a strongly 2-nil-clean ring of bounded index is trinil clean. This provides a large class of rings over which every matrix is the sum of a tripotent and a nilpotent matrices.

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We use $N(R)$ to denote the set of all nilpotent elements in R and $J(R)$ the Jacobson radical of R . \mathbb{N} stands for the set of all natural numbers.

2. Structure theorems

The aim of this section is to investigate general structure of trinil clean rings which will be used in the sequel. We begin several examples of such rings.

Example 2.1. The class of trinil clean rings contains many familiar examples.

- (1) Every weakly nil-clean ring is trinil clean, e.g., strongly nil-clean rings, nil-clean rings, Boolean rings, weakly Boolean rings.
- (2) Every strongly trinil clean ring is trinil clean.
- (3) A local ring R is trinil clean if and only if $R/J(R) \cong \mathbb{Z}_2$ or \mathbb{Z}_3 , $J(R)$ is nil (see [6, Theorem 4.2]).

We also provide some examples illustrating which ring-theoretic extensions of trinil clean rings produce trinil clean rings.

Example 2.2.

- (1) Any quotient of a trinil clean ring is trinil clean.
- (2) Any finite product of trinil clean rings is trinil clean. But $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \dots$ is an infinite product of trinil clean rings, which is not trinil clean. The element $(0, 2, 2, 2, \dots) \in R$ can not written as the sum of a tripotent and a nilpotent element.
- (3) The triangular matrix ring $T_n(R)$ is trinil clean if and only if R is trinil clean.
- (4) If R is trinil clean, then so is the ring $R[[x]]/(x^n)$ ($n \in \mathbb{N}$).

Lemma 2.3. *Let R be trinil clean. Then $6 \in N(R)$.*

Proof. By hypothesis, there exists a tripotent $e \in R$ such that $w := 2 - e \in N(R)$ with $ew = we$, and so $2 - 2^3 = (e + w) - (e + w)^3 \in N(R)$. This shows that $6 \in N(R)$. \square

Theorem 2.4. *Let R be a ring. Then the following are equivalent:*

- (1) R is trinil clean.
- (2) $R = A \times B$, where A and B are trinil clean, $2 \in J(A)$ and $3 \in J(B)$.

Proof. (1) \Rightarrow (2) In view of Lemma 2.3, $6 \in N(R)$. Write $6^n = 0$ ($n \in \mathbb{N}$). Since $2^n R + 3^n R = R$ and $2^n R \cap 3^n R = 0$. By Chinese Remainder Theorem, we have $R \cong A \times B$, where $A = R/2^n R$ and $B = R/3^n R$. Clearly, $2 \in N(A)$ and $3 \in N(B)$. This implies that $2 \in J(A)$ and $3 \in J(B)$, as desired.

(2) \Rightarrow (1) This is obvious, by Example 2.2(2). \square

Lemma 2.5. *Let R be trinil clean. Then $J(R)$ is nil.*

Proof. Let $x \in J(R)$. Then $x = a + b$, $a \in N(R)$, $b = b^3$. Suppose $a^q = 0$. Then $(x - b)^{2q+1} = a^{2q+1} = 0$; hence, $b^{2q+1} \in J(R)$. Clearly, $b^2 \in R$ is an idempotent. Hence, $b^{2q+1} = b(b^2)^q = b(b^2) = b^3 = b$. It follows that $b \in J(R)$, and so $b(1 - b^2) = 0$. This implies that $b = 0$; hence, $x = a \in N(R)$. Accordingly, $J(R)$ is nil, as required. \square

Lemma 2.6 ([13, Lemma 3.5]). *Let R be a ring, let $a \in R$. If $a^2 - a \in N(R)$, then there exists a monic polynomial $f(t) \in \mathbb{Z}[t]$ such that $f(a)^2 = f(a)$ and $a - f(a) \in N(R)$.*

Theorem 2.7. *Let R be a ring and $3 \in J(R)$. Then R is trinil clean if and only if*

- (1) $R/J(R)$ is trinil clean;
- (2) $J(R)$ is nil.

Proof. (\implies) This is obvious, by Example 2.2(1) and Lemma 2.6.

(\impliedby) Let $a \in R$. Then there exists some $\overline{f^3} = \overline{f} \in R/J(R)$ such that $\overline{a - f} \in N(R/J(R))$. As $J(R)$ is nil, we see that $w := a - f \in N(R)$. Let $e = 1 - f$. Then $e^3 - e = (1 - f)^3 - (1 - f) = -3f + 3f^2 - (f^3 - f) \equiv 0 \pmod{J(R)}$. We check that $(-2e^2)^2 = 4e^4 \equiv -2e^4 \equiv -2e^2 \pmod{J(R)}$ as $6 \in J(R)$. Moreover, we have $(e + 2e^2)^2 \equiv e^2 + 4e + 4e^2 = e + 2e^2 + 3(e - e^2) + 6e^2 \equiv e + 2e^2 \pmod{J(R)}$. In light of Lemma 2.6, we can find idempotents $f(e), g(e) \in R$ such that $r := -2e^2 - f(e), s := e + 2e^2 - g(e) \in J(R)$. Here, $f(t), g(t) \in \mathbb{Z}[t]$. Then $e = (-2e^2) + (e + 2e^2) = f(e) + r + g(e) + s$. Hence, $a = 1 - e + w = (1 - f(e)) - g(e) + w - r - s$. Clearly, $(1 - f(e))g(e) = g(e)(1 - f(e))$, we see that $((1 - f(e)) - g(e))^3 = (1 - f(e)) - g(e)$ and $w - r - s \in N(R)$. Therefore R is trinil clean, as asserted. \square

3. Strongly 2-nil clean rings

In this section, we shall establish the connections between trinil clean rings and strongly 2-nil-clean rings. We have:

Lemma 3.1 (see [6, Theorem 2.8]). *Let R be a ring. Then the following are equivalent:*

- (1) R is strongly 2-nil-clean.
- (2) For all $a \in R$, $a - a^3 \in N(R)$.
- (3) Every element in R is the sum of two idempotents and a nilpotent that commute.

Recall that a ring R is right (left) quasi-duo if every right (left) maximal ideal of R is an ideal (see [17]). We come now to the following.

Theorem 3.2. *A ring R is strongly 2-nil-clean if and only if*

- (1) R is trinil clean;
- (2) R is right (left) quasi-duo;

(3) $J(R)$ is nil.

Proof. (\implies) (1) is obvious. As in the proof of Lemma 3.1, $a^3 - a \in N(R)$. It follows by [10, Theorem A1] that $N(R)$ forms an ideal of R . Hence, $N(R) \subseteq J(R)$, and then $R/J(R)$ is tripotent. In view of [9, Theorem 1], $R/J(R)$ is commutative. Let M be a right (left) maximal ideal of R . Then $M/J(R)$ is an ideal of $R/J(R)$. Let $x \in M, r \in R$. Then $\bar{r}\bar{x} \in M/J(R)$, and then $rx \in M + J(R) \subseteq M$. This shows that M is an ideal of R . Therefore R is right (left) quasi-duo. (3) follows from Lemma 2.5.

(\impliedby) Since R is trinil clean, so is $R/J(R)$. As R is right (left) quasi-duo, we can see in the same way as [5, Theorem 2.8] that $R/J(R)$ is abelian, and so it is strongly 2-nil-clean. Let $x \in R$. Then $\overline{x^3 - x} \in N(R/J(R))$ by Lemma 3.1. As $J(R)$ is nil, we see that $x^3 - x \in N(R)$. By using Lemma 3.1 again, R is strongly 2-nil-clean. \square

Corollary 3.3. *A ring R is strongly 2-nil-clean if and only if*

- (1) $R/J(R)$ is tripotent;
- (2) $J(R)$ is nil.

Recall that a ring R is NI if $N(R)$ forms an ideal of R . We now derive:

Theorem 3.4. *A ring R is strongly 2-nil-clean if and only if*

- (1) R is trinil clean;
- (2) R is NI.

Proof. (\implies) Clearly, R is trinil clean. Let $a \in R$. Then $a = e + w$ for some tripotent e and a nilpotent w that commute. Hence, $a - a^3 = w(3e^2 + 3ew + w^2 - 1) \in N(R)$, so by [9, Theorem A1], $N(R)$ is an ideal of R . That is, R is NI.

(\impliedby) Let $a \in R$. Then $a = e + w$ where $e^3 = e \in R$ and $w \in N(R)$. Hence, $a^3 - a \in N(R)$. In light of Lemma 3.1, R is strongly 2-nil-clean, as asserted. \square

Corollary 3.5. *Every 2-primal trinil clean ring is strongly 2-nil-clean.*

Proof. As every 2-primal ring is NI, the result follows from Theorem 3.4. \square

Theorem 3.6. *Let R be a ring. If $N(R)$ is commutative, then R is strongly 2-nil-clean if and only if R is trinil clean.*

Proof. (\implies) This is trivial.

(\impliedby) Step 1. $E(R)N(R) \subseteq N(R)$. This was proved as in [15, Theorem 4.1]. We include a proof for the completeness. Let $e^2 = e \in R$ and $r \in N(R)$. Write $r^{2^n} = 0$ for some $n \in \mathbb{N}$. Since $(er(1-e))^2 = ((1-e)re)^2 = 0$, we see that $er(1-e), (1-e)re \in N(R)$. Hence, $er^2e - (ere)^2 = er(1-e)re = e(er(1-e))(1-e)re = e(1-e)re(er(1-e)) = 0$. Thus, $er^4e = e(r^2)^2e = (er^2e)^2 = (ere)^4$. Repeating this procedure, $(ere)^{2^n} = er^{2^n}e = 0$, and so $ere \in N(R)$. As $N(R)$ is commutative, $(er)^2 = (ere)r \in N(R)$, and so $er \in N(R)$.

Step 2. For any $a \in R$. Then we can find $x^3 = x \in R$ and a nilpotent $w \in R$ such that $a = x + w$ and $xw = wx$. Let $r \in N(R)$. Clearly, $x^2 \in E(R)$. By the Step 1, we have $x^2r \in N(R)$. Write $(x^2r)^k = 0 (k \in \mathbb{N})$. Then $(xrx)^{k+1} = xr(x^2r)^k x = 0$; hence, $xrx \in N(R)$. As $(xr)^2 = (xrx)r$, we see that $xr \in N(R)$. Thus, $ar = (x + w)r = xr + wr \in N(R)$, and so $aN(R) \subseteq N(R)$. Likewise, $N(R)a \subseteq N(R)$. Hence R is NI. According to Theorem 3.4, R is strongly 2-nil-clean. \square

A natural problem is if the matrix ring over a strongly 2-nil-clean ring is strongly 2-nil-clean. The answer is negative as the following shows.

Example 3.7. Let $n \geq 2$. then matrix ring $M_n(R)$ is not strongly 2-nil-clean for any ring R .

Proof. Let R be a ring, and let $A = \begin{pmatrix} 1_R & 1_R \\ 1_R & 0 \end{pmatrix}$. Then $A^3 - A = \begin{pmatrix} 2 & 1_R \\ 1_R & 1_R \end{pmatrix}$. One checks that $\begin{pmatrix} 2 & 1_R \\ 1_R & 1_R \end{pmatrix}^{-1} = \begin{pmatrix} 1_R & -1_R \\ -1_R & 2 \end{pmatrix}$, and so $A^3 - A$ is not nilpotent. If $M_n(R)$ is strongly 2-nil-clean, it follows by Lemma 3.1 that $A^3 - A$ is nilpotent, a contradiction, and we are done. \square

4. Trinil clean matrix rings

The purpose of this section is to investigate when a matrix ring over a strongly 2-nil-clean is trinil clean. We now extend [3, Theorem 3] and [2, Theorem 20] as follows.

Lemma 4.1. *Let K be a field. Then the following are equivalent:*

- (1) $M_n(K)$ is trinil clean.
- (2) $K \cong \mathbb{Z}_2$ or \mathbb{Z}_3 .

Proof. (1) \Rightarrow (2) Let $0 \neq a \in K$. Choose $A = aI_n$. Then $aI_n = E + W$, where $E^3 = E$ and $W \in M_n(K)$ is nilpotent. Clearly, $E = aI_n(I_n - a^{-1}W) \in GL_n(K)$, and so $E^2 = I_n$. As K is commutative, we see that $EW = (aI_n - W)W = W(aI_n - W) = WE$. From this, we get $a^2I_n - E^2 \in M_n(K)$ is nilpotent; hence, $a^2 - 1 \in K$ is nilpotent. This shows that $a^2 = 1$. Thus, $(a - 1)(a + 1) = 0$; whence $a = 1$ or -1 . If $1 = -1$, then $K \cong \mathbb{Z}_2$. If $1 \neq -1$, then $K \cong \mathbb{Z}_3$, as required.

(2) \Rightarrow (1) This follows directly from [1, Theorem 2]. \square

Theorem 4.2. *Let R be tripotent. Then $M_n(R)$ is trinil clean for all $n \in \mathbb{N}$.*

Proof. Let $A \in M_n(R)$, and let S be the subring of R generated by the entries of A . That is, S is formed by finite sums of monomials of the form: $a_1a_2 \cdots a_m$, where a_1, \dots, a_m are entries of A . Since R is a commutative ring in which $6 = 0$, S is a finite ring in which $x = x^3$ for all $x \in S$. Thus, S is isomorphic to finite direct product of \mathbb{Z}_2 or \mathbb{Z}_3 . In light of Lemma 4.1 and Example 2.2, $M_n(S)$ is trinil clean. As $A \in M_n(S)$, A is the sum of two idempotent matrices and a nilpotent matrix over S , as desired. \square

Corollary 4.3. *Let R be regular, and let $n \geq 2$. Then R is tripotent if and only if*

- (1) R is commutative;
- (2) $M_n(R)$ is trinil clean.

Proof. (\implies) In light of [9, Theorem 1], R is commutative. By virtue of Theorem 4.2, $M_n(R)$ is trinil clean, as required.

(\impliedby) Let M be a maximal ideal of R . Then R/M is a field. By hypothesis, $M_n(R)$ is trinil clean, and then so is $M_n(R/M)$. In light of Lemma 4.1, $R/M \cong \mathbb{Z}_2$ or \mathbb{Z}_3 . Thus, R is isomorphic to the subdirect product of \mathbb{Z}_2 's and \mathbb{Z}_3 's. Since \mathbb{Z}_2 and \mathbb{Z}_3 are both tripotent, we then easily check that R is tripotent. \square

We are ready to prove the following main theorems.

Theorem 4.4. *Let R be 2-primal strongly 2-nil-clean ring. Then $M_n(R)$ is trinil clean for all $n \in \mathbb{N}$.*

Proof. According to Theorem 2.4, $R \cong R_1 \times R_2$, where R_1 and R_2 are both strongly 2-nil-clean, $2 \in J(R_1)$ and $3 \in J(R_2)$. By virtue of [6, Theorem 2.11], R_1 is strongly nil-clean. According [12, Theorem 6.1], $M_n(R_1)$ is nil-clean. As R_2 is a homomorphic image of a strongly 2-nil-clean ring, R_2 is strongly 2-nil-clean. It follows by Corollary 3.3 that $J(R_2)$ is nil and $R_2/J(R_2)$ is tripotent. In light of Theorem 4.2, $M_n(R_2/J(R_2))$ is trinil clean. Furthermore, $J(R_2) \subseteq N(R_2) = P(R_2) \subseteq J(R_2)$, we get $J(R_2) = P(R_2)$. Hence, $M_n(J(R_2)) = M_n(P(R_2)) = P(M_n(R_2))$ is nil. Obviously, $3 \in J(M_n(R_2))$. Since $M_n(R_2/J(R_2)) \cong M_n(R_2)/M_n(J(R_2))$, it follows by Theorem 2.7, that $M_n(R_2)$ is trinil clean.

Therefore $M_n(R) \cong M_n(R_1) \times M_n(R_2)$ is trinil clean, as asserted. \square

Corollary 4.5. *Let R be a commutative trinil clean ring. Then $M_n(R)$ is trinil-clean for all $n \in \mathbb{N}$.*

Recall that a ring R is weakly nil-clean if every element in R is the sum or difference of a nilpotent and an idempotent (cf. [14]).

Corollary 4.6. *Let R be a commutative weakly nil-clean ring. Then $M_n(R)$ is trinil clean for all $n \in \mathbb{N}$.*

Proof. As every commutative weakly nil-clean ring is trinil clean 2-primal ring, we obtain the result, by Theorem 4.5. \square

Example 4.7. Let $n \in \mathbb{N}$. Then $M_n(\mathbb{Z}_m)$ is trinil clean for all $n \in \mathbb{N}$ if and only if $m = 2^k 3^l (k, l \in \mathbb{N}^+, k + l \neq 0)$.

Proof. (\implies) In view of Lemma 2.3, $6 \in N(\mathbb{Z}_m)$. Hence, $m = 2^k 3^l (k, l \in \mathbb{N}^+, k + l \neq 0)$.

(\impliedby) By hypothesis, $\mathbb{Z}_m \cong \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{3^l}$. Since $J(\mathbb{Z}_{2^k}) = 2\mathbb{Z}_{2^k}$ and $\mathbb{Z}_{2^k}/J(\mathbb{Z}_{2^k}) \cong \mathbb{Z}_2$, it follows by Corollary 3.3, \mathbb{Z}_{2^k} is trinil clean. Likewise, \mathbb{Z}_{3^l} is trinil clean.

This shows that Z_m is trinil clean. This completes the proof, by Corollary 4.5. \square

Recall that a ring R is of bounded index if there exists some $n \in \mathbb{N}$ such that $x^n = 0$ for all nilpotent $x \in R$.

Lemma 4.8 ([12, Lemma 6.6]). *Let R be of bounded index. If $J(R)$ is nil, then $M_n(R)$ is nil for all $n \in \mathbb{N}$.*

Theorem 4.9. *Let R be of bounded index. If R is strongly 2-nil-clean, then $M_n(R)$ is trinil clean for all $n \in \mathbb{N}$.*

Proof. In view of Theorem 2.4, we easily see that $R \cong R_1 \times R_2$, where R_1 and R_2 are both strongly 2-nil-clean, $2 \in J(R_1)$ and $3 \in J(R_2)$. By virtue of [6, Theorem 2.11], R_1 is strongly nil-clean. It follows by [12, Theorem 6.1] that $M_n(R_1)$ is nil-clean. In light of Corollary 3.3, $J(R_2)$ is nil and $R_2/J(R_2)$ is tripotent. Hence, $M_n(R_2/J(R_2))$ is trinil clean by Theorem 4.2. Clearly, R_2 is of bounded index. In terms of Corollary 3.3, $J(M_n(R_2))$ is nil. Clearly, $3 \in J(M_n(R_2))$. As $M_n(R_2/J(R_2)) \cong M_n(R_2)/M_n(J(R_2))$, it follows by Theorem 2.7, that $M_n(R_2)$ is trinil clean. Therefore $M_n(R) \cong M_n(R_1) \times M_n(R_2)$ is trinil clean. \square

Corollary 4.10. *Let R be a ring, and let $m \in \mathbb{N}$. If $(a - a^3)^m = 0$ for all $a \in R$, then $M_n(R)$ is trinil clean for all $n \in \mathbb{N}$.*

Proof. Let $x \in J(R)$. Then $(x - x^3)^m = 0$, and so $x^m = 0$. This implies that $J(R)$ is nil. In light of [9, Theorem A.1], $N(R)$ forms an ideal of R , and so $N(R) \subseteq J(R)$. Hence, $J(R) = N(R)$ is nil. Further, $R/J(R)$ is tripotent. In light of Lemma 2.7, R is strongly 2-nil-clean. If $a^k = 0 (k \in \mathbb{N})$, then $1 - a, 1 + a \in U(R)$, and so $1 - a^2 = (1 - a)(1 + a) \in U(R)$. By hypothesis, $a^m(1 - a^2)^m = 0$. Hence, $a^m = 0$, and so R is of bounded index. This complete the proof, by Theorem 4.9. \square

A ring R is a 2-Boolean ring provided that a^2 is an idempotent for all $a \in R$.

Corollary 4.11. *Let R be a 2-Boolean ring. Then $M_n(R)$ is trinil-clean for all $n \in \mathbb{N}$.*

Proof. Let $a \in R$. Then $a^2 = a^4$. Hence, $a^2(1 - a^2) = 0$. This shows that $(1 - a^2)^2 a^2 (1 - a^2) a = 0$, i.e., $(a - a^3)^3 = 0$. In light of Corollary 4.10, the result follows. \square

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MARJAN SHEIBANI ABDOLYOUSEFI
 WOMEN'S UNIVERSITY OF SEMNAN (FARZANEGAN)
 SEMNAN, IRAN
Email address: sheibani@fgusem.ac.ir

HUANYIN CHEN
 DEPARTMENT OF MATHEMATICS
 HANGZHOU NORMAL UNIVERSITY
 HANGZHOU, P. R. CHINA
Email address: huanyinchen@aliyun.com